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ISOLS AND GENERALIZED BOOLEAN ALGEBRAS

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ABSTRACT. Let $\Gamma = \langle C, +, \cdot \rangle$ be a finite, hence atomic Boolean algebra. Then Γ is isomorphic to $\langle Q, \cup, \cap \rangle$, where Q is the family of all (finite) subsets of a (finite) set ν, namely the set of all atoms of Γ. Moreover, if ν has cardinality n, the Boolean algebra Γ is determined up to isomorphism by its order, i.e., 2^n, or equivalently by the number n. We shall extend this theorem to atomic generalized Boolean algebras $\Gamma = \langle C, +, \cdot \rangle$ in which the set C is isolated rather than finite. We have to impose some recursivity conditions on Γ which hold trivially, if Γ is finite. If these conditions are satisfied, Γ is effectively isomorphic to $\langle Q, \cup, \cap \rangle$, where Q is the family of all finite subsets of an isolated set ν, namely the set of all atoms of Γ. Moreover, if ν has RET (recursive equivalence type) N, the system Γ is determined up to effective isomorphism by its order, i.e., 2^N, or equivalently by the RET N. This result is of some interest, since the role played in ordinary arithmetic by the family of all (finite) subsets of some finite set ν is played in isolic arithmetic by the family of all finite subsets of some isolated set ν.

1. Algebraic preliminaries. Let $\Delta = \langle D, +, \cdot \rangle$ be a distributive lattice. For $u, v \in D$ we often abbreviate "$u \cdot v$" to "uv." The distributive lattice Δ has a zero-element if there is an element $0 \in D$ such that $x + 0 = x$ for all $x \in D$. Similarly, Δ has a one-element if there is an element $1 \in D$ such that $x \cdot 1 = x$ for all $x \in D$. If $p, q \in D$ we define $p \leq q$ as $pq = p$ or equivalently as $p + q = q$. For $a, b \in D$ we write $[a, b]$ for $\{x \in D \,|\, a \leq x \leq b\}$. A subset S of D is an *interval* of Δ if there are elements $a, b \in D$ such that $a \leq b$ and $S = [a, b]$. Note that, for $a, b, p, q \in D$,

$$(a \leq p \leq b \text{ and } a \leq q \leq b) \Rightarrow (a \leq p + q \leq b \text{ and } a \leq pq \leq b),$$

i.e., that $[a, b]$ is closed under $+$ and $\cdot$. Thus, by restricting the operations $+$ and $\cdot$ of Δ to the interval $[a, b]$ we obtain a distributive lattice with a as zero-element and b as one-element. This is called the lattice *induced by* Δ in $[a, b]$.

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The system $\Gamma = \langle C, +, \cdot \rangle$ is a *Boolean algebra* (abbreviated B.A.) if (a) Γ is a distributive lattice with a zero-element 0 and a one-element 1 and (b) for every $c \in C$ the simultaneous equations $cx = 0$ and $c + x = 1$ have at least (hence exactly) one solution; this solution is denoted by c'. In a B.A. Γ we have, for every interval $[a, b]$,

$$(1) \qquad c \in [a, b] \Rightarrow \begin{cases} \text{the simultaneous equations } cx = a, \\ c + x = b \text{ have exactly one solution.} \end{cases}$$

For, assume $a \leq c \leq b$, Put $x_0 = a + bc'$; then

$$\begin{aligned} cx_0 &= c(a + bc') = ca + 0 = ca = a, && \text{since } a \leq c, \\ c + x_0 &= c + (a + bc') = (c + a) + bc' = c + bc', && \text{since } a \leq c, \\ &= (c + b)(c + c') = (c + b) \cdot 1 = c + b = b, && \text{since } c \leq b. \end{aligned}$$

We now prove the uniqueness of x_0. Assume $cx_1 = a$, $c + x_1 = b$. Then

$$(2) \qquad x_1 = 1 \cdot x_1 = (c + c')x_1 = cx_1 + c'x_1 = a + c'x_1;$$

$$(3) \quad c + x_1 = b \Rightarrow c'(c + x_1) = c'b \Rightarrow c'c + c'x_1 = bc' \Rightarrow c'x_1 = bc';$$

so that (2) and (3) imply $x_1 = a + bc' = x_0$.

Condition (1) can also be phrased as follows. The lattice induced by Γ in any interval $[a, b]$ is a B.A. with a as zero-element and b as one-element. A lattice Γ is called *relatively complemented* if (1) holds in each interval $[a, b]$ of Γ.

Definition. A system $\Gamma = \langle C, +, \cdot \rangle$ is a *generalized* B.A. if it is a relatively complemented, distributive lattice with a zero-element.

Henceforth we use the word "algebra" instead of the words "generalized B.A." We proved above that every B.A. is an algebra. The converse is false. For, let $P_{\text{fin}}(\nu)$ be the family of all finite subsets of a set ν. Then $\langle P_{\text{fin}}(\nu), \cup, \cap \rangle$ is an algebra, but a B.A. if and only if the set ν is finite. Note that $P_{\text{fin}}(\nu)$ has a zero-element for every set ν (namely the empty set), but a one-element (namely ν itself) if and only if ν is finite. In fact, an algebra is a B.A. if and only if it has a one-element. For an algebra $\Gamma = \langle C, +, \cdot \rangle$ we define

$$q - p = \text{ the solution of } px = 0,\ p + x = q, \qquad \text{for } p \leq q.$$

Since $ab \leq b$ in every lattice, we can define $b - a$ in every algebra, namely as $b - ab$. If Γ is a B.A. we can also define $b - a$ as ba'.

2. Notations and terminology. A nonnegative integer is called a *number*, a collection of numbers a *set* and a collection of sets a *class*. We write o for the empty set and ε for the set of all numbers. If f is a function from a subset of ε^n into ε, we denote its domain and its range by δf and ρf respectively. Since we are only concerned with countable algebras, the word "algebra" will from now on refer to an algebra of the type $\Gamma = \langle \gamma, +, \cdot \rangle$, where $\gamma \subset \varepsilon$. We also assume that the zero-element of an algebra is the ordinary number 0 and that if the algebra has a one-element, this is the ordinary number 1. The finite sets, i.e., the members of $P_{\text{fin}}(\varepsilon)$, can be effectively generated without repetitions in an infinite sequence. We shall use a particular sequence of this type, the so-called *canonical enumeration* $\langle \rho_n \rangle$; see [**1**, p. 277]. We define r_n as $\operatorname{card} \rho_n$; it is a recursive function of n. For each finite set τ there is exactly one number i with $\tau = \rho_i$; this number is called the *canonical index* of τ and denoted by $\operatorname{can}(\tau)$ or $\operatorname{can}\tau$.

With an algebra $\Gamma = \langle \gamma, +, \cdot \rangle$ we associate the three functions f, g, h where

$$\delta f = \delta g = \gamma^2, \quad f(a, b) = a + b, \quad g(a, b) = ab,$$

$$\delta h = \{\langle a, c, b\rangle \in \gamma^3 \mid a \leq c \leq b\}, \tag{4}$$

$$h(a, c, b) = \text{ the solution of } cx = a \text{ and } c + x = b.$$

Definition. An algebra $\Gamma = \langle \gamma, +, \cdot \rangle$ is an ω-algebra, if the three functions f, g and h have partial recursive extensions.

It follows that, in an ω-algebra Γ, the function $b - a = b - ab = h(0, ab, b)$ also has a partial recursive *extension.* Thus, in the special case that Γ is a B.A., the function $x' = 1 - x$ has a partial recursive extension. We write γ_0 for $\gamma - (0)$.

Let $\Gamma = \langle \gamma, +, \cdot \rangle$ be an algebra and $p, q \in \gamma$. We call p a *predecessor* of q, if $p \leq q$; p is a *strict* predecessor of q (written $p < q$) if $p \leq q$ and

$p \neq q$. The element p is an *atom* (or is *atomic*) if p is nonzero and 0 is its only strict predecessor. The algebra Γ is *atomic* if each of its nonzero elements has an atomic predecessor. We say that $\Gamma = \langle \gamma, +, \cdot \rangle$ is *isolic*, if the underlying set γ is isolated (i.e., has no infinite r.e. subset). $\mathrm{At}(p)$ denotes the set of all atomic predecessors of p, while $\mathrm{At}(\Gamma)$ stands for the set of all atoms of Γ. Thus $\mathrm{At}(\Gamma) = \mathrm{At}(1)$, in case Γ is a B.A. We abbreviate "At(Γ)" to "At", if Γ is known from the context. A function c is a *special* function of Γ if

$$\delta c = \gamma_0, c(x) = x \text{ if } x \in \mathrm{At}, \quad 0 < c(x) < x \text{ if } x \notin \mathrm{At}.$$

The algebra $\Gamma = \langle \gamma, +, \cdot \rangle$ is *ω-atomic* if there is an effective procedure which, when applied to any $x \in \gamma_0$, yields x if x is atomic, but an atomic predecessor of x if x is not atomic. We claim

(5) an isolic ω-algebra is ω-atomic if and only if it has a special function with a partial recursive extension.

For, let $\Gamma = \langle \gamma, +, \cdot \rangle$ be an isolic ω-algebra and c a special function of Γ with a partial recursive extension. Then we can, given any element $x \in \gamma_0$, compute $x, c(x), c^2(x), \ldots$. Thus $(x, c(x), c^2(x), \ldots)$ is a r.e., hence finite, subset of the isolated set γ_0. We can therefore also compute the numbers

$$m = \min\{y \in \varepsilon \mid c^{y+1}(x) = c^y(x)\} \text{ and } c^m(x).$$

Then $c^m(x)$ is an atomic predecessor of x; moreover, x is an atom if and only if $m = 0$, i.e., if and only if $c(x) = x$. Hence Γ is ω-atomic. The converse is trivial.

For a set ν we denote the class of all finite subsets of ν by $P_{\text{fin}}(\nu)$. We write $\Pi_{\text{fin}}(\nu)$ for $\langle 2^\nu, s, t \rangle$, where

$$(6) \qquad \begin{aligned} 2^\nu &= \{x \in \varepsilon \mid \rho_x \subset \nu\}, \qquad \delta s = \delta t = 2^\nu \times 2^\nu, \\ s(x, y) &= \mathrm{can}(\rho_x \cup \rho_y), \qquad t(x, y) = \mathrm{can}(\rho_x \rho_y). \end{aligned}$$

$\Pi_{\text{fin}}(\nu)$ is an ω-atomic ω-algebra which is isomorphic to $\langle P_{\text{fin}}(\nu), \cup, \cap \rangle$. Its set of atoms is $\{x \in 2^\nu | r_x = 1\}$. It is finite (infinite, r.e., isolic, immune) if and only if the set ν is finite (infinite, r.e., isolated, immune). We call $\Pi_{\text{fin}}(\nu)$ the *standard ω-algebra over the set ν*.

DEFINITIONS. Let $\Gamma_1 = \langle \gamma_1, +, \cdot \rangle$ and $\Gamma_2 = \langle \gamma_2, +, \cdot \rangle$ be ω-algebras. An *isomorphism* from Γ_1 onto Γ_2 is a one-to-one mapping from γ_1 onto γ_2 which preserves $+$ and $\cdot$ (hence $-$). An isomorphism from Γ_1 onto Γ_2 is an *ω-isomorphism* if it has a partial recursive one-to-one extension. We say that Γ_1 is *isomorphic* to Γ_2 (written $\Gamma_1 \cong \Gamma_2$) if there is at least one isomorphism from Γ_1 onto Γ_2. Similarly, Γ_1 is *ω-isomorphic* to Γ_2 (written $\Gamma_1 \cong_\omega \Gamma_2$) if there is at least one ω-isomorphism from Γ_1 onto Γ_2.

Let $\Gamma_1 = \langle \gamma_1, +, \cdot \rangle$ and $\Gamma_2 = \langle \gamma_2, +, \cdot \rangle$ be algebras and φ an isomorphism from Γ_1 onto Γ_2. Then $\varphi(0) = 0$ and, in case Γ_1 and Γ_2 have one-elements (i.e., are B.A. 's), $\varphi(1) = 1$ and $\varphi(x') = [\varphi(x)]'$ for $x \in \gamma_1$. The *order* $o(\Gamma)$ of an ω-algebra $\Gamma = \langle \gamma, +, \cdot \rangle$ is defined as $\operatorname{Req} \gamma$, i.e., the RET of γ. Thus $o(\Gamma)$ has the usual meaning if and only if Γ is finite. Note that two finite algebras are ω-isomorphic if and only if they are isomorphic. Thus $\cong_\omega$ is a generalization of the relation $\cong$.

3. Some propositions. The proof of our main result, namely, the representation theorem for isolic ω-algebras, is a modification of the standard proof of the representation theorem for finite B.A. 's. See, e.g., [**2**, pp. 28–30], [**4**, pp. 136–137] or [**5**, pp. 18–20]. Relations (7)–(15) can be proved as in the finite case. Every algebra $\Gamma = \langle \gamma, +, \cdot \rangle$ has the following four properties:

(7) a atomic $\Rightarrow [a \leq b$ or $ab = 0]$, for $a, b \in \gamma$,

(8) a, b atomic $\Rightarrow [a = b$ or $ab = 0]$, for $a, b \in \gamma$,

(9) $x \in \operatorname{At}(b) \Longleftrightarrow x \leq b \Longleftrightarrow xb \neq 0$, for $x \in \operatorname{At}, b \in \gamma$,

(10) $a \in \operatorname{At} \Longleftrightarrow \operatorname{At}(a) = (a)$, for $a \in \gamma$.

In every atomic algebra $\Gamma = \langle \gamma, +, \cdot \rangle$ the mapping $x \to \operatorname{At}(x)$, for $x \in \gamma$, has the following five properties:

(11) $\operatorname{At}(ab) = \operatorname{At}(a) \cap \operatorname{At}(b)$, for $a, b \in \gamma$,

(12) $ab = 0 \Longleftrightarrow \operatorname{At}(a) \cap \operatorname{At}(b) = o$, for $a, b \in \gamma$,

(13) $\operatorname{At}(a + b) = \operatorname{At}(a) \cup \operatorname{At}(b)$, for $a, b \in \gamma$,

(14) $a \leq b \Rightarrow \operatorname{At}(b - a) = \operatorname{At}(b) - \operatorname{At}(a)$, for $a, b \in \gamma$,

(15) $\operatorname{At}(b - a) = \operatorname{At}(b) - \operatorname{At}(a)$, for $a, b \in \gamma$.

PROPOSITION P1. *Let $\Gamma = \langle \gamma, +, \cdot \rangle$ be an isolic, ω-atomic ω-algebra. Then, for $a \in \gamma_0$,*

(a) *the set* $\mathrm{At}(a)$ *is nonempty and finite,*

(b) *given a, we can compute the* (*canonical index of the*) *set* $\mathrm{At}(a)$*,*

(c) *a is the sum of all its atomic predecessors.*

PROOF. Let $a \in \gamma_0$. From a we can compute an element $a_1 \in \mathrm{At}(a)$. If $a_1 = a$ we are done, for then $\mathrm{At}(a) = (a_1)$. If $a_1 \neq a$ we have $a_1 < a$ and $a - a_1 \in \gamma_0$. From a and a_1 we can compute $a - a_1$, hence an element $a_2 \in \mathrm{At}(a - a_1)$. If $a_2 = a - a_1$ we are done, for then $\mathrm{At}(a) = (a_1, a_2)$ and $a = a_1 + a_2$. If $a_2 < a - a_1$ we can compute the element $(a - a_1) - a_2 = a - (a_1 + a_2)$ and an element $a_3 \in \mathrm{At}[a - (a_1 + a_2)]$ etc., etc. This procedure must terminate, since $(a_1, a_2, \dots)$ is a r.e. subset of $\mathrm{At}(a)$, hence of γ_0. Thus there is an atom a_n such that $a_n = a - (a_1 + \cdots + a_{n-1})$, and this atom can be computed from a. Since $a_1, \dots, a_n$ can be computed from a, so can the (canonical index of the) finite set $\mathrm{At}(a) = (a_1, \dots, a_n)$.

PROPOSITION P2. *Let $\Gamma = \langle \gamma, +, \cdot \rangle$ be an isolic, ω-atomic ω-algebra and $\nu = \mathrm{At}$. Then the mapping $\varphi : x \to \mathrm{At}(x)$, for $x \in \gamma$, is an isomorphism from Γ onto $\langle P_{\mathrm{fin}}(\nu), \cup, \cap \rangle$.*

PROOF. The mapping φ maps γ into $P_{\mathrm{fin}}(\nu)$ by P1. Let $a, b \in \gamma$ and $a \neq b$. First consider the case where exactly one of the two elements a and b is zero. Then exactly one of the two sets $\mathrm{At}(a), \mathrm{At}(b)$ is empty, hence $\mathrm{At}(a) \neq \mathrm{At}(b)$. Now assume that $a, b \in \gamma_0$. Then $\mathrm{At}(a)$ and $\mathrm{At}(b)$ are nonempty finite sets, say $\mathrm{At}(a) = (a_1, \dots, a_p)$ and $\mathrm{At}(b) = (b_1, \dots, b_q)$. Also, $a = a_1 + \cdots + a_p$ and $b = b_1 + \cdots + b_q$, so that $\mathrm{At}(a) = \mathrm{At}(b)$ implies $a = b$. We proved that φ is one-to-one. Let $\delta \in P_{\mathrm{fin}}(\nu)$. If $\delta = o$ we have $\delta = \varphi(0)$. Now assume $\delta \neq o$, say $\delta = (d_1, \dots, d_n)$, where $d_1, \dots, d_n$ are distinct atoms, and put $d = d_1 + \cdots + d_n$. Then

$$\varphi(d) = \mathrm{At}(d_1 + \cdots + d_n) = (d_1) \cup \cdots \cup (d_n) = \delta$$

by (10) and (13). We proved that $\varphi(\gamma) = P_{\mathrm{fin}}(\nu)$. Finally, φ is an isomorphism by (11) and (13).

4. The main result.

PROPOSITION P3. (a) *Every isolic, ω-atomic ω-algebra is ω-isomorphic to the standard ω-algebra over the set of its atoms.*

(b) *An isolic, ω-atomic ω-algebra is up to ω-isomorphism characterized by its order.*

PROOF. (a). Let $\Gamma = \langle \gamma, +, \cdot \rangle$ be an isolic, ω-atomic ω-algebra, ν the set of its atoms, $\delta\varphi = \gamma$ and $\varphi(x) = \mathrm{At}(x)$. Define $\delta\psi = \gamma$ and $\psi(x) = \mathrm{can}[\mathrm{At}(x)]$. Since φ is an isomorphism from Γ onto $\langle P_{\mathrm{fin}}(\nu), \cup, \cap \rangle$, ψ is an isomorphism from Γ onto $\Pi_{\mathrm{fin}}(\nu) = \langle 2^\nu, s, t \rangle$, where 2^ν, s and t are defined as in (6). The mapping ψ has a partial recursive extension by P1(b). Also, for $x \in 2^\nu$,

$$\psi^{-1}(x) = \begin{cases} 0, & \text{if } x = 0, \text{ i.e., } \rho_x = o, \\ a_1 + \cdots + a_n, & \text{if } x \neq 0 \text{ and } \rho_x = (a_1, \ldots, a_n) \end{cases}$$

so that ψ^{-1} also has a partial recursive extension. We conclude that the isomorphism ψ from Γ onto $\Pi_{\mathrm{fin}}(\nu)$ has a partial recursive one-to-one extension. Thus ψ is an ω-isomorphism and $\Gamma \cong_\omega \Pi_{\mathrm{fin}}(\nu)$.

(b). Let Γ and Γ^* be isolic, ω-atomic ω-algebras with ν and ν^* as their respective sets of atoms. Let $\simeq$ denote recursive equivalence. In view of (a) we only need to show

$$\Pi_{\mathrm{fin}}(\nu) \cong_\omega \Pi_{\mathrm{fin}}(\nu^*) \Longleftrightarrow 2^\nu \simeq 2^{\nu^*}.$$

The conditional from the left to the right is trivial. Now assume $2^\nu \simeq 2^{\nu^*}$. Then $\nu \simeq \nu^*$, since the function 2^N from the collection Λ of all isols into itself is one-to-one. Let p be a partial recursive one-to-one function with $\nu \subset \delta p$ and $p(\nu) = \nu^*$. Define the functions q, s, t, s^*, t^* as follows:

$$\delta q = 2^{\delta p}, \quad \rho_{q(x)} = p(\rho_x), \quad \text{i.e., } q(x) = \mathrm{can}[p(\rho_x)],$$

$$\delta s = \delta t = 2^\nu \times 2^\nu, \quad s(x,y) = \mathrm{can}(\rho_x \cup \rho_y), \quad t(x,y) = \mathrm{can}(\rho_x \rho_y),$$

$$\delta s^* = \delta t^* = 2^{\nu^*} \times 2^{\nu^*}, \quad s^*(x,y) = \mathrm{can}(\rho_x \cup \rho_y), \quad t^*(x,y) = \mathrm{can}(\rho_x, \rho_y).$$

Then

$$\begin{aligned} qs(x,y) &= q\ \mathrm{can}(\rho_x \cup \rho_y) = \mathrm{can}[p(\rho_x \cup \rho_y)] = \mathrm{can}[p(\rho_x) \cup p(\rho_y)] \\ &= \mathrm{can}[\rho_{q(x)} \cup \rho_{q(y)}] = s^*[q(x), q(y)] \end{aligned}$$

and similarly $qt(x,y) = t^*[q(x), q(y)]$.

Hence q is a partial recursive one-to-one extension of an isomorphism from $\Pi_{\mathrm{fin}}(\nu) = \langle 2^\nu, s, t\rangle$ onto $\Pi_{\mathrm{fin}}(\nu^*) = \langle 2^{\nu^*}, s^*, t^*\rangle$. Thus we have proved that $\Pi_{\mathrm{fin}}(\nu) \cong_\omega \Pi_{\mathrm{fin}}(\nu^*)$. □

COROLLARY 1. *Let* $\Gamma = \langle \gamma, +, \cdot\rangle$ *be an isolic,* ω*-atomic* ω*-algebra with* $\nu = \mathrm{At}(\Gamma)$ *and* $N = \mathrm{Req}\,\nu$. *Then* $o(\Gamma) = 2^N$.

COROLLARY 2. *An isolic,* ω*-atomic* ω*-algebra is a* B.A. *if and only if it is finite.*

The first corollary is immediate. The second one follows from the fact that $\langle P_{\mathrm{fin}}(\nu), \cup, \cap\rangle$ is B.A. if and only if it has a one-element, i.e., if and only if $\nu \in P_{\mathrm{fin}}(\nu)$, i.e., if and only if ν is finite.

REMARK. Let c denote the cardinality of the continuum. Consider an isolic, ω-atomic ω-algebra $\Gamma = \langle \gamma, +, \cdot\rangle$, where $\gamma \subset \varepsilon$. We can choose the set γ in c ways, since there are c isolated sets ν and we can take $\gamma = 2^\nu$. It follows that, up to recursive equivalence, γ can be chosen in c ways. Thus, up to ω-isomorphism, there are c isolic, ω-atomic ω-algebras $\langle \gamma, +, \cdot\rangle$ of which only denumerably many are finite. Since every finite B.A. is an isolic, ω-atomic ω-algebra, we conclude that P3 is a proper generalization of the representation theorem for finite B.A. s.

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