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WHEN DO TWO GROUPS ALWAYS HAVE ISOMORPHIC EXTENSION GROUPS?

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What is the relationship between abelian groups A and C if $\mathrm{Ext}(A,B) \cong \mathrm{Ext}(C,B)$ for all abelian groups B? (problem 43 in [**5**]). We will address this question, restricting our attention to torsion-free abelian groups A, B and C of finite rank.

Call A and C *related* if $\mathrm{Ext}(A,B) \cong \mathrm{Ext}(C,B)$ for all B. We give a characterization of this relation in §1 and use it to show

THEOREM. *Assume that one of the following hold*: (a) $\mathrm{rank}\, A = 2$; (b) *A has a semi-prime endomorphism ring*; *or* (c) *A is almost completely decomposable. Write $A = D' \oplus F' \oplus G$ with F' free, D' divisible and G reduced with* $\mathrm{Hom}(G, \mathbf{Z}) = 0$.

Then C is related to A if and only if $C = D \oplus F \oplus R$ with F free; *D is divisible and zero if* $\mathrm{OT}(A) \neq$ type $\mathbf{Q}$ *and nonzero if* $\mathrm{OT}(G) \neq$ type $\mathbf{Q}$ *and* $D' \neq 0$; *and R quasi-isomorphic to G*.

Here $\mathbf{Z}$ is the ring of integers and $\mathbf{Q}$ the field of rationals, p will denote a prime of $\mathbf{Z}$. As usual, the p-rank of A, $r_p(A) = \dim A/pA$. We show the

COROLLARY. *Assume that one of the following hold*: (a) $\mathrm{rank}\, A = 2$; (b) *A has a semi-prime endomorphism ring*; *or* (c) *A is almost completely decomposable. Then C is quasi-isomorphic to A if and only if* (i) *$r_p(C) = r_p(A)$ for all p*; (ii) *$r_p(\mathrm{Hom}(C,B)) = r_p(\mathrm{Hom}(A,B))$ for all p and groups B with* $\mathrm{rank}\, B \leq \mathrm{rank}\, A$; (iii) $\mathrm{OT}(C) = \mathrm{OT}(A)$; *and* (iv) $\mathrm{rank}\, C = \mathrm{rank}\, A$.

The notation, if undefined, appears in [**1**], and the basic ideas from [**1**] are assumed. However a few facts about the outer type of $A, \mathrm{OT}(A) =$

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$\sup\{\sigma|\sigma = \text{type}\, A/K$ for some K, A with $\text{rank}\, A/K = 1\}$, and the inner type of B, $\text{IT}(B) = \inf\{\sigma|\sigma = \text{type}\,\langle a\rangle_*, 0 \neq a \in B\}$, are given.

PROPOSITION 0. *Let* $\tau = \text{OT}(A)$ *and* $\sigma = \text{IT}(B)$.

1. *If* $G \leq A$ *and* $C \triangleleft A$, *then* $\text{OT}(G) \leq \tau$ *and* $\text{OT}(A/C) \leq \tau$.
2. $\tau \leq \text{type}\,\mathbf{Z}_p$ *if and only if* $r_p(A) = \text{rank}\, A$.
3. $\text{OT}(\text{Hom}(C, A)) \leq \tau$ *for any* C.
4. $\tau \leq \sigma$ *if and only if* $\text{rank Hom}(A, B) = (\text{rank}\, A)(\text{rank}\, B)$.
5. *If* $\{a_1, \ldots, a_n\} \subseteq A$ *is a basis for* $\mathbf{Q}A$ *then, for* $K_i = \langle a_j | j \neq i\rangle_*$, $\text{OT}(A) = \text{type}\, A/K_1 \vee \cdots \vee \text{type}\, A/K_n$.

Hence, by 5, $\text{OT}(A)$ *is manageable and, by* 4, *is a quasi-isomorphism invariant of* A.

1. Groups with a semi-prime endomorphism ring. By virtue of the fact that A is torsion-free, $\text{Ext}(A, B) = D \oplus T$ where D is torsion-free divisible and T is a divisible torsion group. We excerpt the following from [**8**] with this notation.

THEOREM 1.1. *If* $\text{Ext}(A, B) \neq 0$, *then* $\dim_{\mathbf{Q}} D = 2^{\aleph_0}$ *and* p-rank $T = r_p(A)r_p(B) - r_p(\text{Hom}(A, B))$.

A useful characterization of when $\text{Ext}(A, B) = 0$ is the following. Let $R(B)$ be the subring of $\mathbf{Q}$ generated by 1 and $1/p$ for all p with $pB \neq B$. Note $\text{type}\, R(B) \leq \text{IT}(B)$.

THEOREM 1.2. (WICKLESS [**9**]). $\text{Ext}(A, B) = 0$ *if and only if* $\text{OT}(A) \leq \text{type}\, R(B)$.

We will say that C is n-related to A if $\text{Ext}(A, B) \cong \text{Ext}(C, B)$ for all B of rank $\leq n$. Any group C can be written as $C = F \oplus C'$ with F free and $\text{Hom}(C', \mathbf{Z}) = 0$. Since C' is n-related to A if and only if C is n-related to A, we may as well assume that $\text{Hom}(C, \mathbf{Z}) = 0 = \text{Hom}(A, \mathbf{Z})$ in considering these relations.

THEOREM 1.3. *Assume that* $\mathrm{Hom}(A, Z) = \mathrm{Hom}(C, Z) = 0$ *and* $n > 0$. *The following are equivalent*:

1. C *is* n*-related to* A.

2. (i) $r_p(C) = r_p(A)$ *for all* p; (ii) $r_p(\mathrm{Hom}(C, B)) = r_p(\mathrm{Hom}(A, B))$ *for all* p *and all* B *with* $\mathrm{rank}\ B \leq n$; *and* (iii) $\mathrm{OT}(C) = \mathrm{OT}(A)$.

PROOF. We will first show that if $\mathrm{Ext}(A, B) = 0$, then $r_p(A)r_p(B) - r_p(\mathrm{Hom}(A, B)) = 0$ for all p. Consider $0 \to B \to QB \to T \to 0$ and note that p-rank $T = r_p(B)$ [**1**, Theorem 2]. By Theorem 1.2 and Proposition 0, $\mathrm{rank}\ \mathrm{Hom}(A, B) = (\mathrm{rank}\, A)(\mathrm{rank}\, B)$, and consequently the sequence $0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, QB) \to \mathrm{Hom}(A, T) \to 0$ is exact and $\mathrm{Hom}(A, T)$ is a torsion group. In this case $\mathrm{Hom}(A, QB)$ is the divisible hull of $\mathrm{Hom}(A, B)$, and therefore p-rank $\mathrm{Hom}(A, T) = r_p(\mathrm{Hom}(A, B))$ [**1**, Theorem 2]. But, by [**8**, Theorem 1], p-rank $\mathrm{Hom}(A, T) = r_p(A)(p\text{-rank } T) = r_p(A)r_p(B)$.

Hence the torsion subgroup of $\mathrm{Ext}(A, B)$ always has p-rank equal to $r_p(A)r_p(B) - r_p(\mathrm{Hom}(A, B))$.

$(1 \Rightarrow 2)$. Let $\mathrm{rank}\, B \leq n$. Since the torsion subgroups of $\mathrm{Ext}(A, B)$ and $\mathrm{Ext}(C, B)$ are isomorphic, $(*)$ $r_p(A)r_p(B) - r_p(\mathrm{Hom}(A, B)) = r_p(C)r_p(B) - r_p(\mathrm{Hom}(C, B))$ for every p.

In particular, $r_p(A) = r_p(A)r_p(\mathbf{Z}) = r_p(C)r_p(\mathbf{Z}) = r_p(C)$ for all p. We can solve $(*)$ to get (ii).

Assume that not both $\mathrm{OT}(C)$ and $\mathrm{OT}(A)$ equal $\bar{\infty}(\bar{\infty} = \mathrm{type}\, \mathbf{Q})$. Say $\mathrm{OT}(A) = \tau < \bar{\infty}$. Let p be such that $\tau \leq \mathrm{type}\, \mathbf{Z}_p$. By Theorem 1.2, $\mathrm{Ext}(A, \mathbf{Z}_p) = 0 = \mathrm{Ext}(C, \mathbf{Z}_p)$, and, by the same theorem, $\mathrm{OT}(C) \leq \mathrm{type}\, \mathbf{Z}_p < \bar{\infty}$. By proposition 0, $r_p(C) = \mathrm{rank}\, C$.

Let X be a rank-1 group of type τ. Since, by Proposition 0, $\mathrm{OT}(\mathrm{Hom}(A, X)) \leq \tau, r_p(\mathrm{Hom}(A, X)) = \mathrm{rank}\ \mathrm{Hom}(A, X) = \mathrm{rank}\, A = r_p(A)$. Therefore $r_p(C) = r_p(\mathrm{Hom}(C, X)) = \mathrm{rank}\ \mathrm{Hom}(C, X) = \mathrm{rank}\, C$ and $\mathrm{OT}(C) \leq \mathrm{type}\, X = \tau$.

Since we can repeat the argument to show $\mathrm{OT}(C) \geq \tau$, we have established (iii).

$(2 \Rightarrow 1)$. Let $\mathrm{rank}\, B \leq n$. By (iii) and Theorem 1.2, $\mathrm{Ext}(C, B) = 0$ if and only if $\mathrm{Ext}(A, B) = 0$. If $\mathrm{Ext}(C, B) \neq 0$ then, by (i), (ii) and

Theorem 1.1, the torsion subgroups of $\mathrm{Ext}(C, B)$ and $\mathrm{Ext}(A, B)$ are isomorphic. Whence $\mathrm{Ext}(C, B) \cong \mathrm{Ext}(A, B)$ in this case, too. □

If A and C are quasi-isomorphic, then they are n-related for all n. However if $\mathrm{OT}(A) = \bar{\infty}$, then $A \oplus \mathbf{Q}$ is related to A by the above.

Also if A is a nonzero divisible group, then $\mathrm{Ext}(A, B)$ is a vector space of dimension $2^{\aleph_0}$ over $\mathbf{Q}$. For C to be related to A, C must be a nonzero divisible group plus a free group.

For $E(A) = \mathrm{Hom}(A, A)$, let $N = N(E(A))$ denote the nilradical of $E(A)$ which is the ideal generated by all of the nilpotent right ideals of $E(A)$. We say that $E(A)$ is semi-prime if $N = 0$ or, equivalently, if $\mathbf{Q}E(A) = \mathbf{Q} \otimes E(A)$ is a semi-simple ring. Call A semi-prime if $E(A)$ is a semi-prime ring and note that being semi-prime is a quasi-isomorphism invariant (Chapter 9 in [**1**]).

By a result of J. Reid [**7**, Corollary 4.3], if A is strongly indecomposable, then A is semi-prime if and only if every endomorphism of A is a monomorphism. If C is quasi-isomorphic to A we will write $C \sim A$. Let $S_A(C)$ denote $\langle f(A) \mid f : A \to C \rangle$.

THEOREM 1.4. *Let A be semi-prime of rank n. The following are equivalent*:

1. *C is related to A.*

2. *C is n-related to A.*

3. (i) *If A is a free group plus a divisible group, then $C = D \oplus F$ with F free, and D divisible and $D = 0$ if and only if A is reduced.*

(ii) *Otherwise $C = D \oplus F \oplus R$, where F is free, D is divisible and zero if $\mathrm{OT}(A) \neq \bar{\infty}$ and R is quasi-isomorphic to A.*

PROOF. Write $A \doteq A_1^{n_1} \oplus \cdots \oplus A_k^{n_k}$ ($\doteq$ means quasi equal) with each A_i strongly indecomposable and $A_i \nsim A_j$ if $i \neq j$. Since $A_1^{n_1} \oplus \cdots \oplus A_k^{n_k}$ is semi-prime and C is related to A if and only if C is related to $A_1^{n_1} \oplus \cdots \oplus A_k^{n_k}$, we may assume without loss of generality that $A = A_1^{n_1} \oplus \cdots \oplus A_k^{n_k}$.

Suppose $f : A_i \to A_j$ with $i \neq j$ and regard $f \in E = E(A)$. We will show that $I = fE$ is nilpotent so that $f = 0$.

Let $g \in E$ and, for the natural maps $\pi_i : A \to A_i \subseteq A$ and $\pi_j : A \to A_j \subseteq A$, let $h = \pi_i g \pi_j$. Then $fgf = fhf$ and $(fg)^n = f(hf)^{n-1}g$ with $hf \in E(A_i)$. If hf is a monomorphism, then $\alpha = hf$ is invertible in $\mathbf{Q}E(A_i)$ (Proposition 6.1 in [**1**]). If $k \neq 0$ is such that $k\alpha^{-1} \in E(A_i)$, then $(k\alpha^{-1}h)f = k1_{A_i}$, and, for $u = k\alpha^{-1}h, A_j \supseteq f(A_i) \oplus \ker u \supseteq kA_j$. Since A_j is strongly indecomposable and f is a monomorphism, $\ker u = 0$, which contradicts the assumption that $A_i \nsim A_j$. Whence $hf \in NE(A_i)$ is nilpotent. If $(hf)^n = 0$, then $(fg)^{n+1} = f(hf)^n g = 0$ so that fg is nilpotent. Since $\mathbf{Q}I$ is finite dimensional and contains only nilpotent elements, it is easy to check that $\mathbf{Q}I$ hence I is nilpotent.

Therefore $E(A) = E(A_1^{n_1}) \times \cdots \times E(A_k^{n_k})$. Since $E(A_i^{n_i})$ could have no nilpotent ideals, $0 = N(E(A_i^{n_i})) = \mathrm{Mat}_{n_i}(N(A_i))$ and A_i is semi-prime and strongly indecomposable. If A is divisible, the theorem follows from Theorem 1.1.

$(2 \Rightarrow 3)$. Assume that C is n-related to A. If A is free, then C is clearly free by Theorem 1.3. Otherwise, if $A = F' \oplus A'$ with F' free and $\mathrm{Hom}(A', Z) = 0$, then $\mathrm{Hom}(F', A')$ is a nilpotent ideal in $E(A)$. Hence $F' = 0$ and $\mathrm{Hom}(A, Z) = 0$. Similarly A is reduced since A is not divisible.

Writing $C = B \oplus F$ with F free and $\mathrm{Hom}(B, Z) = 0$ we see that B is n-related to A. Write $B = D \oplus R$ with D divisible and R reduced. We will show that $R \sim A$.

Now, by Theorem 1.3, $r_p(R) = r_p(A)$ and $r_p(\mathrm{Hom}(R, G)) = r_p(\mathrm{Hom}(A, G))$ for every p and G with $\operatorname{rank} G \leq n$. We will show, by induction on $\operatorname{rank} R$, that if, for some summand $K = A_1^{m_1} \oplus \cdots \oplus A_k^{m_k}$ of A we have $r_p(\mathrm{Hom}(R, G)) = r_p(\mathrm{Hom}(K, G))$ for all p and groups G with $\operatorname{rank} G \leq n$, then $R \sim A_1^{e_1} \oplus \cdots \oplus A_k^{e_k} \oplus R'$ with R' satisfying $S_A(R') \leq R'[A]$.

If $\operatorname{rank} R = 0$, then there is nothing to show. Otherwise let $K = A_1^{m_1} \oplus \cdots \oplus A_k^{m_k}$ with $r_p(\mathrm{Hom}(R, G)) = r_p(\mathrm{Hom}(K, G))$ for all p and G having $\operatorname{rank} G \leq n$. If $S_A(R) \leq R[A]$, then we are finished. Assume $S_A(R) = \sum_i S_{A_i}(R) \not\leq R[A]$.

If $pK \neq K$, then $r = \operatorname{rank} \mathrm{Hom}(R, A) \geq r_p(\mathrm{Hom}(R, K)) = r_p(\mathrm{Hom}(K, K)) > 0$. Using standard arguments, we may embed

$R/R[A]$ into $A^r = A_i^{rn_1} \oplus \cdots \oplus A_k^{rn_k}$ and identify $R/R[A]$ with its image. Take $\pi_j : A^r \to A_j^{rn_j}$ and $\pi : R \to R/R[A]$ to be the natural maps.

Since $S_{A_j}(R) \not\leq R[A]$ for some j, there is an $f : A_j \to R$ such that $f(A_j) \not\leq R[A]$. Since $\mathrm{Hom}(A_j, A_i) = 0$ if $i \neq j$, then $\pi_i \pi f(A_j) = 0$ and $\pi_j \pi f(A_j) = (f(A);\ +R[A])/R[A] \neq 0$. Therefore there is a $g \in \mathrm{Hom}(R, A_j)$ such that $0 \neq gf \in E(A_i)$. But A_j is strongly indecomposable and semi-prime so that gf is a monomorphism. As before, gf is invertible in $\mathbf{Q}E(A_j)$ and we get a quasi-splitting of $R \xrightarrow{g} A_j$, i.e., $R \sim A_j \oplus R_1$. Now $r_p(\mathrm{Hom}(R,G)) = r_p(\mathrm{Hom}(A_j \oplus R_1, G))$ for every p and G of rank $\leq n$. If $pA_j \neq A_j$, then $r_p(\mathrm{Hom}(A_j \oplus R_1, A_j)) = r_p(\mathrm{Hom}(K, A_j)) = m_j r_p(E(A_j)) \geq r_p(E(A_j)) > 0$ and $m_j \neq 0$.

Hence $r_p(\mathrm{Hom}(R,G)) = r_p(\mathrm{Hom}(A_j,G)) + r_p(\mathrm{Hom}(R_1,G)) = r_p(\mathrm{Hom}(A_j,G)) + r_p(\mathrm{Hom}(A_1^{l_1} \oplus \cdots \oplus A_k^{l_k}, G)) = r_p(\mathrm{Hom}(K,G))$ for all p where $l_i = m_i$ if $i \neq j$; $l_j = m_j - 1$ and induction applies to R_1.

Whence $R \sim A_1^{e_1} \oplus \cdots \oplus A_k^{e_k} \oplus R'$ with $S_A(R') \leq R'[A]$ as desired. Returning to the proof, $r_p(\mathrm{Hom}(R, A_i)) = r_p(\mathrm{Hom}(A, A_i)) = n_i r_p(E(A_j)) \geq e_i r_p(E(A_j))$ from which we infer that $n_i \geq e_i$ for every i.

Consider $K = A_1^{n_1 - e_1} \oplus \cdots \oplus A_k^{n_k - e_k}$. Clearly $r_p(\mathrm{Hom}(R', G)) = r_p(\mathrm{Hom}(K,G))$ for all p and G of rank $\leq n$. If $\operatorname{rank} R' > 0$, then R' has a quasi-summand A_j for some j and $S_{A_j}(R') \leq S_A(R') \leq R'[A]$ is impossible. Thus $R' = 0$ which implies that $r_p(\mathrm{Hom}(K,K)) = 0$ for all p. Since K is reduced, $K = 0$, and $n_i = e_i$ for all i. Whence $R \sim A$.

If $\mathrm{OT}(A) \neq \bar{\infty}$, then $\mathrm{OT}(C) \neq \bar{\infty}$ by Theorem 1.3 and D must be zero.

$(3 \Rightarrow 1)$. The case when 3(i) holds is covered by the remark preceding the statement of the theorem. If $C = D \oplus R \oplus F$ as stated, then $\mathrm{OT}(D \oplus R) = \bar{\infty}$ if $\mathrm{OT}(A) = \bar{\infty}$ and $\mathrm{OT}(D \oplus R) = \mathrm{OT}(R) = \mathrm{OT}(A)$ if $\mathrm{OT}(A) \neq \bar{\infty}$. Since $r_p(\mathrm{Hom}(D \oplus R, G)) = r_p(\mathrm{Hom}(R,G)) = r_p(\mathrm{Hom}(A,G))$ for all p and G, $D \oplus R$, hence C, is related to A. □

Corollary 1.5. *If A is semi-prime, then C is quasi-isomorphic to A if and only if* (i) $r_p(C) = r_p(A)$ *for all* p; (ii) $r_p(\mathrm{Hom}(C,G)) =$

$r_p(\operatorname{Hom}(A,G))$ *for all* p *and* G *with* $\operatorname{rank} G \leq \operatorname{rank} A$; (iii) $\operatorname{OT}(C) = \operatorname{OT}(A)$; *and* (iv) $\operatorname{rank} C = \operatorname{rank} A$.

PROOF. ($\Leftarrow$) If A is free, then $\operatorname{OT}(A) = \operatorname{OT}(C)$ implies C is free and (iv) implies $A \cong C$. If A is divisible, then (i) and (iv) imply $A \cong C$. Otherwise, if $C \sim F \oplus D \oplus A$ then, by (iv), $F = D = 0$. □

Any finite rank group A can be decomposed $A = D \oplus F \oplus A'$ with D divisible, F free and A' reduced with $\operatorname{Hom}(A', \mathbf{Z}) = 0$. Call A' a *free* and *divisible complementary* summand of A, or *fdc-summand* for short.

2. When is C related to an almost completely decomposable group A? The simplest example of a group without a semi-prime endomorphism ring is $A = X \oplus Y$ where $X, Y \leq Q$ with type $X <$ type Y. The nilradical of $E(A)$ is $\operatorname{Hom}(X,Y)$.

THEOREM 2.1. *Let A be completely decomposable with a linearly ordered typeset and $n = \operatorname{rank} A$. The following are equivalent*:

1. *C is related to A.*
2. *C is n-related to A.*
3. (i) *If A is a free group plus a divisible, then $C = D \oplus F$ with F free and D divisible and zero if and only if A is reduced.*

(ii) *Otherwise write $C = D \oplus F \oplus R$ and $A = D' \oplus F' \oplus R'$ with R and R' respective* fdc-*summands, D and D' divisible groups and F and F' free groups. Then R is isomorphic to R', and D is zero if* $\operatorname{OT}(A) < \bar{\infty}$, *and nonzero if both* $\operatorname{OT}(R') < \bar{\infty}$ *and* $D' \neq 0$.

PROOF. If $A = D' \oplus F'$ with D' divisible and F' free, then $C = D \oplus F$ with D divisible and F free with the further restriction that $D = 0$ if and only if $D' = 0$. We will exclude this case in the following.

($2 \Rightarrow 3$). Suppose C is n-related to A. We may assume that $\operatorname{Hom}(A, \mathbf{Z}) = 0$. Write $A = G \oplus D'$ with D' divisible and G reduced, and write $C = D \oplus F \oplus R$ with D divisible, F free and R an fdc-summand. We will show that $R \cong G$.

Write $G = G_1 \oplus \cdots \oplus G_n$ with $G_i = X_i^{r_i}$ where X_i is a rank-1 group of type τ_i. Assume that $\tau_i < \tau_j$ if $i < j$. We will construct an embedding of R into G below.

We note that $r_p(\operatorname{Hom}(R, B)) = r_p(\operatorname{Hom}(C, B)) = r_p(\operatorname{Hom}(A, B)) = r_p(\operatorname{Hom}(G, B))$ and that $r_p(R) = r_p(G)$ for all p and all B of rank $\leq n$. For a group K and a rank-1 group X it is easy to check that $f_1, \ldots, f_l \in \operatorname{Hom}(K, X)$ are independent if and only if $\operatorname{rank} K - \operatorname{rank}(\cap_{i=1}^{l} \ker f_i) = l$ (Proposition 0).

Since, for $pX_1 \neq X_1, r_p(\operatorname{Hom}(R, X_1)) = r_p(\operatorname{Hom}(G, X_1)) = r_1$ (Proposition 0), there are linearly independent maps $g_1, \ldots, g_{r_1} \in \operatorname{Hom}(R, X_1)$. Define $\theta_1 : R \to G_1$ by $\theta_1(x) = (g_1(x), \ldots, g_{r_1}(x))$. Then $\operatorname{rank} \operatorname{Im} \theta_1 = \operatorname{rank} R - \operatorname{rank} \ker \theta_1 = r_1$ and $\operatorname{coker} \theta_1$ is torsion.

Assume that a map $\theta : R \to G_1 \oplus \cdots \oplus G_k$ has been constructed so that $\operatorname{coker} \theta$ is a torsion group T, for $k < n$. Now $\operatorname{OT}(\operatorname{Im} \theta) \leq \operatorname{OT}(G_1 \oplus \cdots \oplus G_k) = \tau_k < \tau_{k+1}$ by Proposition 0. This implies rank $\operatorname{Hom}(\operatorname{Im} \theta, X_{k+1}) = \operatorname{rank} \operatorname{Im} \theta = \operatorname{rank} \operatorname{Hom}(G_1 \oplus \cdots \oplus G_k, X_{k+1}) = r_1 + \cdots + r_k$.

From $0 \to \ker \theta \to R \to \operatorname{Im} \theta \to 0$ we derive $0 \to \operatorname{Hom}(\operatorname{Im} \theta, X_{k+1}) \to \operatorname{Hom}(R, X_{k+1}) \xrightarrow{\alpha} \operatorname{Hom}(\ker \theta, X_{k+1})$. If $pX_{k+1} \neq X_{k+1}$, then $r_p(\operatorname{Hom}(R, X_{k+1})) = \operatorname{rank} \operatorname{Hom}(R, X_{k+1}) = r_p(\operatorname{Hom}(G, X_{k+1})) = \operatorname{rank} \operatorname{Hom}(G, X_{k+1}) = r_1 + \cdots + r_{k+1}$. Hence there are linearly independent maps $f_1, \ldots, f_{r_{k+1}} \in \operatorname{Hom}(R, X_{k+1})$ so that $\alpha f_1, \ldots, \alpha f_{r_{k+1}}$ are linearly independent in $\operatorname{Hom}(\ker \theta, X_{k+1})$. Define $\phi : R \to G_1 \oplus \cdots \oplus G_{k+1}$ by $\phi(x) = (\theta(x), (f_1(x), \ldots, f_{r_{k+1}}(x)))$. Since $\ker \phi = \ker \theta \cap \cap_i \ker f_i$, $\operatorname{rank}(R/\ker \phi) = \operatorname{rank}(R/\ker \theta) + \operatorname{rank}(\ker \theta / \ker \phi) = \operatorname{rank}(R/\ker \theta) + \operatorname{rank}(\ker \theta / \cap_i \ker \alpha f_i) = r_1 + \cdots + r_k + r_{k+1} = \operatorname{rank}(G_1 \oplus \cdots \oplus G_{k+1})$ and $\operatorname{coker} \phi$ is torsion.

We have constructed a map $\theta : R \to G$ with $T = \operatorname{coker} \theta$ a torsion group. By [**1**, Theorem 2], $r_p(R/\ker \theta) + \dim T/pT = r_p(G) + \dim T[p]$ for all p where $T[p] = \{x \in T \mid px = 0\}$. Since $\dim T/pT \leq \dim T[p]$ and $r_p(R) = r_p(R/\ker \theta) = r_p(\ker \theta) = r_p(G)$ for all p, the inequality $r_p(G) \leq r_p(R/\ker \theta) \leq r_p(R)$ implies $r_p(\ker \theta) = 0$ for all p. Whence $\ker \theta = 0$ since R is reduced.

Let p satisfy $pX_n \neq X_n$. From $0 \to R \xrightarrow{\theta} G \to T \to 0$ we derive $0 \to \operatorname{Hom}(G, R) \to \operatorname{Hom}(R, R)$. Since $\operatorname{OT}(R) \leq \operatorname{OT}(G) = \operatorname{type} X_n$, $r_p(R) = \operatorname{rank} R$. By Proposition 0, $r_p(\operatorname{Hom}(R, R)) = \operatorname{rank}$

Hom$(R, R) =$ rank Hom$(G, R) = r_p($Hom$(G, R))$. Let $m \neq 0$ satisfy $m1_r = f \in$ Hom(G, R). Since f is clearly 1-1, $G \sim R$ [**1**, Corollary 6.2]. By a theorem of Beaumont-Pierce [**1**, Theorem 2.3] $G \cong R$.

Now $R \oplus D$ is related to $A = G \oplus D'$ so that $\mathrm{OT}(R \oplus D) = \mathrm{OT}(R) \vee \mathrm{OT}(D) = \mathrm{OT}(A)$ by Theorem 1.3. Clearly $D = 0$ if $\mathrm{OT}(A) < \bar{\infty}$. If $\mathrm{OT}(A) = \bar{\infty}$ but $\mathrm{OT}(G) < \bar{\infty}$, then $D \neq 0$ in order for $\mathrm{OT}(C) = \bar{\infty}$.

$(3 \Rightarrow 1)$. Assume $\mathrm{Hom}(C, \mathbf{Z}) = \mathrm{Hom}(A, \mathbf{Z}) = 0$ and suppose $C = R \oplus D$ with D divisible, R reduced and $R \oplus D' = A$ for some divisible group D'. Moreover $D = 0$ if and only if $D' = 0$ since the hypothesis that R is completely decomposable with linearly ordered typeset implies $\mathrm{OT}(R) < \bar{\infty}$. Therefore $\mathrm{OT}(C) = \mathrm{OT}(A)$, the hypotheses of Theorem 1.3 hold, and C is related to A. □

Remark. The n in part 2 could be taken to be the maximum rank of a fdc-summand of A in the case that A is not a free plus a divisible.

Let X be a rank-1 group which is neither free nor divisible. It is easy to construct a strongly indecomposable group C of rank-2 such that any rank-1 image of C is isomorphic to X. Then, from Theorem 1.3, C is 1-related to $A = X \oplus X$, but not 2-related. If, however, both C and A are presumed almost completely decomposable, then C is related to A if and only if C is 1-related to A.

Unfortunately the proof of Theorem 2.1 does not go through under the assumption that A is almost completely decomposable. To prove the analogue in this case we must use several results about Butler groups. Recall that A is a Butler group if A is a pure subgroup of a completely decomposable group G (see [**2**] and [**3**]).

For a set S of primes, let A_S be the localization of A at S. That is, for $\mathbf{Z}_S$ the subring of $\mathbf{Q}$ generated by 1 and $1/p$ if $p \in S$, $A_S = \mathbf{Z}_S \otimes A$. We identify $A \leq A_S$ as usual. Using the notation from [**9**] let $\mathrm{supp}\, A = \{p \,|\, pA \neq A\}$. Note that if C and A are n-related and $\mathrm{Hom}(C, \mathbf{Z}) = \mathrm{Hom}(A, \mathbf{Z}) = 0$, then $r_p(A) = r_p(C)$ for all p. In this case $\mathrm{supp}\, C = \mathrm{supp}\, A$.

LEMMA 2.2. *Assume* $\mathrm{Hom}(A, \mathbf{Z}) = \mathrm{Hom}(C, \mathbf{Z}) = 0$. *Then C is n-related to A if and only if C_S is n-related to A_S for every set S of primes.*

PROOF. We will only consider the necessity since, if S is the set of all primes, $C_S = C$ and $A_S = A$. Let S be a set of primes. Then $\mathrm{OT}(A_S) = \mathrm{OT}(A) + \mathrm{type}\,\mathbf{Z}_S = \mathrm{OT}(C) + \mathrm{type}\,\mathbf{Z}_S = \mathrm{OT}(C_S)$ (see Exercise 1.2 in [**1**]). Therefore $\mathrm{Ext}(A_S, B) = 0$ if and only if $\mathrm{Ext}(C_S, B) = 0$ by Theorem 1.2.

If $\mathrm{supp}\, B \subseteq S$, then QB/B has a zero p-component if $p \notin S$ and both A_S/A and C_S/C have a zero p-component if $p \in S$. This implies $0 = \mathrm{Hom}(A_S/A, \mathbf{Q}B/B) = \mathrm{Hom}(C_S/S, \mathbf{Q}B/B) = \mathrm{Ext}(A_S/A, B) = \mathrm{Ext}(C_S/C, B)$ so that $\mathrm{Ext}(A_S, B) \cong \mathrm{Ext}(A, B) \cong \mathrm{Ext}(C, B) \cong \mathrm{Ext}(C_S, B)$.

Note $\mathrm{supp}\, A_S = (\mathrm{supp}\, A) \cap S = (\mathrm{supp}\, C) \cap S = \mathrm{supp}\, C_S$ since $r_p(A) = r_p(C)$ for all p. For B with $\mathrm{supp}\, B \not\subseteq S$ take $P = (\mathrm{supp}\, B) \setminus \mathrm{supp}\, A_S$. Let $P^\omega B$ be the pure subgroup $\{b \in B \mid b \in p^m B$ for all $p \in P$ and $m \in \mathbf{Z}\}$. Then $P^\omega(B/P^\omega B) = 0$ (defined analogously). If $f : A_S \to B$ then $\mathrm{Im}\, f$ is p-divisible for all $p \in P$. Furthermore, $\mathrm{Hom}(A_S, B/P^\omega B) = 0$. A similar arrangement holds for C. Thus

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}(C_S, P^\omega B) & \to & \mathrm{Ext}(C_S, B) & \to & \mathrm{Ext}(C_S, B/P^\omega B) & \to 0 \\
& \| & & & & \| & \\
0 \to & \mathrm{Ext}(A_S, P^\omega B) & \to & \mathrm{Ext}(A_S, B) & \to & \mathrm{Ext}(A_S, B/P^\omega B) & \to 0
\end{array}$$

has split exact rows.

The first isomorphism is due to $\mathrm{supp}\, P^\omega B = (\mathrm{supp}\, B) \setminus P \subseteq S$. The latter isomorphism is due to $r_p(A_S) r_p(B/p^\omega B) = 0 = r_p(C_S) r_p(B/p^\omega B)$ for all p. This implies the torsion subgroups of $\mathrm{Ext}(A_S, B/p^\omega B)$ and $\mathrm{Ext}(C_S, B/p^\omega B)$ are isomorphic. By the opening remark and Theorem 1.1, the two groups are isomorphic. Therefore $\mathrm{Ext}(A_S, B) \cong \mathrm{Ext}(C_S, B)$. $\square$

Some key properties are preserved by our relation.

PROPOSITION 2.3. *Assume that* C *is* n*-related to* A *for* $n = \operatorname{rank} A$.

(i) *If* A *is locally completely decomposable*, *then* C *is locally completely decomposable*.

(ii) *If* A *is a Butler group*, *then* C *is a Butler group*.

PROOF. In the proof we may assume that $\operatorname{Hom}(C, \mathbf{Z}) = \operatorname{Hom}(A, \mathbf{Z}) = 0$.

(i). By Lemma 2.2, C_p is n-related to A_p. Hence, by Theorem 2.1, C_p is completely decomposable.

(ii). By Theorem 1.12 in [**2**], A is a Butler group if and only if there is a partition $S_1, \dots, S_m$ of the set of primes such that A_S is completely decomposable with a linearly ordered typeset for all $S \in \{S_1, \dots, S_m\}$.

By Lemma 2.2, C_S is n-related to A_S for any $S \in \{S_1, \dots, S_m\}$ and, by Theorem 2.1, C_S is completely decomposable with linearly ordered typeset. □

To prove a generalization of Theorem 2.1 we need a few results from [**3**]. Let A be a Butler group. If X is a rank-1 group of type τ, then $A[\tau] = \cap\{\ker f | f \in \operatorname{Hom}(A, X)\}$ and $A^*[\tau] = \cap\{A[\sigma] | \sigma < \tau\}$. It is easy to see that if K is a pure subgroup of A with $K \subseteq A[\sigma]$ for all $\sigma < \tau$, then $(A/K)[\sigma] = A[\sigma]/K$ and $(A/K)^*[\tau] = A^*[\tau]/K$.

THEOREM 2.4. (ARNOLD-VINSONHALER [**3**]). *The exact sequence* $0 \to A^*[\tau]/A[\tau] \to A/A[\tau] \to A/A^*[\tau] \to 0$ *is split exact.*

Recall that $A^*[\tau]/A[\tau]$ is homogeneous completely decomposable of type τ (see [**3**]).

THEOREM 2.5. *Let* A *be an almost completely decomposable group and* $n = \operatorname{rank} A$. *The following are equivalent.*

1. C *is related to* A.

2. C *is* n*-related to* A.

3. (i) *If* A *is a free group plus a divisible group, then* $C = D \oplus F$ *with* D *divisible and zero if and only if* A *is reduced, and* F *is free.*

(ii) *Otherwise write* $A = D' \oplus F' \oplus G$ *and* $C = D \oplus F \oplus R$ *with* D *and* D' *divisible,* F *and* F' *free, and* R *and* G fdc-*summands. Then* R *is quasi-isomorphic to* G*, and* $D = 0$ *if* $\mathrm{OT}(A) < \bar{\infty}$ *and* $D \neq 0$ *if* $D' \neq 0$ *but* $\mathrm{OT}(G) < \bar{\infty}$.

PROOF. $(2 \Rightarrow 3)$. If A is a free group plus a divisible group, then clearly the conclusion of 3(i) must hold. Otherwise assume $\mathrm{Hom}(A, \mathbf{Z}) = \mathrm{Hom}(C, \mathbf{Z}) = 0$ and $A = D' \oplus G$ with G an fdc-summand. Without loss of generality, assume $G = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, where each A_i is homogeneous completely decomposable of type τ_i and rank r_i and $\tau_i \not\geq \tau j$ if $i < j$.

We will construct the map $\theta : C \to A_1 \oplus \cdots \oplus A_k$ for $1 \leq k \leq n$ such that $C/\ker\theta \sim A_1 \oplus \cdots \oplus A_k$ by induction on k. Since C is a Butler group, by Proposition 2.3, there is a splitting map $f_i : C/C[\tau_i] \to C^*[\tau_i]/C[\tau_i]$, for each i, by Theorem 2.4.

If X is a rank-1 group of type $< \tau_1$, then $\mathrm{Hom}(G, X) = 0$. By Theorem 1.3, $\mathrm{Hom}(C, X) = 0$ so that $C^*[\tau_1] = C$. By Theorem 2.4, $C/C[\tau_1] \cong C^*[\tau_1]/C[\tau_1] \oplus C/C^*[\tau_1] = C^*[\tau_1]/C[\tau_1]$ is homogeneous completely decomposable of type τ_1. Since, for $pX_1 \neq X_1$, $\mathrm{rank}\, C/C[\tau_1] = r_p(\mathrm{Hom}(C, X_1)) = r_p(\mathrm{Hom}(G, X_1)) = r_1$, we have $C/C[\tau_1] \cong A_1$.

Assume for the sake of induction, that the map $\theta : C \to \oplus_{i=1}^{k-1} C^*[\tau_1]/C[\tau_i]$, given by $\theta(c) = (f(c), \ldots, f_{k-1}(c))$, satisfies $\mathrm{Im}\,\theta \sim \oplus_{i=1}^{k-1} C^*[\tau_i]/C[\tau_i]$ and that $A_i \cong C^*[\tau_i]/C[\tau_i]$. Identify A_i with $C^*[\tau_i]/C[\tau_i]$.

Since our characterization is only up to quasi-isomorphism, we may assume without loss of generality that $\mathrm{Im}\,\theta = A_1 \oplus \cdots \oplus A_{k-1}$. Let $K = \ker\theta$, and consider $0 \to K \to C \to A_1 \oplus \cdots \oplus A_{k-1} \to 0$. Applying $\mathrm{Hom}(-, X)$ for a rank-1 group X we have $0 \to \mathrm{Hom}(A_1 \oplus \cdots \oplus A_{k-1}, X) \to \mathrm{Hom}(C, X) \xrightarrow{t} \mathrm{Hom}(K, X)$. Let $\tau = \mathrm{type}\, X$.

If $\tau < \tau_k$ and $pX \neq X$, then $\mathrm{rank}\ \mathrm{Hom}(C, X) = r_p(\mathrm{Hom}(C, X)) = r_p(\mathrm{Hom}(G, X)) = r_p(\mathrm{Hom}(A_1 \oplus \cdots \oplus A_{k-1}, X)) = \mathrm{rank}\ \mathrm{Hom}(A_1 \oplus \cdots \oplus A_{k-1}, X)$ so that $t = 0$ since $\mathrm{Hom}(K, X)$ is torsion-free. This implies $K \subseteq C[\tau]$ and consequently $K \subseteq C^*[\tau_k]$. Moreover, $(C/K)[\tau] =$

$C[\tau]/K$ here so that $(C/K)^*[\tau_k] = C^*[\tau_k]/K$ by the remark preceding Theorem 2.4. Therefore $C/C^*[\tau_k] = (C/K)/((C/K)^*[\tau_k]) = \oplus\{A_i|\tau_i < \tau_k\} = A/A^*[\tau_k]$.

If $pX_k \neq X_k$, then $r_p(\text{Hom}(C, X_k)) = \text{rank Hom}(C, X_k) = \text{rank}\, C/C[\tau_k] = \text{rank}\, A/A[\tau_k] = r_p(\text{Hom}(A, X_k))$. Since $C/C^*[\tau_k] \cong A/A^*[\tau_k]$ we must have $\text{rank}\, C^*[\tau_k]/C[\tau_k] = \text{rank}\, A^*[\tau_k]/A[\tau_k]$ by Theorem 2.4. Hence $C^*[\tau_k]/C[\tau_k] \cong A^*[\tau_k]/A[\tau_k]$ since both are homogeneous completely decomposable of type τ_k.

Hence $C/C[\tau_k] = C^*[\tau_k]/C[\tau_k] \oplus C/C^*[\tau_k] = C^*[\tau_k]/C[\tau_k] \oplus \{C^*[\tau_i]/C[\tau_i] \mid \tau_i < \tau_k\}$ and the map $f: C \to \oplus\{C^*[\tau_i]/C[\tau_i]|\tau_i < \tau_k\}$ defined by $f(c) = (\theta(c), f_k(c))$ is an isomorphism.

Hence, by induction, there is a map $\theta : C \to G$ such that $C/ker\theta \sim G$. Since $r_p(C) = r_p(G) + r_p(\ker\theta) = r_p(G)$ for all p, $\ker\theta$ is divisible. Then $C \cong \ker\theta \oplus \text{Im}\,\theta \sim \ker\theta \oplus G$. Since $\text{OT}(C) = \text{OT}(A)$, if $\text{OT}(A) < \bar{\infty}$, then $\ker\theta$ must be zero. If $\text{OT}(A) = \bar{\infty}$ but $\text{OT}(G) < \bar{\infty}$, then $\text{OT}(C) = \bar{\infty}$ so $\ker\theta$ cannot be zero.

$(3 \Rightarrow 1)$. This follows along the line of the proof of $(3 \Rightarrow 1)$ in theorem 2.1. □

In the proof of the theorem we only used the full strength of the relation in deducing that C was a Butler group.

COROLLARY 2.6. *Assume C is a Butler group and A is almost completely decomposable. If $r_p(C) = r_p(A)$ and $r_p(\text{Hom}(C, X)) = r_p(\text{Hom}(A, X))$ for all p and* rank-1 *groups X, then $C \sim D \oplus C'$, where D is divisible and $A = D' \oplus C'$ with D' divisible and C' reduced. In particular, C is almost completely decomposable.*

PROOF. If A is divisible, then so is C. Otherwise repeat along the lines of Theorem 2.5 noting that $\text{rank Hom}(A, Z) = \text{rank Hom}(C, Z)$. □

3. Rank-2 groups. We can apply the results from §1 and §2 to settle the question for rank-2 groups. By doing so we determine a complete system of quasi-invariants for rank-2 groups.

In [**7**] James Reid gave a classification of rank-2 groups in terms of their quasi-endomorphism rings. This classification is similar to the Beaumont-Pierce classification (Theorem 3.3 in [**1**]). Of the strongly indecomposable rank-2 groups $A, \mathbf{Q}E(A)$ is either (i) a quadratic number field; (ii) $\mathbf{Q}$; or (iii) the ring of upper trianglular 2×2 matrices over $\mathbf{Q}$ with equal diagonal entries.

THEOREM 3.1. *Let* $\operatorname{rank} A = 2$. *Then* C *is* 2*-related to* A *if and only if*

(i) *If* A *is a free group plus a divisible, then* $C = D \oplus F$ *with* D *divisible and zero if and only if* A *is reduced, and* F *free.*

(ii) *Otherwise write* $C = D \oplus F \oplus R$ *with* F *free,* R *quasi-isomorphic to an* fdc*-summand of* A, *and* D *divisible and zero if* $\mathrm{OT}(A) < \bar{\infty}$ *and nonzero if* $\mathrm{OT}(R) < \bar{\infty}$ *and* A *is not reduced.*

PROOF. We will only consider the necessity, and assume $\operatorname{Hom}(C, \mathbf{Z}) = \operatorname{Hom}(A, \mathbf{Z}) = 0$.

Theorem 2.5 handles the case that A is almost completely decomposable. The case when $\mathbf{Q}E(A)$ is a field is covered by Theorem 1.4. The final case is where $\mathbf{Q}E(A) = \left\{ \left[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right] \middle| x, y \in \mathbf{Q} \right\}$ and A is strongly indecomposable.

Let $f : A \to A$ satisfy $\operatorname{rank} \operatorname{Im} f = 1$. Clearly $f^2 = 0$. Let $a \in A$ be such that $b = f(a) \neq 0$. Since $0 \neq b \in \ker f, a$ and b are independent and thus, for $\bar{A} = A/\langle b\rangle_*$ and $\bar{B} = A/\langle a\rangle_*, \mathrm{OT}(A) = \operatorname{type} \bar{A} \vee \operatorname{type} \bar{B}$. Since $\bar{A}$ embeds in $\bar{B}$ as $x + \langle b\rangle_* \to f(x) + \langle a\rangle_*$, $\operatorname{type} \bar{A} \leq \operatorname{type} \langle b\rangle_* \leq \operatorname{type} \bar{B} = \mathrm{OT}(A)$.

Let p satisfy $p\langle b\rangle_* \neq \langle b\rangle_*$. Embed $\langle a\rangle_*$ into $\bar{A}$ and $\langle b\rangle_*$ into $\bar{B}$ naturally. Then $\bar{A}/\langle a\rangle_* \cong A/(\langle a\rangle_* + \langle b\rangle_*) \cong \bar{B}/\langle b\rangle_*$. Since $\bar{A}/\langle a\rangle_*$ is p-reduced, $\bar{B}/\langle b\rangle_*$ is p-reduced. Since $p\langle b\rangle_* \neq \langle b\rangle_*$, $p\bar{B} \neq \bar{B}$ and $OT(C) = OT(A) < \bar{\infty}$. Therefore $r_p(\operatorname{Hom}(C, A)) = \operatorname{rank} \operatorname{Hom}(C, A) = r_p(\operatorname{Hom}(A, A)) = \operatorname{rank} E(A) = 2$, and $\mathrm{OT}(C) < \bar{\infty}$ so that C must be reduced and $\operatorname{rank} C = r_p(C) = r_p(A) = 2$.

Let $f, g \in \operatorname{Hom}(C, A)$ be indepedent. To show that there is a map in $\operatorname{Hom}(C, A)$ with a rank-2 image assume $\operatorname{rank} \operatorname{Im} f = \operatorname{rank} \operatorname{Im} g = 1$.

If $\ker f = \ker g = K$, then $f, g \in \operatorname{Hom}(C/K, A)$ and therefore, for $U = g(C) + f(C)$, $\operatorname{rank} U = 2$. This implies that any $x \in A$ has type $\geq \operatorname{type} C/K$. But $\operatorname{rank} \operatorname{Hom}(A, C/K) = \operatorname{rank} \operatorname{Hom}(C, C/K) \neq 0$ so any $0 \neq h : A \to C/K$ is quasi-split, a contradiction.

If $\ker f \neq \ker g$ but $\langle f(C) \rangle_* = \langle g(C) \rangle_* = W$, then we must have $\operatorname{OT}(C) \leq \operatorname{type} W$ (C embeds in W^2). But $\operatorname{OT}(A) \leq \operatorname{type} W$ implies that W is a summand of A [**6**, Corollary 1.8], a contradiction.

Define $\theta : C \to A$ by $\theta(c) = f(c) + g(c)$. If $x \in \ker \theta$, then $f(x) = -g(x) \in \langle \operatorname{Im} f \rangle_* \cap \langle \operatorname{Im} g \rangle_* = 0$ so that $x \in \ker f \cap \ker g = 0$. By the above, $\operatorname{rank} \operatorname{Im} \theta = 2$ so that $\operatorname{coker} \theta = T$ is torsion.

From $0 \to C \xrightarrow{\theta} A \to T \to 0$ we have $0 \to \operatorname{Hom}(A, C) \xrightarrow{\theta^*} \operatorname{Hom}(C, C)$. Since $\operatorname{rank} \operatorname{Hom}(C, C) = \operatorname{rank} \operatorname{Hom}(A, C)$, there is an $m \neq 0$ so that $\phi = m1_C : A \to C$. Clearly ϕ is 1-1 and therefore $A \sim C$.

Corollary 3.2. *Let* $\operatorname{rank} A = 2$. *The following are equivalent*:

1. *C is quasi-isomorphic to A.*

2. (i) $\operatorname{rank} C = 2$; (ii) $r_p(C) = r_p(A)$ *for all* p; (iii) $r_p(\operatorname{Hom}(C, B)) = r_p(\operatorname{Hom}(A, B))$ *for all p and B with* $\operatorname{rank} B \leq 2$; *and* (iv) $\operatorname{OT}(C) = \operatorname{OT}(A)$.

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