ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 1, Winter 1990

THE AUTOMORPHISM GROUPS OF THE HYPERELLIPTIC SURFACES

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1. Introduction. In this paper we will compute the automorphism groups of the so-called hyperelliptic surfaces. These compact complex surfaces are characterized by having invariants $p_g = 0$, $q = 1$, and $12K = 0$. References for the elementary properties of these surfaces may be found in [**2**] (where they are called "bielliptic surfaces") or in [**1**]. They may all be constructed as the quotient $X = (E \times F)/G$, where E and F are elliptic curves, and G is a finite group of translations of E acting also on F *not* only as a group of translations; the action on $E \times F$ is the diagonal action.

There are seven non-isomorphic groups G which can act on $E \times F$ as above, two of which act on any $E \times F$, the other five requiring F to be a specific elliptic curve. In the following table the reader will find a list of the seven groups G, together with the elliptic curves E and F, and the action of G on $E \times F$.

Write $E = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau_1)$ and $F = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau_2)$. Throughout this article we will use the notation $i = \sqrt{-1}, \omega = e^{2\pi i/3}$, and $\zeta = e^{\pi i/3}$; note that $\omega = \zeta^2$.

In the last three cases it is technically more convenient to consider $X = (E \times F)/G$ as the quotient of $Y = (E \times F)/\langle\psi\rangle$ by a cyclic group of order $r(= 2, 3, 4$, or $6)$, generated by the automorphism $\bar{\phi}$ induced by ϕ. Since ψ is a translation of $E \times F, Y$ is also a complex torus of dimension two. For uniformity of notation we will define $Y = E \times F$ and $\psi =$ identity in the first four cases, so that in each case $X = Y/\langle\bar{\phi}\rangle$. Note that r is the order of the canonical class K_X in $\operatorname{Pic}(X)$ and Y is the etale cyclic cover of X defined by $K_X : Y = \operatorname{Spec}(\oplus_{i=0}^{r-1}\varphi_X(iK_X))$, with the multiplication in φ_Y defined by a chosen isomorphism $\theta : \varphi_X \to \varphi_X(rK_X)$. The formation of Y

Both authors gratefully acknowledge the support of the NSF while this research was being completed.
Received by the editors on August 8, 1986 and in revised form on April 4, 1987.

from X is functorial: if $p : T \to X$ is a scheme over X, a morphism from T to Y over X corresponds to an φ_T-map $\alpha : p^*K_X \to \varphi_T$ such that the composition $\varphi_T \xrightarrow{p^*\theta} \varphi_T(rp^*K_X) \xrightarrow{\alpha^{\otimes r}} \varphi_T^{\otimes r} \xrightarrow{\text{mult}} \varphi_T$ is the identity. This description allows us to readily conclude the lemma,

TABLE 1.1.

The seven groups G used to construct the hyperelliptic surfaces.
In all cases τ_1 is arbitrary.

τ_2	G	action of the generators of G on $E \times F$
arbitrary	$\mathbf{Z}/2 = \langle \phi \rangle$	$\phi\binom{e}{f} = \binom{e+1/2}{-f}$
ζ	$\mathbf{Z}/3 = \langle \phi \rangle$	$\phi\binom{e}{f} = \binom{e+1/3}{\omega f}$
i	$\mathbf{Z}/4 = \langle \phi \rangle$	$\phi\binom{e}{f} = \binom{e+1/4}{if}$
ζ	$\mathbf{Z}/6 = \langle \phi \rangle$	$\phi\binom{e}{f} = \binom{e+1/6}{\zeta f}$
arbitrary	$\mathbf{Z}/2 \times \mathbf{Z}/2 = \langle \phi, \psi \rangle$	$\phi\binom{e}{f} = \binom{e+1/2}{-f}; \psi\binom{e}{f} = \binom{e+\tau 1/2}{f+1/2}$
ζ	$\mathbf{Z}/3 \times \mathbf{Z}/3 = \langle \phi, \psi \rangle$	$\phi\binom{e}{f} = \binom{e+1/3}{\omega f}; \psi\binom{e}{f} = \binom{e+\tau 1/3}{f+(1+\zeta)/3}$
i	$\mathbf{Z}/4 \times \mathbf{Z}/2 = \langle \phi, \psi \rangle$	$\phi\binom{e}{f} = \binom{e+1/4}{if}; \psi\binom{e}{f} = \binom{e+\tau 1/2}{f+(1+i)/2}$

LEMMA 1.2. *Every automorphism of X lifts to Y.*

PROOF. Let $\pi : Y \to X$ be the quotient map and assume σ is an automorphism of X. Let $p : Y \to X$ be the composition $p = \sigma \circ \pi$. We require a lifting, $f : Y \to Y$ such that $\pi \circ f = p = \sigma \circ \pi$. Since σ is an automorphism of $X, \sigma^*K_X \cong K_X$; since π is unramified, $\pi^*K_X \cong K_Y$. Moreover since Y is an abelian surface, $K_Y \cong \varphi_Y$; hence $p^*K_X \cong \varphi_Y$. We may then choose an isomorphism $\alpha : p^*K_X \to \varphi_Y$ so that the composition $\text{mult} \circ \alpha^{\otimes r} \circ p^*\theta$ is the identity; in fact there are r choices for α, differing from each other by a factor which is an r^{th} root of unity. Each of these choices for α provide a lift to Y of the automorphism σ. $\square$

Since every automorphism of X lifts to Y, the standard theory of covering spaces [**3**] implies that $\mathrm{Aut}(X) \cong N/\langle\bar{\phi}\rangle$, where N is the normalizer of $\langle\bar{\phi}\rangle$ in $\mathrm{Aut}(Y)$. It is this group we will calculate in the first four cases where $Y = E \times F$; in the last three we can in fact lift automorphisms to $E \times F$ also, and make the analysis there.

There does not seem to be any standard notation for the hyperelliptic surfaces. We will use $X_r(\tau_2)$ for the first four surfaces in Table 1.1, for which Y is the product $E \times F$, and $\bar{X}_r(\tau_2)$ for the last three; if $r \neq 2$ then we will drop the τ_2, which is determined. Hence the hyperelliptic surfaces are $X_2(\tau_2), X_3, X_4, X_6, \bar{X}_2(\tau_2), \bar{X}_3$, and $\bar{X}_4$ in the order in which they appear in Table 1.1. Note that they all of course depend on τ_1 also, which we omit from the notation.

2. The lifting to $\mathbf{E} \times \mathbf{F}$. Since Y is an abelian surface, $\mathrm{Aut}(Y)$ is an extension of $\mathrm{Aut}_0(Y)$ (the subgroup of automorphisms fixing 0) by the translation subgroup. $\mathrm{Aut}_0(Y)$ has a natural representation into $\mathrm{GL}(2, \mathbf{C})$, inducing a homomorphism from $\mathrm{Aut}(Y)$ to $\mathrm{GL}(2, \mathbf{C})$; we will denote the image of an automorphism α of Y by $\alpha_* \in \mathrm{GL}(2, \mathbf{C})$. By composing with the determinant we have a homomorphism $\det : \mathrm{Aut}(Y) \to \mathbf{C}^*$. These same constructions apply to $E \times F$ as well, and we will use the same notation for them.

LEMMA 2.1. *Let N be the normalizer of $\langle\bar{\phi}\rangle$ in* $\mathrm{Aut}(Y)$. *Then $\alpha \in N$ if and only if α is induced from an element of* $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ *which normalizes G.*

PROOF. Let $\alpha \in N$. Then $\alpha\bar{\phi}\alpha^{-1} = \bar{\phi}^k$, and applying det to both sides forces $k = 1$, showing that α and $\bar{\phi}$ must in fact commute. Therefore α_* commutes with $\bar{\phi}_* = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$, where $\varepsilon = e^{2\pi i/r}$. Therefore α_* must be diagonal, since $\varepsilon \neq 1$; but this is equivalent to α lifting to an element of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$, which must normalize G, since it descends to α, which descends to X. Conversely, if β is in $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ and normalizes G, then $\beta\psi\beta^{-1} = \phi^i\psi^j$, and applying det to both sides forces $i = 0$, so β normalizes $\langle\psi\rangle$ and descends to some $\alpha \in \mathrm{Aut}(Y)$. Since β normalizes $G = \langle\phi, \psi\rangle, \alpha$ will normalize $\langle\bar{\phi}\rangle$. □

The elements of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ can be conveniently represented by 4-tuples $[p, q; a, d]$, which will denote the map sending (e, f) to $(ae + p, df + q)$; here $p \in E, q \in F, a \in \mathrm{Aut}_0(E)$, and $d \in \mathrm{Aut}_0(F)$. Note that, in this notation, $\phi = [1/r, 0; 1, e^{2\pi i/r}]$ and $\psi = [u, v; 1, 1]$ for appropriate u, v. It is easy to verify the following formulas:

$$[p_1, q_1; a_1, d_1][p_2, q_2; a_2, d_2] = [p_1 + a_1 p_2, q_1 + d_1 q_2; a_1 a_2, d_1 d_2], \tag{2.1}$$

$$[p, q; a, d]^{-1} = [-a^{-1}p, -d^{-1}q; a^{-1}, d^{-1}], \tag{2.2}$$

$$[p, q; a, d][u, v; 1, 1][p, q; a, d]^{-1} = [au, dv; 1, 1], \tag{2.3}$$

$$[p, q; a, d]\phi[p, q; a, d]^{-1}\phi^{-1} = [(a - 1)/r, (1 - e^{2\pi i/r})q; 1, 1]. \tag{2.4}$$

These allow us to prove the following refinement of Lemma (2.1):

Lemma 2.6. *Any element of* $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ *which normalizes* G *in fact centralizes* G, *i.e., commutes with* ϕ *and* ψ.

Proof. Let $\beta \in \mathrm{Aut}(E) \times \mathrm{Aut}(F)$ normalize G. Then $\beta\psi\beta^{-1} = \phi^i\psi^j$, and applying det to both sides forces $i = 0$, so $\beta\psi\beta^{-1} = \psi^j$. Similarly $\beta\phi\beta^{-1} = \phi^i\psi^k$, and applying det forces $i = 1$, so that $\beta\phi\beta^{-1}\phi^{-1} = \psi^k$ for some k. We want to show that $k = 0$ and $j = 1$. In the first four cases when Y is a product, ψ is the identity and there is nothing to show; hence we must analyze only the last three cases. In these cases $\psi = [u, v; 1, 1]$, where $u = n\tau_1/r$; here $n = 1$ if $r = 2$ or 3 and $n = 2$ if $r = 4$. Write $\beta = [p, q; a, d]$ and assume $\beta\psi\beta^{-1} = \psi^j$ and $\beta\phi\beta^{-1}\phi^{-1} = \psi^k$. Then, from (2.3) and (2.4), we must have

$$(a - 1)/r = kn\tau_1/r \quad \text{and} \quad an\tau_1/r = jn\tau_1/r \tag{2.5}$$

by only considering the first coordinate in the two equalities. Recalling that $a \in \mathrm{Aut}_0(E)$ and $0 \le j, k < r/n$, one checks easily that the only solutions to (2.5) are $a = j = 1,\ k = 0,\ r = 2, 3, 4$ and $a = -1,\ j = 1,\ k = 0,\ r = 2$. In all cases $k = 0$ and $j = 1$, proving the lemma. □

3. The computation of AutX). Let M denote the centralizer of G in $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$. By the above lemma, $\mathrm{Aut}(X) \cong N/\langle\bar{\phi}\rangle \cong M/G$. It is a simple matter to calculate M using formulas (2.1)–(2.4); we present the results below

PROPOSITION 3.1.

(a) $M(X_2(\tau_2)) = \{[p,q;a,d] | a = \pm 1, d \in \mathrm{Aut}(F)$, and $2q = 0$, i.e., $q = 0, 1/2, \tau_2/2$, or $(1+\tau_2)/2 \mod \Lambda_2\}$,

(b) $M(X_3) = \{[p,q;a,d] | a = 1, d \in \mathrm{Aut}(F)$, and $(\omega - 1)q = 0$, i.e., $q = 0, (1+\zeta)/3$, or $2(1+\zeta)/3 \mod \Lambda_2\}$,

(c) $M(X_4) = \{[p,q;a,d] | a = 1, d \in \mathrm{Aut}(F)$, and $(i-1)q = 0$, i.e., $q = 0$ or $(1+i)/2 \mod \Lambda_2\}$,

(d) $M(X_6) = \{[p,q;a,d] | a = 1, d \in \mathrm{Aut}(F)$, and $q = 0\}$,

(e) $M(\bar{X}_2(\tau_2)) = \{[p,q;a,d] | a = \pm 1, d = \pm 1$, and $2q = 0$, i.e., $q = 0, 1/2, \tau_2/2$, or $(1+\tau_2)/2 \mod \Lambda_2\}$,

(f) $M(\bar{X}_3) = \{[p,q;a,d] | a = 1, d = 1, \omega$, or ω^2, and $(\omega - 1)q = 0$, i.e., $q = 0, (1+\zeta)/3$, or $2(1+\zeta)/3 \mod \Lambda_2\}$,

(g) $M(\bar{X}_4) = M(X_4)$.

It is evident from the above proposition that in every case M is generated by its E-translations, its F-translations, its E-automorphisms (elements of $\mathrm{Aut}_0(E)$), and its F-automorphisms. It may be convenient to the reader to present these generators for M, which we do in Table 3.1.

Note that, in every case, $p \in E$ is arbitrary, so that $E \subseteq M$ as the subgroup $\{[p,0;1,1]\}$; moreover $E \cap G = \{\mathrm{id}\}$. Hence E also embeds in the quotient $M/G \cong \mathrm{Aut}(X)$ as a normal subgroup and we will consider our task complete if we identify the quotient of M/G by E which is a finite group. We will also give generators for $\mathrm{Aut}(X)/E$, lifted to M. We present this information in Table 3.2.

TABLE 3.1.
Generators for M

X	translations of E	translations of F	automorphisms of E	automorphisms of F
$X_2(i)$	E	$\{0, 1/2, i/2, (1+i)/2\}$	$\{\pm 1\}$	$\{1, i, -1, -i\}$
$X_2(\zeta)$	E	$\{0, 1/2, \zeta/2, (1+\zeta)/2\}$	$\{\pm 1\}$	$\{1, \zeta, \zeta^2, -1, -\zeta, -\zeta^2\}$
$X_2(\tau_2)$	E	$\{0, 1/2, \tau_2/2, 1+\tau_2/2\}$	$\{\pm 1\}$	$\{\pm 1\}$
	(for τ_2 general, i.e., Λ_2 is neither square nor hexagonal)			
X_3	E	$\{0, (1+\zeta)/3, (2+2\zeta)/3\}$	$\{1\}$	$\{1, \zeta, \zeta^2, -1, -\zeta, -\zeta^2\}$
X_4	E	$\{0, (1+i)/2\}$	$\{1\}$	$\{1, i, -1, -i\}$
X_σ	E	$\{0\}$	$\{1\}$	$\{1, \zeta, \zeta^2, -1, -\zeta, -\zeta^2\}$
$\bar{X}_2(\tau_2)$	E	$\{0, 1/2, \tau_2/2, (1+\tau_2)/2\}$	$\{\pm 1\}$	$\{\pm 1\}$
$\bar{X}_3$	E	$\{0, (1+\zeta)/3, (2+2\zeta)/3\}$	$\{1\}$	$\{1, \omega, \omega^2\}$
$\bar{X}_4$	E	$\{0, (1+i)/2\}$	$\{1\}$	$\{1, i, -1, -i\}$

TABLE 3.2.

X	$\lvert\mathrm{Aut}\,(X)/E\rvert$	$\mathrm{Aut}\,(X)/E$	generators for $\mathrm{Aut}\,(X)/E$ in M
$X_2(i)$	16	$\mathbf{Z}/2 \times D_8$	$[0,0;-1,1]$ generates the $\mathbf{Z}/2$
		(D_8 is the dihedral	$[0,1/2;1,i]$ has order 4 in D_8
		group of order 8	$[0,0;1,i]$ has order 2 in D_8
$X_2(\zeta)$	24	$\mathbf{Z}/2 \times A_4$	$[0,0;-1,1]$ generates the $\mathbf{Z}/2$
		(A_4 is the	$[0,0;1,\zeta]$ has order 3 in A_4
		alternating group	$[0,1/2;1,1]$ and $[0,\zeta/2;1,1]$
		of order 12)	generate the 2-part of A_4
$X_2(\tau_2)$	8	$(\mathbf{Z}/2)^3$	$[0,0;-1,1], [0,1/2;1,1]$, and
	(τ_2 general)		$[0,\tau_2/2;1,1]$ generate $(\mathbf{Z}/2)^3$
X_3	6	S_3	$[0,(1+\zeta)/3;1,1]$ has order 3
		(the symmetric group)	$[0,0;1,\zeta]$ has order 2
X_4	2	$\mathbf{Z}/2$	$[0,(1+i)/2;1,1]$ generates
X_σ	1	$\{1\}$	
$\bar{X}_2(\tau_2)$	4	$\mathbf{Z}/2 \times \mathbf{Z}/2$	$[0,0;-1,1]$ and $[0,\tau_2/2;1,1]$
			generate the $\mathbf{Z}/2 \times \mathbf{Z}/2$
$\bar{X}_3$	1	$\{1\}$	
$\bar{X}_4$	1	$\{1\}$	

With this table we consider our description of $\mathrm{Aut}(X)$ complete. We note the following interesting corollary:

Every automorphism of $\bar{X}_r(\tau_2)$ *lifts to* $X_r(\tau_2)$.

Indeed, we have proven that every automorphism of $\bar{X}_r(\tau_2)$ lifts to $E \times F$, in fact to an automorphism which commutes with ϕ. Hence that lifting descends to $X_r(\tau_2)$.

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