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THE SQUARE CLASS INVARIANT

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A. Solow introduced the square class invariant in [**6**] and showed in [**6, 7, 8**] that the square class invariant or the square class invariant plus the determinant classify quadratic forms over some particular fields. It was hoped that this new invariant might be helpful in the classification problem. However, in this paper we will determine all non-pythagorean fields over which the square class invariant classifies quadratic forms and will see that the answer is tied very closely to value sets of forms. In what appears below, the notation follows that in [**4**].

DEFINITION. Let q be a quadratic form over a field F. The square class invariant for q is a function $m_q : \dot{F}/\dot{F}^2 \to \mathbf{Z}$ given by $m_q(a\dot{F}^2) = n$ where $q \cong n\langle a\rangle + p$ and $a \notin D(p)$.

It is clear that m_q is related closely to the value set of q. Indeed, $D(q) = \{a \in \dot{F} | m_q(a) \geq 1\}$ where we write $m_q(a)$ for $m_q(a\dot{F}^2)$. We will show that when the square class invariant classifies forms for non-pythagorean fields, the field's anisotropic forms are determined uniquely by their value sets. Such fields are called C-fields and were introduced in [**2**]. Although there exist C-fields when the level, $s(F)$, is greater than 2, the square class invariant fails to classify forms over any non-pythagorean field when the level is greater than 2. We suspect the complete answer is that the square class invariant classifies quadratic forms over a non-pythagorean field if and only if F is a C-field and $s(F) \leq 2$. Below we will verify this except for one direction in the case when $s(F) = 2$ and both $u(F)$ and the index of $D\langle 1, 1\rangle$ in $\dot{F}$ are infinite.

PROPOSITION 1. *Let F be a field with $s(F) = 1$. Then the square class invariant classifies forms over F if and only if F is a C-field.*

PROOF. Assume the square class invariant classifies forms over F.

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Suppose f and g are anisotropic forms with $D(f) = D(g)$. If $m_f(\alpha) \geq 2$ for some $\alpha \in \dot{F}$, then $\langle \alpha, \alpha \rangle \cong \langle \alpha, -\alpha \rangle \cong \langle 1, -1 \rangle$ is a subform of f and thus f is isotropic. So $m_f \leq 1$, and similarly $m_g \leq 1$ (that is, $m_f(\alpha) \leq 1$ and $m_g(\alpha) \leq 1$, for all $\alpha \in \dot{F}$). Thus $D(f) = \{\alpha \in \dot{F} | m_f(\alpha) = 1\}$ and $D(g) = \{\alpha \in \dot{F} | m_g(\alpha) = 1\}$. But $D(f) = D(g)$. Hence, $m_f = m_g$, which implies $f \cong g$. Thus F is a C-field.

Now assume F is a C-field. Suppose f and g are forms over F with $m_f = m_g$. Write $f \cong 2n\langle 1 \rangle + f'$, where n is a non-negative integer and $\dim f' = 0$ or $\dim f' \geq 1$ and f' is anisotropic. If $\dim f' = 0$, then we write $m_{f'} = 0$. If $\dim f' \geq 1$, then $m_{f'} \leq 1$ since $s = 1$ and f' is anisotropic. Similarly write $g \cong 2k\langle 1 \rangle + g'$, where k is a non-negative integer and $\dim g' = 0$ or $\dim g' \geq 1$ and g' is anisotropic. Again $m_{g'} \leq 1$. Without loss of generality, assume $k \leq n$. Then, for every $\alpha \in \dot{F}$, $m_f(\alpha) = 2n + m_{f'}(\alpha)$ and $m_g(\alpha) = 2k + m_{g'}(\alpha)$. So $0 = m_f(\alpha) - m_g(\alpha) = 2(n-k) + m_{f'}(\alpha) - m_{g'}(\alpha)$ and $0 \leq m_{g'}(\alpha) = 2(n-k) + m_{f'}(\alpha) \leq 1$. Hence $n = k$ and $m_{g'}(\alpha) = m_{f'}(\alpha)$ for every $\alpha \in \dot{F}$. Thus $D(f') = D(g')$. But F is a C-field, so $f' \cong g'$. Since $n = k$ and $f' \cong g'$, and the square class invariant classifies forms over F. □

PROPOSITION 2. *Let F be a C-field with $s(F) = 2$ and either $u(F) < \infty$ or $|\dot{F}/D\langle 1, 1\rangle| < \infty$. Then the square class invariant classifies quadratic forms over F.*

PROOF. Suppose ψ and ϕ are forms over F with $m_\psi = m_\phi$. First, suppose ψ and ϕ are both anisotropic. Since $D(\psi) = \{a \in \dot{F} | m_\psi(a) \geq 1\}$ and $D(\phi) = \{a \in \dot{F} | m_\phi(a) \geq 1\}, D(\psi) = D(\phi)$. Thus $\psi \cong \phi$ since F is a C field.

Now suppose ψ is isotropic and ϕ is anisotropic. Since ψ is isotropic, ψ is universal and $m_\psi(\alpha) \geq 1$ for every $\alpha \in \dot{F}$. If $m_\phi(\alpha) \geq 3$ for some $\alpha \in \dot{F}$, then $\langle \alpha, \alpha, \alpha \rangle$ is a subform of ϕ. But $\langle \alpha, \alpha, \alpha \rangle \cong \langle \alpha, -\alpha, -\alpha \rangle$, and this contradicts ϕ being anisotropic. Thus $m_\phi(\alpha) \leq 2$ for every $\alpha \in \dot{F}$. So $1 \leq m_\psi(a) = m_\phi(a) \leq 2$ for every $a \in \dot{F}$. Now suppose there exists a non-zero form ψ' such that $\psi \cong \langle 1, -1 \rangle + \psi'$ and let $a \in D(\psi')$. Then $\langle 1, -1, a \rangle \cong \langle a, -a, a \rangle \cong \langle -a, -a, -a \rangle$ is a subform of ψ. But $m_\psi(-a) \leq 2$, and so $\psi \cong \langle 1, -1 \rangle$ and $m_\psi = m_\phi = 1$. Since F

is a C-field, ϕ must be the unique anisotropic universal form. So $\langle 1,1\rangle$ is a subform of ϕ because either $u(F)<\infty$ or $D\langle 1,1\rangle$ has finite index in $\dot{F}$. This follows from Proposition 5.3 in [**2**] for the first case and from its proof in the second case. But this contradicts $m_\phi = 1$. Thus it cannot occur that ψ is isotropic and ϕ is anisotropic.

Finally, suppose both ψ and ϕ are isotropic. Write $\psi \cong 2n\langle 1,-1\rangle + \psi'$ and $\phi \cong 2k\langle 1,-1\rangle + \phi'$, where n and k are non-negative integers, $\dim\psi' = 0$ or $\dim\psi' \geq 1$ and $m_{\psi'}(1) \leq 3$, and $\dim\phi' = 0$ or $\dim\phi' \geq 1$ and $m_{\phi'}(1) \leq 3$. If $\dim\psi' = 0$ let $m_{\psi'} = 0$, and if $\dim\phi' = 0$ let $m_{\phi'} = 0$. In any case, $m_{\psi'}(\alpha) \leq 3$ and $m_{\phi'}(\alpha) \leq 3$ for $\alpha \in \dot{F}$ and $m_\psi(\alpha) = 4n + m_{\psi'}(\alpha)$ and $m_\phi(\alpha) = 4k + m_{\phi'}(\alpha)$ for all $\alpha \in \dot{F}$. Without loss of generality, assume $k \leq n$. Then, for every $\alpha \in \dot{F}$, $0 = m_\psi(\alpha) - m_\phi(\alpha) = 4(n-k) + m_{\psi'}(\alpha) - m_{\phi'}(\alpha)$ and $0 \leq m_{\phi'}(\alpha) = 4(n-k) + m_{\psi'}(\alpha) \leq 3$. Hence $n = k$ and $m_{\psi'} = m_{\phi'}$. If ψ' and ϕ' are anisotropic, then $\psi' \cong \phi'$ by the first part of this proof. Hence $\psi \cong \phi$. Also by the first part of this proof, it cannot occur that one of ψ' and ϕ' is isotropic and the other is anisotropic. Finally suppose that ψ' and ϕ' are both isotropic. Since $m_{\psi'} \leq 3, m_{\phi'} \leq 3$ and $s(F) = 2, \psi' \cong \langle 1,-1\rangle + f$ and $\phi' \cong \langle 1,-1\rangle + g$, where f and g are anisotropic (or $\dim f = 0$ or $\dim g = 0$). Now, $a \in D(f) \iff \langle 1,-1\rangle + \langle a\rangle$ is a subform of $\psi' \iff \langle a,-a\rangle + \langle a\rangle \cong \langle -a,-a,-a\rangle$ is a subform of ψ'. Thus $D(f) = \{a \in \dot{F} | m_{\psi'}(-a) = 3\}$. Similarly, $D(g) = \{a \in \dot{F} | m_{\phi'}(-a) = 3\}$. But since $m_{\psi'} = m_{\phi'}, D(f) = D(g)$. So $f \cong g$ since F is a C-field. Thus $\psi' \cong \phi'$ and $\psi \cong \phi$. □

Note that the condition, $u(F) < \infty$ or $|\dot{F}/D\langle 1,1\rangle| < \infty$, is used only to guarantee there exists no anisotropic universal form ϕ with $m_\phi = 1$. We believe this to be the case for any C-field with $s(F) = 2$ but have been unable to find a proof.

PROPOSITION 3. *Let F be a field with $s(F) = 2$. If the square class invariant classifies the quadratic forms over F, then F is a C-field.*

PROOF. Suppose F is not a C-field. Then there exist anisotropic forms $f \ncong g$ with $D(f) = D(g)$. Let $f' = \langle 1,-1\rangle + f$ and $g' = \langle 1,-1\rangle + g$. Consider $m_{f'}$. If there exists $a \in \dot{F}$ such that $m_{f'}(a) \geq 4$, then $4\langle a\rangle \cong 2\langle 1,-1\rangle$ is a subform of f'. Hence, by Witt's Cancellation Theorem, $\langle 1,-1\rangle$ is a subform of f. But f is anisotropic. Thus $m_{f'} \leq 3$.

Now, $m_{f'}(a) = 3 \iff f' \cong 3\langle a\rangle + \phi$, for some form ϕ (possibly the 0-form) $\iff f' \cong \langle a, -a, -a\rangle + \phi \cong \langle 1, -1\rangle + \langle -a\rangle + \phi \iff f \cong \langle -a\rangle + \phi \iff -a \in D(f)$.

Also, $m_{f'}(a) = 2 \iff f' \cong 2\langle a\rangle + \phi$ and $a \notin D(\phi)$, for some form $\phi \iff \langle -a\rangle + f \cong \langle a\rangle + \phi$ and $a \notin D(\alpha) \iff a \in D[\langle -a\rangle + f]$ and $-a \notin D(f)$.

Finally, $m_{f'}(a) = 1 \iff a \notin D[\langle -a\rangle + f]$. Thus

$$m_{f'}(a) = \begin{cases} 3, & \text{if } -a \in D(f), \\ 2, & \text{if } -a \notin D(f) \text{ and } a \in D[\langle -a\rangle + f], \\ 1, & \text{if } a \notin D[\langle -a\rangle + f]. \end{cases}$$

Similarly,

$$m_{g'}(a) = \begin{cases} 3, & \text{if } -a \in D(g), \\ 2, & \text{if } -a \notin D(g) \text{ and } a \in D[\langle -a\rangle + g], \\ 1, & \text{if } a \notin D[\langle -a\rangle + g]. \end{cases}$$

Since $D(f) = D(g), D[\langle -a\rangle + f] = D[\langle -a\rangle + g]$ for every $a \in \dot{F}$. So $m_{f'} = m_{g'}$, but $f' \ncong g'$ since $f \ncong g$. But this contradicts the fact that the square class invariant classifies forms over F. Hence F must be a C-field. □

Solow [**6**] showed that the square class invariant classifies forms over the following Pythagorean fields:

(1) Pythagorean fields F which satisfy the Strong Approximation Property and for which $q(F) < \infty$.

(2) Superpythagorean fields.

(3) Iterated power series fields over a field K, where K is of type 1 or 2 above.

Shapiro and Lam [**5**] showed that Solow's results include all the Pythagorean fields with finite square class group for which the square class invariant classifies forms. Furthermore, Shapiro and Lam extended the results to Pythagorean fields with finitely many real-valued places. They proved that if K is a Pythagorean field with only finitely many real-valued places, then the square class invariant classifies forms over K if and only if K is equivalent to an iterated power series field over k, where k is some SAP Pythagorean field.

The first part of this paper considered fields where the level was at most 2. The following result applies to all non-Pythagorean fields with level greater than 2.

PROPOSITION 4. *Let F be a non-Pythagorean field with $s(F) \geq 4$. Then the square class invariant does not classify forms over F.*

PROOF. Since F is not Pythagorean, $D\langle 1,1\rangle \supsetneq \dot{F}^2$. Let $\alpha \in D\langle 1,1\rangle - \dot{F}^2$. Consider $f = \langle 1,-1,-1\rangle$ and $g = \langle 1,-1,-\alpha\rangle$. Both f and g are isotropic and thus universal, so $m_f \geq 1$ and $m_g \geq 1$. Suppose $m_f(a) = 3$ for some $a \in \dot{F}$. Then $\langle 1,-1,-1\rangle \cong \langle a,a,a\rangle$ and clearly $a \in \dot{F}^2$ by comparing determinants. So $\langle 1,-1,-1\rangle \cong \langle 1,1,1\rangle$ and $-1 \in D\langle 1,1\rangle$. This contradicts $s(F) \geq 4$. Thus $1 \leq m_f(a) \leq 2$, for every $a \in \dot{F}$.

Now

$$\begin{aligned} m_f(a) = 2 &\iff \langle 1,-1,-1\rangle \cong \langle a,a,1\rangle \iff \langle -1,-1\rangle \cong \langle a,a\rangle \\ &\iff -a \in D\langle 1,1\rangle. \end{aligned}$$

Thus

$$m_f(a) = \begin{cases} 2, & \text{if } -a \in D\langle 1,1\rangle, \\ 1, & \text{otherwise.} \end{cases}$$

Now suppose $m_g(a) = 3$, for some $a \in \dot{F}$. Then $\langle 1,-1,-\alpha\rangle \cong \langle a,a,a\rangle$. Again, by determinants, $a \in \alpha\dot{F}^2$, and $\langle 1,-1,-\alpha\rangle \cong \langle \alpha,\alpha,\alpha\rangle$. This implies that $\langle \alpha,-\alpha,-\alpha\rangle \cong \langle \alpha,\alpha,\alpha\rangle$ and that $-1 \in D\langle 1,1\rangle$, another contradiction to $s(F) \geq 4$. Thus $1 \leq m_g(a) \leq 2$, for every $a \in \dot{F}$. Also

$$\begin{aligned} m_g(a) = 2 &\iff \langle 1,-1,-\alpha\rangle \cong \langle a,a,\alpha\rangle \iff \langle \alpha,-\alpha,-\alpha\rangle \cong \langle a,a,\alpha\rangle \\ &\iff \langle -\alpha,-\alpha\rangle \cong \langle a,a\rangle \iff -a\alpha \in D\langle 1,1\rangle \\ &\iff -a \in D\langle 1,1\rangle \quad \text{since } \alpha \in D\langle 1,1\rangle. \end{aligned}$$

Thus

$$m_g(a) = \begin{cases} 2, & \text{if } -a \in D\langle 1,1\rangle, \\ 1, & \text{otherwise.} \end{cases}$$

Hence $m_f = m_g$. But $f \not\cong g$ since $\det f \neq \det g$. So the square class invariant does not classify forms over F. □

The following theorem summarizes the propositions above and, along with the results in [**5**], completely answers the question of when the square class invariant classifies quadratic forms, at least in the presence of certain relatively weak finiteness conditions.

THEOREM. *Let F be a field. If F is non-Pythagorean and $s(F) \geq 4$, then the square class invariant does not classify forms over F. If $s(F) \leq 2$ and if the square class invariant classifies forms over F, then F is a C-field. If F is a C-field with $s(F) = 1$ or with $s(F) = 2$ and either $u(F)$ or $|\dot{F}/D\langle 1,1\rangle|$ finite, then the square class invariant classifies forms over F.*

One might ask whether the square class invariant and another set of invariants might be sufficient to classify forms. Solow considered adding the determinant and found that these two invariants did work for local fields. In fact, using the same techniques she applied to the dyadic case [**7**], one can extend her results to all generalized Hilbert fields [**3**] (i.e., fields having exactly two quaternion algebras). The proof of the next proposition is a lengthy explanation of cases and is omitted. $R(F)$ denotes the Kaplansky radical of F.

PROPOSITION 5. *Let F be a generalized Hilbert field. Assume also that if $s(F) = 1$, then $R(F) = \dot{F}^2$ and that if $s(F) = 2$, then $-1 \notin R(F)$. Then the square class invariant and determinant classify quadratic forms over F.*

The assumptions in Proposition 5 are necessary. For example, if F is a generalized Hilbert field with $s(F) = 1$ and $R(F) \neq \dot{F}^2$, let $d \in R(F) - \dot{F}^2$ and set $f = \langle 1,1\rangle$. If g is the unique four-dimensional anisotropic form of determinant d, then $m_f = m_g = 1$. If we choose such a field with $s(F) = 2$ and $-1 \in R(F)$, then $f = \langle 1, -1, \rangle$ and the unique four-dimensional anisotropic form g of determinant -1 also yield $m_f = m_g = 1$.

Once the number of quaternion algebras increases to four, however, the square class invariant and determinant are no longer always sufficient as the next result illustrates.

PROPOSITION 6. *If F is a nonreal field having more than two quaternion algebras and satisfying $u(F) = 4$, then the square class invariant and determinant do not classify quadratic forms over F.*

PROOF. If ϕ is a quaternion form, then $m_\phi(\alpha) = m_\phi(1)$ for every $\alpha \in \dot{F}$ since ϕ is universal and $G(\phi) = D(\phi)$. Thus m_ϕ is a constant function, and, in particular, $m_\phi = m_\phi(1)$ which is a power of 2.

It is well known that if $u(F) = 4$, then the quaternion algebras form a subgroup of the Brauer group. Let ϕ_1, ϕ_2, ϕ_3 be the anisotropic quaternion forms corresponding to distinct quaternion algebras A_1, A_2, A_3 which satisfy $A_1A_2A_3 = 1$. Then $\det \phi_i = 1$ for $i = 1, 2, 3$. If $s(F) \leq 2$, then $m_{\phi_i} \leq 2$; and so $m_{\phi_i}(1) = m_{\phi_j}(1)$ for some $i \neq j$. Thus $m_{\phi_i} = m_{\phi_j}$, and the square class invariant and determinant do not classify quadratic forms. Now suppose $s(F) = 4$. Then $m_{\phi_i} = 1, 2,$ or 4 for $1 \leq i \leq 3$. Say $m_{\phi_1} = 1, m_{\phi_2} = 2$, and $m_{\phi_3} = 4$. Since $m_{\phi_2}(1) = 2$, $A_2 = [-1, -\alpha]$ for some $\alpha \in \dot{F} - D\langle 1, 1\rangle$. Since $m_{\phi_3}(1) = 4$, $A_3 = [-1, -1]$. But $A_1 = A_2A_3 = [-1, -\alpha][-1, -1] = [-1, \alpha]$. Thus $m_{\phi_1}(1) \geq 2$ which contradicts $m_{\phi_1} = 1$. Thus $m_{\phi_i} = m_{\phi_j}$ for some $i \neq j$. So the square class invariant and determinant do not classify quadratic forms over F. □

COROLLARY. *Let F be a field with $R(F) = \dot{F}^2, s(F) = 4$, and $u(F) = 4$. If the square class invariant and determinant classify quadratic forms over F, then F is a C-field.*

PROOF. By the Proposition, $m(F)$, the number of quaternion algebras over F, must be 2. By Proposition 1 in [**1**] the number of anisotropic four-dimensional forms of determinant d which represent 1 is equal to $m(F) - |\dot{F}/D\langle 1, -d\rangle|$. In this case then the above number will be 0 if $d \notin R(F)$ and 1 if $d \in R(F)$. Since $u(F) = 4$, every four-dimensional form represents 1. Hence, there exists a unique anisotropic u-dimensional form; and, by Proposition 5.3 in [**2**], F is a C-field. □

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