ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 1 Winter 1990

A NOTE ON COMPACTNESS IN L-FUZZY PRETOPOLOGICAL SPACES

A.S. MASHHOUR, A.A. RAMADAN AND M.E.Abd. EL-MONSEF

ABSTRACT. The main task of this paper is to introduce and study the concept of countable compactness, Lindelöf, almost compactness and near compactness in L-fuzzy pretopological spaces. Also, the images of such spaces are investigated. Finally, some examples of the above spaces are given.

1. Introduction. Throughout this paper, the symbol L will denote a complete lattice, with a smallest element 0 and a largest element 1, that is equipped with an order-reversing involution; for such a lattice the DeMorgan laws hold for arbitrarily indexed suprema and infima.

Let X be a nonempty set and $L^X = \{A : X \to L\}$. The elements of L^X are called *L-fuzzy subsets* of X[**4**]. If $A \in L^X$, then $A^c = 1 - A$. We denote 0_X and 1_X for the functions on X identically equal to 0 and 1 respectively. If $f : X \to Y$ and $B \in L^Y$, then $f^{-1}(B) = B \circ f$. One proves that $f^{-1} \vee_{j\in J} B_j = \vee_{j\in J} f^{-1}(B_j)$ and $\wedge_{j\in J} B_j) = \wedge_{j\in J} f^{-1}(B_j)$. If $f : X \to Y$ and $A \in L^X$, then $f(A) : Y \to L$ is defined by setting $f(A)(y) = 0$ if $f^{-1}(y) = \phi$ and $f(A)(y) = \vee_{y=f(x)} A(x)$ otherwise. One proves that $f(f^{-1}(A)) \leq A$, and if f is surjective, then $f(f^{-1}(A)) = A$. Yet, $f^{-1}(f(A)) \geq A, f(\wedge_{j\in J} A_j) \leq \wedge_{j\in J} f(A_j)$ and $f(\vee_{j\in J} A_j) = \vee_{j\in J} f(A_j)$. A *fuzzy point* P in X is an L-fuzzy set in X defined by: $P(x) = t$ for $x = x_0$ and $P(x) = 0$ otherwise. The point x_0 is the support of P, and $0 < t < 1$. For a fuzzy point P in X and $A \in L^X, P \in A$ if $P(x_0) < A(x_0)$ [**9**]. A collection $\{A_j\}_{j\in J}$, where $A_j \in L^X, \forall A_j \in L^X \forall_j \in J$, is a cover of X if and only if $\vee_{j\in J} A_j = 1_X$.

An L-fuzzy pretopology [**2**] on X is a function $a : L^X \to L^X$ which satisfies:

(P1). $a(\phi) = \phi$,

(P2). $a(A) \geq A$, for every $A \in L^X$.

The pair (X, a) is said to be an L-fuzzy pretopological space (for short, L-fps). We will consider the following particular L-fps:

Received by the editors on April 30, 1985 and in revised Form on April 21, 1986.
Keywords: L-Fuzzy sets, Fuzzy pretopological spaces.

(P3). For every $A, B \in L^X$, we have $A \leq B$ implies $a(A) \leq a(B)$; (X, a) is said to be of type I.

(P4). For every $A, B \in L^X$, we have $a(A \vee B) = a(A) \vee a(B)$: (X, a) is said to be of type D.

(P5). For every $A \in L^X$, we have $a^2(A) \equiv a \circ a(A) = a(A)$; (X, a) is said to be of type S.

If (X, a) is of type I, D and S, then (X, a) is an L-fuzzy topological space [**4**] and a is its Kuratowsky closure (cl.). We define the interior function $i_a : L^X \to L^X$ by

$$i_a(A) = (a(A^c))^c.$$

Then it is clear that the properties (P1) to (P5) become, for the interior i_a:

(P1). $i_a(\phi) = \phi$;

(P2). For every $A \in L^X$, $i_a(A) \leq A$;

(P3). For every A, $B \in L^X$, we have $A \leq B$ implies $i_a(A) \leq i_a(B)$;

(P4). For every $A, B \in L^X$, we have $i_a(A \wedge B) = i_a(A) \wedge i_a(B)$;

(P5). For every $A \in L^X$, we have $i_a^2(A) = i_a(A)$.

R. Badard in [**2**] introduced and studied the concept of continuity and compactness in L-fps.

In the present paper, we introduce and study the concept of countable compactness, Lindelöf, almost and near compactness. Also, we define strong continuity, λ-continuity, and θ-continuity since these are useful for studying the images of almost compact and nearly compact L-fps.

2. Covering axioms: compactness, Lindelöf and countable, almost, and near compactness.

DEFINITION 2.1. [**2**]. The function $\psi : L^X \to L$ is said to be a *degree of non-vacuity* if it satisfies:

(1) $\psi(\phi) = \phi$;

(2) $\psi(A) = 1$, if there exists $x \in X$ such that $A(x) = 1$;

(3) $A \leq B$ implies $\psi(A) \leq \psi(B)$.

In particular, $\psi(A) = \vee_{x \in X} A(x)$ is a degree of nonvacuity and we use this formula in the sequel.

DEFINITION 2.2. [**2**]. A type I L-fps (X, a) is 1-*compact* (respectively 2-*compact*) if, for every family $\{A_j\}_{j \in J}$ of L-fuzzy subsets of X such that $\wedge_{j \in J_0} A_j \neq 0_X$ (respectively $\psi(\wedge_{j \in J_0} A_j) \geq \alpha$), where Jo is a finite subset of J, we have $\wedge_{j \in J} a(A_j) \neq 0_X$ (respectively $\psi(\wedge_{j \in J} a(A_j)) \geq \alpha$).

DEFINITION 2.3. A function $\tilde{\psi} : L^X \to L$ is said to be the *dual* of ψ if $\tilde{\psi}(A) = \xi(\psi(A^c)$, or $\psi(A) = \xi(\tilde{\psi}(A^c))$, where $A \in L^X$ and ξ is an order-reversing involution.

DEFINITION 2.4. Let (X, a) be an L-fps. A family $\{A_j\}_{j \in J}$ of L-fuzzy subsets of X is said to be an *a-cover of* X if $\{i_a(A)\}_{j \in J}$ covers X.

DEFINITION 2.5. A type I L-fps (X, a) is 1-*Lindelöf* (respectively 2-*Lindelöf*) if, for every family $\{A_j\}_{j \in J}$ of L-fuzzy subsets of X such that $\wedge_{j \in c_0} A_j \neq 0_X$ (respectively $\psi(\wedge_{j \in c_0} A_j) \geq \alpha$), where c_0 is a countable family of J, we have $\wedge_{j \in J} a(A_j) \neq 0_X$ (respectively $\psi(\wedge_{j \in J} a(A_j)) \geq \alpha$).

DEFINITION 2.6. A type I L-fps (X, a) is *countable* 1-*compact* (respectively *countable* 2-*compact*) if, for every countable family $\{A_j\}_{j \in c_0}$ of L-fuzzy subsets of X such that $\wedge_{j \in \grave{c}_0} A \neq 0_X$ (respectively $\psi(\wedge_{j \in \grave{c}_0} A_j) \geq \alpha$), where $\grave{c}_0$ is a finite subset of c, we have $\wedge_{j \in c_o} ia(a(A_j)) \neq 0_X$ (respectively $\psi(\wedge_{j \in c_o} ia(a(A_j)) \geq \alpha)$.

DEFINITION 2.7. A type I L-fps (X, a) is *almost* 1-*compact* (respectively *almost* 2-*compact*) if, for every family $\{A_j\}_{j \in J}$ of L-fuzzy subsets of X such that $\wedge_{j \in J_0} i_a(A_j) \neq 0_X$ (respectively $\psi(\wedge_{j \in J_0} i_a(A_j)) \geq \alpha$), where J_0 is a finite subset of J, we have $\wedge_{j \in J} a(A_j) \neq 0_X$ (respectively $\psi(\wedge_{j \in J} a(A_j)) \geq \alpha$).

DEFINITION 2.8. A type I L-fps (X, a) is said to be *nearly* 1-

compact (respectively *nearly* 2-*compact*) if, for every family $\{A_j\}_{j\in J}$ of L-fuzzy subsets of X such that $\wedge_{j\in J_0} ia(A_j) \neq 0_X$ (respectively $\psi(\wedge_{j\in J_0} ia(Aj)) \geq \alpha$), where J_0 is a finite subset of J, we have $\wedge_{j\in J} ia(a(A_j)) \neq 0_X$ (respectively $\psi(\wedge_{j\in J} ia(a(A_j)) \geq \alpha)$.

Clearly, in any L-fps of type I, 1-compact (respectively 2-compact) $\rightarrow$ nearly 1-compact (respectively nearly 2-compact) $\rightarrow$ almost 1-compact (respectively almost 2-compact).

THEOREM 2.1. *Let (X, a) be a type* I *L-fps. Each of the following pairs of statements is an equivalence*:

I. (1) *(X, a) is* 1-*compact* (*respectively almost* 1-*compact, nearly* 1-*compact.*

(2) *For each a-cover $\{A_j\}_{j\in J}$ of X, there exists a finite $J_0 \subset J$ such that $\{A_j\}_{j\in J_0}$ covers X(respectively$\{a(A_j)\}_{j\in J_0}$ covers X, $\{a(A_j)\}_{j\in J_0}$ is an a-cover of X).*

II. (1) *(X, a) is* 2-*compact* (*respectively almost* 2-*compact, nearly* 2-*compact*).

(2) *For each family $\{A_j\}_{j\in J}$ of L-fuzzy subsets of X such that $\tilde{\psi}(V_{j\in J} i_a(A_j)) > \xi(\alpha)$, there exists a finite $J_0 \subset J$ such that $\tilde{\psi}(V_{j\in J_0} A_j) > \xi(\alpha)$ (respectively $\tilde{\psi}(V_{j\in J_0} a(A_j) > \xi(\alpha), \tilde{\psi}(V_{j\in J_0} i_a(a(A_j)) > \xi(\alpha))$.*

III. (1) *(X, a) is* 1-*Lindelöf.*

(2) *For each a-cover $\{A_j\}_{j\in J}$ of X, there exists a countable family c_0 of J such that $\{A_j\}_{j\in c_0}$ covers X.*

IV. (1) *(X, a) is* 2-*Lindelöf.*

(2) *For each family $\{A_j\}_{j\in J}$ of L-fuzzy subsets of X such that $\tilde{\psi}(\vee_{j\in J} i_a(A_j)) > \xi(\alpha)$, there exists a countable $c_0 \subset J$ such that $\tilde{\psi}(V_{j\in c_0} A_j) > \xi(\alpha)$.*

V. (1) (X, a) *is countable* 1-*compact.*

(2) *For each countable* a-*cover* $\{A_j\}_{j\in c_0}$ *of* X, *there exists a finite* $c_0' \subset c_0$ *such that* $\{A_j\}_{j\in c_0'}$ *covers* X.

VI. (1) (X, a) *is countable* 2-*compact.*

(2) *For each countable family* $\{A_j\}_{j\in c_0}$ *of* L-*fuzzy subsets of* X *such that* $\tilde{\psi}(V_{j\in c_0} i_a(A_j)) > \xi(\alpha)$, *there exists a finite* $c_0' \subset c_0$ *such that* $\tilde{\psi}(\vee_{j\in c_0'} A_j) > \xi(\alpha)$.

PROOF. We prove only I, II since III, V (respectively IV, VI) are analogous to I (respectively II), and, for I (respectively II), we prove only the 1-compact case (respectively 2-compact case).

I(1)⇒I(2). Let $\{A_j\}_{j\in J}$ be an a-cover of X. Assume that there is no finite $J_0 \subset J$ such that $\{A_j\}_{j\in J_0}$ covers X. Then, for every finite $J_0 \subset J$, we have $\wedge_{j\in J_0} A_j^c \neq 0_X$. Since (X, a) is 1-compact we have $\wedge_{j\in J} a(A_j^c) \neq 0_X$. Thus, $V_{j\in J} i_a(A_j) \neq 1_X$ which contradicts our assumption.

I(2)⇒I(1). Let $\{A_j\}_{j\in J}$ be a family of L-fuzzy subsets of X such that $\wedge_{j\in J_0} A_j \neq 0_X$ and $\wedge_{j\in J} a(A_j) = 0_X$. This implies that $\{A_j^c\}_{j\in J}$ is an a-cover of X. By I(2), we have $\{A_j^c\}_{j\in J}$ covers X. Thus $\wedge_{j\in J_0} A_j = 0_X$, a contradiction.

II(1)⇒II(2). Let $\{A_j\}_{j\in J}$ be a family of L-fuzzy subsets of X such that $\tilde{\psi}(\vee_{j\in J} i_a(A_j)) > \xi(\alpha)$. Assume that, for every finite $J_0 \subset J$, $\tilde{\psi}(\vee_{j\in J_0} A_j) \not> \xi(\alpha)$. This implies that $\tilde{\psi}(V_{j\in J_0} A_j) \leq \xi(\alpha)$ and hence $\xi(\tilde{\psi}(V_{j\in J_0} A_j)) = (\wedge_{j\in J_0} A_j^c) \geq \alpha$. By II(1), $\psi(\wedge_{j\in J} a(A_j^c)) \geq \alpha$. So $\xi(\tilde{\psi}(\wedge_{j\in J} a(A_j^c)) \leq \xi(\alpha)$ and $\psi(V_{j\in J} i_a(A_j^c)) \leq \xi(\alpha)$, a contradiction.

II(2)⇒II (1). Let $\{A_j\}_{j\in J}$ be a family of L-fuzzy subsets of X such that, for every finite $J_0 \subset J$, we have $\psi(\wedge_{j\in J_0} A_j) \geq \alpha$. Assume that $\psi(\wedge_{j\in J} a(A_j)) \not\geq \alpha$. So, $\psi(\wedge_{j\in J} a(A_j)) < \alpha$. We have $\xi(\psi(\wedge_{j\in J} a(A_j))) = \tilde{\psi}(\vee_{j\in J} i_a(A_j^c)) > \xi(\alpha)$. There is a finite $J_0' \subset J$ such that $\tilde{\psi}(\vee_{j\in J_0'} A_j^c) > \xi(\alpha)$. But $\psi(\wedge_{j\in J_0'} A_j) \geq \alpha$, and then $\tilde{\psi}(V_{j\in J_0'} A_j^c) \leq \xi(\alpha)$, a contradiction. □

DEFINITION 2.8. [**2**]. Let (X, a) be an L-fps and $A \in L^X$. *The trace of a on A*, denoted by a_A, is defined

$$a_A(B) = a(B) \wedge A,$$

for every subset B of A.

THEOREM 2.2. *Let (X, a) be a type* I 1-*Lindelöf* (*respectively* 2-*Lindelöf*) L-fps. *Then every closed L-fuzzy subset of X (A is said to be a closed L-fuzzy subset of X if $a(A) = A$) is* 1-*Lindelöf* (*respectively* 2-*Lindelöf*).

PROOF. This follows from proposition 5 in [**2**].

THEOREM 2.3. *Let (X, a) be a type* I *and* S L-fps. *Then the following are equivalent*:

(1) (X, a) *is* 1-*compact* (*respectively* 2-*compact*)

(2) (X, a) *is* 1-*Lindelöf and countable* 1-*compact* (*respectively* 2-*Lindelöf and countable* 2-*compact*).

PROOF. (1)$\Rightarrow$(2). Let $\{A_j\}_{j \in C_0}$ be a countable family of L-fuzzy subsets of X such that $\psi(\wedge_{j \in c_0'} A_j) \geq \alpha$, where c_0' is a finite family of c_0. By (1), we have $\psi(\wedge_{j \in C_0} a(A_j)) \geq \alpha$. Then (X, a) is countable 2-compact. This implies that $\psi(\wedge_{j \in J} a^2(A_j)) \geq \alpha$. Since (X, a) is of type S, then $\psi(\wedge_{j \in J} a(A_j)) \geq \alpha$. Hence (X, a) is 2-Lindelöf.

(2)$\Rightarrow$(1). Let (X, a) be 2-Lindelöf and countable 2-compact. Let $\{A_j\}_{j \in J}$ be a family of L-fuzzy subsets of X such that $\psi(\wedge_{j \in J_0} A_j) \geq \alpha$, where J_0 is a finite subset of J. Hence J_0 is countable and $\psi(\wedge_{j \in J_0'} a(A_j)) \geq \alpha$, where J_0' is a finite subset of J. Then

$$\psi(\wedge_{j \in J} a^2(A_j)) = \psi(\wedge_{j \in J} a(A_j)) \geq \alpha,$$

since a is of type S. Hence (X, a) is 2-compact. □

DEFINITION 2.9. An L-fps (X, a) is said to be *regular* if, for every fuzzy point P and $A \in L^X$ such that $P \in i_a(A)$, there exists $B \in L^X$ such that $P \in i_a(B) \subset a(B) \subset i_a(A)$.

THEOREM 2.4. *Let* (X, a) *be a type* I *and regular* L-fps. *The following are equivalent*:

(1) (X, a) *is* 1-*compact*;

(2) (X, a) *is almost* 1-*compact*.

PROOF. (1)⇒(2). Obvious.

(2)⇒(1). Let $\{A_j\}_{j \in J}$ be an a-cover of X. For each fuzzy point P in X, there is an A_{j_p} such that $P \in i_a(A_{j_p})$. By regularity, there exists $B_p \in L^X$ such that $P \in i_a(B_p) \subset a(B_p) \subset i_a(A_{j_p})$. Thus, the family $\{B_p\}$ is an a-cover of X. Since (X, a) is almost 1-compact, there is a finite number of fuzzy points $P_1, P_2, \ldots, P_n$ in X such that $\{a(B_{p_1}), \ldots, a(B_{p_n})\}$ covers X. It follows that $\{A_{j_{p_1}}, \cdots, A_{j_{p_n}}\}$ is a finite a-subcover of $\{A_j\}_{j \in J}$. By Theorem (2.1), (X, a) is 1-compact. □

3. Countinuity axioms: strong, λ-, and θ-continuity.

DEFINITION 3.1. [**2**]. Let (X, a) and (Y, b) be two L-fps's. A function $f : (X, a) \longrightarrow (Y, b)$ is said to be *continuous* if $f(a(A)) \subset b(f(A)$, for every $A \in L^X$.

DEFINITION 3.2. Let (X, a) and (Y, b) be two L-fps's. A function $f : (X, a) \longrightarrow (Y, b)$ is said to be *strongly continuous* if $f(a(A)) \subset f(A)$, for every $A \in L^X$.

DEFINITION 3.3. Let (X, a) and (Y, b) be two L-fps's. A function $f : (X, a) \longrightarrow (Y, b)$ is said to be λ-*continuous* (respectively θ-*continuous*) if, for each fuzzy point P in X and $B \in L^Y$ such that $f(p) \in i_b(B)$, there exists $A \in L^X$ such that $p \in i_a(A)$ *and* $f(a(A)) \subset b(B)$ (respectively $f(A) \subset i_b(b(B))$.

One can easily deduce that if f is strongly continuous, then f is continuous. Also, if f is continuous, then f is λ-continuous (respectively θ-continuous) but not conversely.

THEOREM 3.1. *Let* (X,a) *and* (Y,b) *be two* L-fps's *and* $f:(X,a)\longrightarrow(Y,b)$ *be continuous and surjective. If* (X,a) *is* 1-*Lindelöf* (*respectively* 2-*Lindelöf*), *so is* (Y,b).

The proof is as in Proposition 6 in [**2**].

The following theorem, which can be easily verified, characterizes strongly continuous functions.

THEOREM 3.2. *If* $f:(X,a)\longrightarrow(Y,b)$, *where* (X,a) *and* (Y,b) *are* L-fps's, *then the following are equivalent*:

(1) f *is strongly continuous.*

(2) $f(i_a(A))=f(A)=f(a(A)$, *for every* $A\in L^X$.

(3) $i_a(F^{-1}(B))=f^{-1}(B)=a(f^{-1}(B))$, *for every* $B\in L^Y$.

THEOREM 3.3. *Let* (X,a) *and* (Y,b) *be two type* I L-fps's *and* $f:(X,a)\longrightarrow(Y,b)$ *be strongly continuous and surjective. If* (X,a) *is almost* 1-*compact* (*respectively almost* 2-*compact*), *then* (Y,b) *is* 1-*compact* (*respectively* 2-*compact*).

PROOF. We prove only the case of 2-compactness. Let $\{A_j\}_{j\in J}$ be a family of L-fuzzy subsets of Y such that $\psi(\wedge_{j\in J_0}A_j)\geq\alpha$, where J_0 is a finite subset of J. Then $\psi(\wedge_{j\in J}ia(f^{-1}(A_j)))=\psi(\wedge_{j\in J}(f^{-1}(A_j)))$. Since (X,a) is almost 2-compact, $\psi\wedge_{j\in J}f^{-1}(A_j))=\psi(\wedge_{j\in J}a(f^{-1}(A_j))\geq\alpha$. Consequently, $\psi(\wedge_{j\in J}(b(A_j))=\psi(\wedge_{j\in J}A_j)=\psi(\wedge_{j\in J}f(f^{-1}(A_j))\geq\alpha$. Hence (Y,b) is 2-compact. □

THEOREM 3.4. *Let* (X,a) *and* (Y,b) *be two type* I L-fps's *and* $f:(X,a)\longrightarrow(Y,b)$ *be* λ-*continuous* (*respectively* θ-*continuous*) *and surjective. If* (X,a) *is almost* 1-*compact* (*respectively* 1-*compact*), *then* (Y,b) *is almost* 1-*compact* (*respectively, nearly* 1-*compact*).

PROOF. We prove only the λ-case, the θ-case being perfectly analogous. Let $\{A_j\}_{j\in J}$ be a b-cover of Y. For each fuzzy point P in X, there is an A_{j_q} such that $q = f(p) \in i_b(A_{j_q})$. Since f is λ-continuous, there is a $B_p \in L^X$ such that $P \in i_a(B_p)$ and $f(a(B_p) \subset b(A_{j_q})$. Now, $\{B_p\}$ is an a-cover of X and (X, a) is almost 1-compact. Then, there is a finite number of fuzzy points $P_1, P_2, \dots, P_n$ in X such that $\{a(B_{p_1}), \dots a(B_{p_n})\}$ covers X. Hence $\{b(A_{j_{q_1}}), \dots, b(A_{j_{q_n}})\}$, where $q_i = f(p_i)$, covers Y, i.e., (Y, b) is almost 1-compact.

4. Examples. In the following examples, let (X, a) be of types I, D and S, then the collection $\mathcal{T} = \{A \in L^X : i_a(A) = A\}$ is a fuzzy topology on X and $i_a(A) = \text{Int}(A)$.

EXAMPLE 4.1. If $a =$ identity, then 1-compactness is just compactness in the sense of [**1**, **5** and **8**]. In this case the fuzzy unit interval [**6, 7**] is 1-compact.

EXAMPLE 4.2. If $a =$ c1., then almost 1-compactness is just almost compact in the sense of [**3**], where $L = [0, 1]$.

EXAMPLE 4.3. If $a =$ c1. int, then nearly 1-compactness is just nearly compactness.

Acknowledgements. We are grateful to Prof. Robert Badard who read the manuscript and made valuable comments.

REFERENCES

1. C.L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.

2. Robert Badard, *Fuzzy pretopological spaces and their representation*, J. Math. Anal. Appl. **81** (1981), 378–390.

3. A. Di Concilio and G. Gerla, *Almost compactness in fuzzy topological spaces*, Fuzzy sets and systems **13** (1984), 187–192.

4. J.A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145–174.

5. ———, *The fuzzy Tychonoff theorem*, J. Math. Appl. **43** (1973), 734–742.

6. T.E. Santner, R.C. Steinlage and R.H. Warren, *Compactness in fuzzy topological spaces*, J. Math Anal. Appl. **62** (1978), 547–562.

7. Bruce Hutton, *Normality in fuzzy topological spaces*, J. Math. Anal. Appl. **50** (1975), 74–79.

8. C.K. Wong, *Fuzzy topology; product and quotient theorems*, J. Math. Anal. Appl. **45** (1974), 512–521.

9. ———, *Fuzzy points and local properties of fuzzy topology*, J. Math. Anal. Appl. **46** (1974), 316–328.

10. L.A. Zadeh, *Fuzzy sets*, Inform. Cont. **8** (1965), 338–353.

Mathematics Department, Faculty of Science, Assuit University, Assuit, Egypt

Mathematics Department, Faculty of Science, Assuit University, Assuit, Egypt

Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt