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THE SPATIAL FORM OF ANTIAUTOMORPHISMS OF VON NEUMANN ALGEBRAS

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1. Introduction. There are three problems which have been studied concerning antiautomorphisms of von Neumann algebras; the existence problem, the conjugacy problem, and their description. The latter problem includes whether they are spatial of a particular form, i.e., of the form $x \to w^* x^* w$ with w a conjugate linear isometry of a prescribed type. In the present paper we shall study the spatial problem, with main emphasis on antiautomorphisms α leaving the center elementwise fixed, called *central* in the sequel, and with α an *involution*, i.e., $\alpha^2 = 1$. This problem with variations has previously been studied in [**2, 6**]. E.g., it was shown in [**6**] that a central involution α is automatically spatial with w^2 a selfadjoint unitary operator in the center of the von Neumann algebra.

It turns out that the general problem of whether a central antiautomorphism is spatial has a solution similar to that of automorphisms, with proof also quite similar. We include these results for the sake of completeness. The main new ingredient in the paper is that if α is a central involution of the von Neumann algebra M then α is necessarily of the form $\alpha(x) = Jx^*J$ with J a conjugation, unless the commutant M' of M has a direct summand of type I_n with n odd. In the latter case it may happen that α can only be written in the form $\alpha(x) = -jx^*j$ with $j^2 = -1$.

2. The results. Recall that two projections e and f in a von Neumann algebra M acting on a Hilbert space H are said to be equivalent, written $e \sim f(\operatorname{mod} M)$, or just $e \sim f$ if there is a partial isometry $v \in M$ such that $v^*v = e$, $vv^* = f$. e is said to be cyclic, written $e = [M'\xi]$ if there is a vector $\xi \in H$ such that e is the projection onto the space spanned by vectors of the form $x'\xi,\ x' \in M'$. If w is a conjugate linear operator we denote by w^* its adjoint, viz, $(w^*\xi, \eta) = (w\eta, \xi)$. We denote by ω_ξ the positive functional $\omega_\xi(x) = (x\xi, \xi)$ on M.

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Lemma. *Let M be a von Neumann algebra acting on a Hilbert space H. Suppose α is a central antiautomorphism of M. Let ξ be a unit vector in H.*

(i) *If $[M'\xi] \sim \alpha([M'\xi])(\mathrm{mod}\, M)$ then there exists a unit vector $\eta \in [M\xi]H$ satisfying*

(1) $\omega_\eta = \omega_\xi o \alpha$

(2) $[M\xi] \sim [M\eta]$ $(\mathrm{mod}\, M')$

(ii) *If η is a unit vector in H satisfying* (1)*, then there exists a conjugate linear partial isometry w on H such that $w^*w = [M\xi], ww^* = [M\eta]$, and $w^*x^*w[M\xi] = \alpha(x)[M\xi], x \in M$.*

(iii) *Suppose ξ is cyclic and ω_ξ is α-invariant. If $\alpha^{2n} = \iota$, the identity map, then w can be chosen so that $w^{2n} = 1$.*

Proof. Let $e = [M'\xi]$ be the support of the vector state ω_ξ. Let $f = \alpha^{-1}(e)$. By assumption $e \sim f$, so there exists a partial isometry $v \in M$ such that $v^*v = e$, $vv^* = f$. Since α and α^{-1} are order isomorphisms of M,

$$\mathrm{supp}\, \omega_\xi o \alpha = \alpha^{-1}(\mathrm{supp}\, \omega_\xi) = \alpha^{-1}(e) = f = [M'v\xi].$$

By [**8**, Theorem 5.23] there is $\eta \in [M\xi]H$ such that $\omega_\xi o \alpha = \omega_\eta$. This proves (1). Since $[M'\eta] = \alpha^{-1}(e) \sim e = [M'\xi]$, (2) follows by [**1**; Chapter III, §1.3, Corollary].

With η as above define a conjugate linear operator $w : M\xi \to M\eta$ by $wx\xi = \alpha^{-1}(x^*)\eta$. Then $||wx\xi|| = ||\alpha^{-1}(x^*)\eta||^2 = (\alpha^{-1}(x^*x)\eta, \eta) = (x^*x\xi, \xi) = ||x\xi||^2$, so that w extends to a conjugate linear isometry of $[M\xi](H)$ onto $[M\eta](H)$. Extend w to all of H by defining it to be 0 on $[M\xi](H)$. Since $w^*w = [M\xi]$ we have, for $x, y \in M$,

$$\begin{aligned} w^*x^*wy\xi &= w^*x^*\alpha^{-1}(y^*)\eta \\ &= w^*\alpha^{-1}(y^*\alpha(x^*))\eta = w^*w(y^*\alpha(x^*))^*\xi = \alpha(x)y\xi. \end{aligned}$$

Thus (iii) follows.

Finally, if ξ is cyclic and ω_ξ is α-invariant, then w is a conjugate linear isometry such that $w^*x^*w = \alpha(x)$, $x \in M$. By definition of

$w, w^{2k}x\xi = \alpha^{-2k}(x)\xi, k \in N$; hence, in particular, $w^{2n}x\xi = x\xi$ for all x, so that $w^{2n} = 1$. □

THEOREM 1. *Let M be a von Neumann algebra and α an antiautomorphism such that $\alpha(e) \sim e$ for all projections $e \in M$. Then α is spatial.*

PROOF. We first note that if e' is a projection in M' then the map $\alpha_{e'} : Me' \to Me'$ defined by

$$\alpha_{e'}(xe') = \alpha(x)e'$$

is an antiautomorphism. Indeed, if $x \in M$ let c_x denote the central projection which is the intersection of all central projections q in M with $qx = x$. Since the assumption on α implies α is central, $c_x = c_{\alpha(x)}$. By [**5**; Lemma 3.1.1] $xe' = 0$ if and only if $0 = c_x c_{e'} = c_{\alpha(x)} c_{e'}$, if and only if $\alpha(x)e' = 0$. Thus $\alpha_{e'}$ is well defined and injective. Since it is clearly surjective, the assertion follows.

To prove the theorem let, by Zorn's lemma, p' be a projection in M' maximal with respect to the property that $\alpha_{p'}$ is spatial on Mp'. Suppose $p' \neq 1$ and let $q' = 1 - p'$. Let ξ be a unit vector in $q'(H)$ and let, by Lemma, (i), η be a unit vector in $q'(H)$ such that $\omega_\eta = \omega_\xi o \alpha$ on Mq'. Let $w : [M\xi](H) \to [M\eta](H)$ be as in Lemma, (ii). By Lemma, (i) $[M\xi] \sim [M\eta] \pmod{M'}$ so there is $u \in M'$ such that $u^*u = [M\eta], uu^* = [M\xi]$. Then uw is a conjugate linear partial isometry which is 0 on $[M\xi](H)^\perp$ and isometric on $[M\xi](H)$ onto itself, such that if $x \in M[M\xi]$ then

$$(uw)^*x^*(uw) = w^*u^*xuw = w^*xw = \alpha_{[M\xi]}(x),$$

using $u \in M'$ and $[M\xi]u = u$. Thus $\alpha_{p'} + \alpha_{[M\xi]} = \alpha_{p'+[M\xi]}$ is spatial, contradicting the maximality of p'. Thus $p' = 1$, completing the proof. □

THEOREM 2. *Let M be von Neumann algebra with no direct summand of type II_∞ with finite commutant. Then each central antiautomorphism of M is spatial.*

PROOF. Let α be a central antiautomorphism of M. We may consider the different types separately. The type I portion is taken care of by [**6**, Lemma 4.3]. Suppose M is finite. Let Φ be the centervalued trace on M which is the identity on the center. By uniqueness of $\Phi, \Phi o\alpha = \Phi$, hence $\Phi(\alpha(e)) = \Phi(e)$ for all projections e. It follows that $e \sim \alpha(e)$ for all projections, hence α is spatial by Theorem 1.

Assume M is of type II_∞ with II_∞ commutant. Since the identity is the sum of central projections which are countably decomposable with respect to the center, we may assume the center is countably decomposable. By [**5**, Lemma 3.3.6] there is a cyclic projection $e = [M'\xi], \xi$ a unit vector, in M with central support 1 such that eq is infinite for all central projections $q \neq 0$ in M. Since α maps infinite projections onto infinite projections, $f = \alpha^{-1}(e)$ is infinite and is the support of $\omega_\xi o\alpha$. Since M' is infinite there is a unit vector η such that $\omega_\xi o\alpha = \omega_\eta$ [**1**; Chapter III, §8.6, Corollary 10 and Chapter III, §1.4, Theorem 4]. Thus $f = [M'\eta]$ is countably decomposable, and fq is infinite for all central projections $q \neq 0$, and the central support of f equals that of e since α is central. By [**1**; Chapter III,§8.6, Corollary 5] $f \sim e$.

The proof is completed with a maximality argument similar to that used in Theorem 1. Let p' be a projection in M' maximal with respect to the property that $\alpha_{p'}$ is spatial on Mp'. Suppose $q' = 1-p' \neq 0$. Apply the previous paragraph to Mq' and find a cyclic projection $e = [M'\xi]$ in M with ξ a unit vector in $q'M$ with the described properties. Then $f = \alpha^{-1}(e) = [M'\eta] \sim e$, where $\omega_\xi o\alpha = \omega_\eta$ on Mq'. The proof is now completed exactly like that of Theorem 1.

Finally, assume M is of type III. Then each normal state is a vector state [**1**; Chapter III, §8.6, Corollary 10] so the conclusion of Lemma, (i) holds. Since any two countably decomposable projections with the same central supports are equivalent in M, the argument from the proof of Theorem 1 case applies to conclude that α is spatial. □

Remark 1. The above theorem reflects the situation for automorphisms of von Neumann algebras. For a factor M of type II_∞ with finite commutant it was shown by Kadison [**4**] that an automorphism is spatial if and only if it preserves the trace, or equivalently the dimension of projections. By Theorem 1 the latter condition is sufficient for an anti-automorphism α to be spatial. Conversely, if α is spatial the argument of Kadison on [**4**, p. 324] can be repeated word by word to conclude that α preserves the dimension of projections.

The difficulty in the above situation can be avoided if α is periodic.

THEOREM 3. *Let M be a von Neumann algebra and α a periodic central antiautomorphism. Then α is spatial. Furthermore, if each normal state on M is a vector state* (*e.g., if M has a separating vector, or M' is properly infinite*) *then there exists a conjugate linear isometry w such that $\alpha(x) = w^*x^*w$ with $w^{2n} = 1$, where $2n$ is the period of α.*

PROOF. Let e be a projection in M. In order to show $\alpha(e) \sim e$ we may, since α is central, assume by the Comparison Theorem that $\alpha(e) \prec e$. Iterating, we have $e = \alpha^{2n}(e) \prec \alpha^{2n-1}(e) \prec \cdots \prec \alpha(e) \prec e$. Thus $\alpha(e) \sim e$, and α is spatial by Theorem 1.

Now assume each normal state is a vector state. Let ϕ be a unit vector. Then the state

$$\omega = \frac{1}{2n} \sum_{k=1}^{2n} \omega_\phi o \alpha^k$$

is a normal α-invariant state. Thus $\omega = \omega_\xi$ for a unit vector ξ, and $\omega_\xi o \alpha = \omega_\xi$. By the proof of Lemma, (iii) there exists a conjugate linear partial isometry w with support and range $[M\xi]$ such that $w^{2n} = [M\xi]$, and $w^*x^*w[M\xi] = \alpha(x)[M\xi]$. A maximality argument like that employed in the proof of Theorem 1 now completes the proof. $\square$

The above theorem states that, for a periodic α with M' large, then w can be chosen with $w^{2n} = 1$. Our last result gives a sharper statement if α is an involution. Special cases of this result appeared in [**6**]. Recall that a conjugation is a conjugate linear isometry J such that $J^2 = 1$.

THEOREM 4. *Let M be a von Neumann algebra whose commutant has no direct summand of type I_n with n an odd integer. If α is a central involution on M then there exists a conjugation J such that $\alpha(x) = Jx^*J,\ x \in M$.*

PROOF. Let M act on a Hilbert space H and assume first that M has no direct summand of type I. By [**6**, Theorem 3.7] there exist central projections p and q in M such that $\alpha|pM$ is implemented by a conjugation on $p(H)$ and $\alpha|qM$ by a conjugate isometry j with $j^2 = -q$. To prove the theorem it suffices to modify j so that $\alpha|qM$ is implemented by a conjugation. We therefore assume $\alpha(x) = -jx^*j$ for $x \in M$, where $j^2 = -1$. In particular, α extends to an involution α of $B(H)$ implemented by j, which leaves M' globally invariant. Since M' has no direct summand of type I, neither does the fixed point algebra A of α in M' [**3**, 7.4.3], hence the Halving Lemma for Jordan algebras [**3**, 5.2.14] yields the existence of projections $e, f \in A$ with sum 1 and a symmetry $s \in A$ such that $ses = f$. Let $e_{11} = e, e_{12} = es, e_{21} = se = fs, e_{22} = f$. Then $\{e_{ij} : i, j = 1, 2\}$ is a set of matrix units which generates an I_2-factor M_2. Since $\alpha(e_{12}) = e_{21}, \alpha(e_{ii}) = e_{ii}, \alpha$ leaves M_2 globally invariant. Thus $B(H) = B(H_0) \otimes M_2$, and $\alpha = \alpha_1 \otimes \alpha_2$ with α_1 an involution of $B(H_0)$, and $\alpha_2 = \alpha|M_2$ an involution of M_2. For simplicity of notation we identify M with $M \otimes 1$, and consider M as a subalgebra of $B(H_0)$. Since an involution of a factor is implemented by a conjugate linear isometry v with $v^2 = 1$ or -1, [**6**, Theorem 3.7], it follows that $j = j_1 \otimes j_2$ with $j_i^2 = \pm 1$, and $\alpha|M = \alpha_1|M$ is implemented by j_1. If $j_1^2 = -1$ replace j_2 by a conjugate linear isometry v with square -1, and if $j_1^2 = +1$ by v with square $+1$. In either case $J = j_1 \otimes v$ is a conjugation implementing α_1, and hence α on M.

It remains to consider the case when M is of type I. Since α is central we may consider the different direct summands separately, hence we may assume M is homogeneous of type $\mathrm{I}_n, n \in \mathbf{N} \cup \{\infty\}$, with M' homogeneous of type I_r, $r \in \mathbf{N} \cup \{\infty\}$, see, e.g., [**1**; Chapter III, §3.1, Proposition 2] applied to M and M'. For a Hilbert space K let t denote the transpose on $B(K)$ with respect to some orthonormal basis, and let q be the involution

$$q\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

on the complex 2×2 matrices. By [**7**, Theorem 2.6] M is a direct sum $M = M_1 \oplus M_2$ such that α leaves each M_i invariant; $M_1 = B(H_1) \otimes Z_1$, $M_2 = B(H_2) \otimes B(\mathbf{C}^2) \otimes Z_2$, where in both cases Z_i is an abelian von Neumann algebra with Z_i' of type I_r. In the first case $\alpha|M_1 = t \otimes \iota$, hence $\alpha|M_1$ is implemented by a conjugation, see, e.g. [**3**, §7.5]. In the second case $\alpha|M_2 = t \otimes q \otimes \iota$. Now q is implemented by a conjugate linear isometry j such that $j^2 = -1$, while t is implemented by a conjugation J. Since, by assumption M', is of type I_r with r even or $r = \infty$, there exists a conjugate linear isometry j_r with $j_r^2 = -1$ which implements a central involution on Z_2', see [**3**, §7.5]. Thus $J \otimes j \otimes j_r$ is a conjugation which implements α on M_2. This completes the proof of the theorem.

Remark 2. The conclusion of Theorem 4 is false if M' is of type I_n with $n \in \mathbf{N}$ odd. Let, for example, $M = M_m(C) \otimes C1_n$, so that $M' = C1_m \otimes M_n(C)$, with m even and n odd. Then there exists j on C^m such that $j^2 = -1$, while each involution on $M_n(C)$ is conjugate to the transpose map. Let $\alpha(x \otimes 1) = (-jx^*j) \otimes 1_n$ on M. Then α is not implemented by a conjugation. Indeed, if J is a conjugation on $C^m \otimes C^n$ implementing α, then J also implements an involution on $M' = C1_m \otimes M_n(C)$; hence there would exist a conjugation J' on C^n such that $JxJ = -(j\otimes J')x(j\otimes J')$ for all $x \in B(C^m \otimes C^n) = M_m(C)\otimes M_n(C)$. Since $J^2 = 1$ and $(j \otimes J')^2 = -1$, this is impossible by [**6**, Lemma 3.9], hence α is not implemented by a conjugation. This example also shows that the assumption on the normal states being vector states is necessary in Theorem 3.

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