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INVERTIBILITY AND TOPOLOGICAL STABLE RANK FOR SEMI-CROSSED PRODUCT ALGEBRAS

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0. Introduction. The computation of invariants for C^*-algebras such as the K-groups and topological stable rank, which has attracted so much attention in recent years, can be fruitful for nonselfadjoint operator algebras as well, though relatively little has been done in that direction. In the present discussion we will compute these invariants for a special class of nonselfadjoint, norm closed operator algebras, which we have called semi-crossed products [**4**]. These algebras include the subalgebras of C^* crossed products of $C(X)$ by a homeomorphism of X which are generated by the nonnegative powers of the homeomorphism.

Denote the semi-crossed product of $C(X)$ with respect to a homeomorhpism φ by $\mathbf{Z}^+ \times_\varphi C(X)$. As the invertible elements are never dense, the topological stable rank is greater than one, and we show it is in fact equal to two in case X is zero or one dimensional. (In particular, we show that the right and left stable ranks coincide, which is not automatic since the algebra is not involutive.) On the other hand, the K-theory for these algebras is "contractible" to K-theory of $C(X)$.

As the class of algebras we will be discussing is somewhat more general than the class of those algebras which arise naturally as subalgebras of C^*-crossed products, we will need some preliminaries. By a dynamical system we mean a pair (X, φ) where X is a compact Hausdorff space and $\varphi : X \to X$ is a continuous surjection. Denote by $K(\mathbf{Z}^+, C(X))$ the algebra over $\mathbf{C}$ which is the free product of $C(X)$ with a single operator, U together with the relations $fU = Uf \circ \varphi, f \in C(X)$. A typical $F \in K(\mathbf{Z}^+, C(X))$ thus has the form of a polynomial, $F = f_0 + Uf_1 + \cdots + U^n f_n$ where $n \in \mathbf{Z}^+, f_0, \dots, f_n \in C(X)$. For $x \in X$ define a representation of $K(\mathbf{Z}^+, C(X))$ on ℓ_2^+ by means of

$$\Pi_x(U)(\xi_0, \xi_1, \xi_2, \dots) = (0, \xi_0, \xi_1, \xi_2, \dots)$$

(the unilateral shift), and

$$\Pi_x(f)(\xi_0, \xi_1, \xi_2, \dots) = (f(x)\xi_0, f \circ \varphi(x)\xi_1, f \circ \varphi^2(x)\xi_2, \dots), \quad f \in C(X).$$

For $F = \sum_0^n U^k f_k$, set $\Pi_x(F) = \sum_0^n \Pi_x(U)^k \Pi_x(f_k)$. One checks that this defines a representation of $K(\mathbf{Z}^+, C(X))$. Obtain a norm on

$K(\mathbf{Z}^+, C(X))$ by $||F|| = \sup_{x\in X} ||\Pi_x(F)||$. The semi-crossed product $\mathbf{Z}^+ \times_\varphi C(X)$ is the completion of $K(\mathbf{Z}^+, C(X))$ in this norm. Alternatively, the semi-crossed product could have been defined directly as a norm-closed algebra of operators in Hilbert space.

As we will see in Corollary 0.3, to each F in $\mathbf{Z}^+ \times_\varphi C(X)$ we may associate a unique Fourier series, $F \sim \sum_0^\infty U^n f_n$. Just as with ordinary Fourier series, the partial sums of the Fourier series of F may not converge in norm to F. (In fact, the disk algebra can be realized as a semi-crossed product by taking X to be a singleton and φ the identity map.)

Define a one-parameter group τ_t of automorphisms of $K(\mathbf{Z}^+, C(X))$ by $\tau_t(\sum_{n=0}^N U^n f_n) = \sum_{n=0}^N U^n e^{\mathrm{int}} f_n$.

Lemma 0.1. *$\tau_t, t \in \mathbf{R}$, is isometric, and hence extends to an isometric automorphism of the semi-crossed product $\mathbf{Z}^+ \times_\varphi C(X)$.*

Proof. Let $\Lambda_t : \ell_2^+ \to \ell_2^+$ be a one-parameter family of Hilbert space isomorphisms given by $\Lambda_t((\xi_n)_{n=0}^\infty = (e^{-\mathrm{int}}\xi_n)_{n=0}^\infty$. Let $F = \sum_{n=0}^N U^n f_n \in K(\mathbf{Z}^+, C(X)), x \in X$, and observe

$$\begin{aligned}\Pi_x(\tau_t(F))(\xi_n)_{n=0}^\infty &= \Pi_x\Big(\sum_{k=0}^N U^k e^{ikt} f_k\Big)(\xi_n)_{n=0}^\infty \\ &= (f_0(x)\xi_0, f_0 \circ \varphi(x)\xi_1 + e^{it} f_1(x)\xi_0, \\ &\qquad f_0\circ\varphi^2(x)\xi_2 + e^{it} f_1 \circ \varphi(x)\xi_1 + e^{2it} f_2(x)\xi_0, \dots).\end{aligned}$$

Consequently,

$$\begin{aligned}&||\Pi_x(\tau_t(F))(\xi_n)_{n=0}^2||^2 \\ &\quad = |f_0(x)\xi_0|^2 + |f_0\circ\varphi(x)e^{-it}\xi_1 + f_1(x)\xi_0|^2 \\ &\qquad |f_0\circ\varphi^2(x)e^{-2it}\xi_2 + f_1\circ\varphi(x)e^{-it}\xi_1 + f_2(x)\xi_0|^2 \\ &\qquad + \dots \\ &\quad = ||\Pi_x(F)\Lambda_t((\xi_n)_{n=0}^\infty)||^2.\end{aligned}$$

Thus $||\Pi_x(\tau_t(F))|| = ||\Pi_x(F)||, x \in X$, and $||\tau_t(F)|| = ||F||$. Since τ_t is isometric, it extends to an automorphism of $\mathbf{Z}^+ \times_\varphi C(X)$, also denoted τ_t. $\square$

LEMMA 0.2. *The automorphism group $t \to \tau_t$ is continuous in the topology of pointwise-norm convergence on $\mathbf{Z}^+ \times_\varphi C(X)$.*

PROOF. Let $F \in \mathbf{Z}^+ \times_\varphi C(X)$, $\varepsilon > 0$ be given. Let $G \in K(\mathbf{Z}^+, C(X))$ with $||F - G|| < \varepsilon/3$. Since G has only finitely many nonzero Fourier coefficients, $t \mapsto \tau_t(G)$ is norm continuous, so there is $\delta > 0$ such that $|t - t_0| < \delta$ implies $||\tau_t(G) - \tau_{t_0}(G)|| < \varepsilon/3$. So if $|t - t_0| < \delta$,

$$\begin{aligned} &||\tau_t(F) - \tau_{t_0}(F)|| \\ &\qquad \leq ||\tau_t(F - G)|| + ||\tau_t(G) - \tau_{t_0}(G)|| + ||\tau_{t_0}(G - F)|| \\ &\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

where the first and third inequalities follow from 0.1. □

COROLLARY 0.3. *For $n = 0, 1, \ldots,$ there is a linear mapping $P_n : \mathbf{Z}^+ \times_\varphi C(X) \to C(X)$ satisfying*

(i) $$||P_n(F)|| \leq ||F||, \quad F \in \mathbf{Z}^+ \times_\varphi C(X)$$

(ii) $$P_n(fF) = f \circ \varphi^n P_n(F),$$

and

$$P_n(Ff) = P_n(F)f, \quad F \in \mathbf{Z}^+ \times_\varphi C(X), \quad f \in C(X);$$

(iii) $$P_n\Big(\sum_{k=0}^{N} U^k f_k\Big) = \begin{cases} f_n, & 0 \leq n \leq N \\ 0, & n > N \end{cases}.$$

PROOF. Define $P_n(F)$ by $U^n P_n(F) = \int_0^{2\pi} e^{-\mathrm{int}} \tau_t(F) \frac{dt}{2\pi}$. Thus

$$\begin{aligned} ||P_n(F)|| &\leq \Big|\Big| \int_0^{2\pi} e^{-\mathrm{int}} \tau_t(F) \frac{dt}{2\pi} \Big|\Big| \\ &\leq ||F||. \end{aligned}$$

Property (iii) is a straightforward calculation and (ii) follows immediately for F in the dense subalgebra $K(\mathbf{Z}^+, C(X))$, but then by continuity holds for $F \in \mathbf{Z}^+ \times_\varphi C(X)$. □

In §I.1 we show that the invertible elements in a semi-crossed product are not dense. Next we introduce a notion of local invertibility, which could also be done in the larger context of triangular operator algebras. Here the main result is that in the case of a free action, local invertibility of an element F is equivalent to saying its zero$^{\text{th}}$ Fourier coefficient is nowhere vanishing, which holds iff F belongs to no maximal ideal. In §II the topological stable rank of $\mathbf{Z}^+ \times_\varphi C(X)$ is shown to equal two in case φ is a homeomorphism and the invertible elements of $C(X)$ are dense. Finally §III begins with a discussion of summability theory for Fourier series, which parallels the classical theory, and uses this to show that the study of the K-theory of the semi-crossed product $\mathbf{Z}^+ \times_\varphi C(X)$ can be reduced to that of $C(X)$. The author acknowledges a useful discussion with Y.T. Poon.

I. Invertibility and local invertibility.

Proposition I.1. *The invertible elements in the semi-crossed product* $\mathfrak{U} = \mathbf{Z}^+ \times_\varphi C(X)$ *are not dense. In particular, if* $F \in \mathfrak{U}, ||F - U|| < 1/2$, *then* F *is not invertible.*

Proof. Suppose to the contrary that F is invertible and let $G = F^{-1}$. Now $||F - U|| < 1/2$ implies

$$(*) \qquad ||FG - UG|| \le ||F - U||||G||,$$

or

$$||1 - UG|| < \frac{1}{2}||G||.$$

Since the norm of any element of $\mathfrak{U}$ dominates the norms of its Fourier coefficients, and since the zero$^{\text{th}}$ Fourier coefficient of $1 - UG$ is 1, it follows from $(*)$ that $1 < \frac{1}{2}||G||$, or

$$(**) \qquad ||G|| > 2.$$

On the other hand, since U is an isometry, $||UG|| = ||G||$, and so

$$||1 - UG|| \geq |\,||1|| - ||UG||\,| = |1 - ||G||\,| = ||G|| - 1.$$

Therefore it also follows from $(*)$ that

$$||G|| - 1 < \frac{1}{2}||G||,$$

or

$$||G|| < 2,$$

contradicting $(**)$. □

PROPOSITION I.2. *An element of $\mathfrak{U}$ is left invertible if and only if it is right invertible.*

PROOF. Let $F \in \mathfrak{U}$ be left invertible, and let $G \in \mathfrak{U}$ satisfy $GF = 1$. Since

$$1 = P_0(GF) = P_0(G)P_0(F) = P_0(FG),$$

it follows that the Fourier series of $H = FG$ has the form $1 + \sum_{n=1}^{\infty} U^n h_n$. As $(H-1)H = (FG-1)FG = F(GF)G - FG = 0$, we obtain $h_1 = P_1[(H-1)H] = 0$. Continuing inductively, suppose $h_1 = \cdots = h_n = 0$; then $0 = P_{n+1}[(H-1)H] = h_{n+1}$. We conclude $H = 1$, so that F is invertible. A parallel argument shows that right invertibility implies left invertibility.

I.3. Though the invertible elements of $\mathfrak{U}$ fail to be dense, one can ask if they are nonetheless dense "locally" in some sense. The motivation for the following definition comes from commutative Banach algebras.

DEFINITION. An element $F \in \mathbf{Z}^+ \times_\varphi C(X)$ will be called locally left invertible at $x_0 \in X$ if there is a neighborhood V of x_0 and an invertible $G \in \mathbf{Z}^+ \times_\varphi C(X)$ such that, for some $u \in C(X)$ satisfying $u|_V \equiv 1, Fu = Gu$. We will say that F is locally left invertible if it is locally left invertible at every point of X.

Let $\mathfrak{U} = \mathbf{Z}^+ \times_\varphi C(X)$.

LEMMA I.4. *Let $u \in C(X), 0 \leq u \leq 1, u \equiv 1$ in a neighborhood of $x_0 \in X$. If $F \in \mathfrak{U}, ||Fu - u|| < 1$, then F is locally left invertible at x_0.*

PROOF. Let $\{u = u_1, u_2, \dots, u_m\}$ be a partition of unity on X, and set $F_1 = Fu_1 + u_2 + \cdots + u_m$. Then $||F_1 - 1|| = ||Fu - u|| < 1$, so F_1 is invertible. Let $v \in C(X)$, $0 \leq v \leq 1, v \equiv 1$ in some neighborhood of x_0 and such that $\text{supp}(v) \subset \{x : u(x) = 1\}$. Then $Fv = Fuv = F_1 v$, so F is locally left invertible at x_0. □

COROLLARY I.5. *Let $F \in \mathfrak{U}$ be locally left invertible at $x_0, u \in C(X), 0 \leq u \leq 1, u \equiv 1$ in a neighborhood of x_0, and let $G \in \mathfrak{U}$ be invertible with $Fu = Gu$. If $F' \in \mathfrak{U}$ satisfies*

$$||Fu - F'u|| < (||G^{-1}||)^{-1},$$

then F' is locally left invertible at x_0.

PROOF. Since $G^{-1}Fu = u$,

$$\begin{aligned} ||u - G^{-1}F'u|| &= ||G^{-1}Fu - G^{-1}F'u|| \\ &\leq ||G^{-1}||\ ||Fu - F'u|| \\ &< 1, \end{aligned}$$

so, by the lemma, $G^{-1}F'$ is locally left invertible at x_0, Thus there is a $v \in C(X), 0 \leq v \leq 1, v \equiv 1$ in a neighborhood of x_0, and an invertible $H \in \mathfrak{U}$ such that $G^{-1}F'v = Hv$. Then $F'v = GHv$, and F' is locally left invertible at x_0 since GH is invertible. □

LEMMA I.6. *Let (X, φ) be a free dynamical system. Let $F \in K(\mathbf{Z}^+, C(X)), F = \sum_{n=0}^{N} U^n f_n$, and suppose $f_0 \equiv 1$ in a neighborhood of $x_0 \in X$. Then F is locally left invertible at x_0. Furthermore, there exist $v \in C(X), 0 \leq v \leq 1, v \equiv 1$ in a neighborhood of x_0, and $G \in \mathfrak{U}$ invertible such that $Fv = Gv$ and $||G^{-1}|| \leq 2 + ||F||$.*

PROOF. Since φ acts freely, there is a neighborhood W_0 of x_0 such that $W_0, \varphi^{-1}(W_0), \dots, \varphi^{-N}(W_0)$ are pairwise disjoint. Let $u \in C(X), 0 \leq u \leq 1, u \equiv 1$ in a neighborhood of x_0, and $\text{supp}(u) \subset W_0 \cap \{x : f_0(x) = 1\}$.

Setting $g_k = f_k u, 1 \leq k \leq N$, we see that $\sum_{k=1}^N U^k g_k$ is rank two nilpotent, since $(U^k g_k)(U^\ell g_\ell) = U^{k+\ell} g_k \circ \varphi^\ell g_\ell = 0$, $g_k \circ \varphi^\ell g_\ell$ is supported on $\varphi^{-\ell}(W_0) \cap W_0 = \varnothing$. Thus $\exp(\sum_1^N U^k g_k) = 1 + \sum_1^N U^k g_k$. Set $G = 1 + \sum_1^N U^k g_k \in \mathfrak{U}^{-1}$; then

$$\begin{aligned} G^{-1} &= \exp\Big(-\sum_1^N U^k g_k\Big) \\ &= 1 - \sum_1^N U^k g_k, \end{aligned}$$

so

$$\begin{aligned} ||G^{-1}|| &\leq 1 + \Big|\Big|\sum_1^N U^k f_k u\Big|\Big| \\ &\leq 1 + ||(F-1)u|| \\ &\leq 2 + ||F||. \quad \square \end{aligned}$$

Remark I.7. Let $f \in C(X), f(x_0) \neq 0$. Then an element $F \in \mathfrak{U}$ is locally left invertible at x_0 iff Ff is locally left invertible at x_0. The proof is straightforward.

Proposition I.8. *Let (X, φ) be free. Then $F \in \mathfrak{U}$ is locally left invertible at x_0 iff $P_0(F)$, the zero*$^{\text{th}}$ *Fourier coefficient, satisfies $P_0(F)(x_0) \neq 0$.*

Proof. The map $E_{x_0} : \mathfrak{U} \to \mathbf{C}\,, F \mapsto E_{x_0}(F) = P_0(F)(x_0)$ is a nonzero algebra homomorphism, and in fact every maximal ideal of $\mathfrak{U}$ has the form $\ker(E_x)$ for some $x \in X$ [**4**, IV.9]. Suppose F is locally left invertible at x_0, and let $u \in C(X)$ be $\equiv 1$ in a neighborhood of x_0 such that $Fu = Gu$ for some invertible $G \in \mathfrak{U}$. Then

$$E_{x_0}(Fu) = E_{x_0}(Gu),$$

whence

$$E_{x_0}(F)E_{x_0}(u) = E_{x_0}(G)E_{x_0}(u),$$

or

$$E_{x_0}(F) = E_{x_0}(G) \neq 0,$$

since $E_{x_0}(u) = u(x_0) = 1$ and G cannot by virtue of its invertibility belong to any maximal ideal. Thus the condition is necessary.

Now assume $P_0(F)(x_0) \neq 0$; replacing F by Fg, where $g = P_0(F)^{-1}$ in some neighborhood of x_0, does not affect local left invertibility by I.7, so we may assume $P_0(F) \equiv 1$ in some neighborhood of x_0. Let $F' \in K(\mathbf{Z}^+, C(X))$ with $||F' - F|| < 1/(3 + ||F||)$, and we may assume that $P_0(F') \equiv 1$ in some neighborhood of x_0. Then $||F' - F|| < 1/(2 + ||F'||) < ||G^{-1}||^{-1}$, where G is invertible with $F'v = Gv$ for some $v \in C(X), v \equiv 1$ in a neighborhood of x_0, as in Lemma I.6. Consequently, by Corollary I.5, F is locally invertible at x_0. □

Corollary I.9. *Let (X, φ) be free. Then F is locally left invertible iff $P_0(F)$, the zero*$^{\text{th}}$ *Fourier coefficient, is invertible in $C(X)$.*

Suppose now that φ is a homeomorphism. Then every $F \in K(\mathbf{Z}^+, C(X))$ has a unique expression of the form $F = f_0 + f_1 U + \cdots + f_N U^N$. Also, each $F \in \mathfrak{U} = \mathbf{Z}^+ \times_\varphi C(X)$ has a unique Fourier series $\sum_0^\infty f_n U^n$. Thus, everything that has been said about local left invertibility could be said about local right invertibility; so from the "right" version of I.9 we can conclude

Corollary I.10. *Let φ be a freely acting homeomorphism. Then $F \in \mathfrak{U}$ is locally left invertible iff F is locally right invertible iff F does not belong to any maximal two sided ideal of $\mathfrak{U}$.*

Remark I.11. Note that, in view of the above Corollary, one can speak of the "locally invertible" elements. It follows that the locally invertible elements in $\mathfrak{U}$ will be dense iff the invertible elements in $C(X)$ are dense. In any case, there are always locally invertible elements which are not invertible, e.g., $F = 1/3 + U$.

Remark I.12. The situation in case (X, φ) has periodic points is not clear. We can, however, show the following: If $x_0 = \varphi(x_0)$ is a fixed point, then the locally invertible elements of $\mathbf{Z}^+ \times_\varphi C(X)$ are not dense.

Proof. There is a continuous homomorphism $\psi : \mathbf{Z}^+ \times_\varphi C(X) \to \mathcal{A}(D)$ (the disk algebra), given as follows: if $F \in \mathbf{Z}^+ \times_\varphi C(X)$ has Fourier series $\sum_{n=0}^\infty U^n f_n$, then $\psi(F) = h \in \mathcal{A}(D)$ has Fourier series $\sum_{n=0}^\infty a_n z^n, a_n = f_n(x_0)$. (The fact that this is a homomorphism uses $\varphi(x_0) = x_0$.) If $\varphi^{-1}(x_0) = \{x_0\}$, the homomorphism is isometric and hence onto [**5**, Corollary IV.1], but in any case the image is dense as it contains the polynomials, which is all that is needed. Now if $F \in \mathbf{Z}^+ \times_\varphi C(X)$ has local inverse G near x_0, so $Fu = Gu$ for some $u \in C(X)$ with $u(x_0) = 1$; then since $\psi(u) = 1$, we have $\psi(F) = \psi(G)$ so $\psi(F)$ is invertible in $\mathcal{A}(D)$. Now if the locally invertible elements in $\mathbf{Z}^+ \times_\varphi C(X)$ were dense, that would imply in particular that, since $h(z) = z \in \text{image}\psi$, every neighborhood of $h(z)$ contains invertible elements of $\mathcal{A}(D)$, which contradicts Hurwitz's Theorem. $\square$

If x_0 has least period $k_0 > 1$, then there is a homomorphism of $\mathbf{Z}^+ \times_\varphi C(X)$ into a $k_0 \times k_0$ matrix algebra of analytic functions [**4**].

The question of local invertibility in $\mathbf{Z}^+ \times_\varphi C(X)$ seems to be related to the question of topological stable rank in this matrix algebra of analytic functions.

II. Topological stable rank. Let $\mathfrak{U}$ be a topological ring with identity, and denote by $\mathcal{R}g_n(\mathfrak{U})$ the set of n-tuples of elements of $\mathfrak{U}$ which generate $\mathfrak{U}$ as a right ideal: $\mathcal{R}g_n(\mathfrak{U}) = \{(F_i)_{i=1}^n \in \mathfrak{U}^n$: there exist $G_1, \dots, G_n \in \mathfrak{U}$ such that $\sum_{i=1}^n F_i G_i = 1\}$. Recall that the right topological stable rank of $\mathfrak{U}$, denoted $\text{rtsr}(\mathfrak{U})$, is the least integer n such that $\mathcal{R}g_n(\mathfrak{U})$ is dense in $\mathfrak{U}^n$ (for the product topology). Left topological stable rank is defined analogously. If left and right topological stable rank coincide, their common value is called the topological stable rank, $\text{tsr}(\mathfrak{U})$.

It seems to be unknown for Banach algebras in general (without involution) whether left and right topological stable rank must coincide. We will show that if $\mathfrak{U} = \mathbf{Z}^+ \times_\varphi C(X)$ where φ is a homeomorphism of X and $\text{tsr}(C(X)) = 1$, then $\text{ltsr}(\mathfrak{U}) = \text{rtsr}(\mathfrak{U}) = 2$.

We will need some notation. Let $[F_1, F_2] \in \mathfrak{U}^2$; if one of F_1, F_2 has infinitely many nonzero terms in its Fourier series, define $\ell([F_1, F_2])$, the length of $[F_1, F_2]$, to be $+\infty$. If $F_i = \sum_{k=0}^{n_i} U^k f_k^{(i)},\ n_i < \infty,\ i = 1, 2$, set $\ell([F_1, F_2]) = n_1 + n_2$.

$\mathrm{GL}(2,\mathfrak{U})$ is the group of 2 by 2 invertible matrices with entries in $\mathfrak{U}$. Note that, for any $F \in \mathfrak{U}$, the matrix $E = \begin{bmatrix} 1 & 0 \\ F & 1 \end{bmatrix}$ belongs to $\mathrm{GL}(2,\mathfrak{U})$; indeed, $E^{-1} = \begin{bmatrix} 1 & 0 \\ -F & 1 \end{bmatrix}$.

LEMMA II.1. *Let* $F_i = \sum_{k=0}^{n_i} U^k f_K^{(i)}$, *and assume* $f_{n_i}^{(i)}$ *is invertible in* $C(X), i = 1, 2$. *If* $n_1 + n_2 = \ell([F_1, F_2]) > 0$, *then there is an* $E \in \mathrm{GL}(2,\mathfrak{U})$ *such that*

$$\ell([F_1, F_2]E) < \ell([F_1, F_2]).$$

PROOF. Let $n_1 = n, n_2 = m$ and suppose $0 \leq m \leq n$, where one of the two inequalities is strict. If E is taken to be

$$E = \begin{bmatrix} 1 & 0 \\ -U^{n-m} f_n^{(1)} (f_m^{(2)} \circ \varphi^{(n-m)})^{-1} & 1 \end{bmatrix}$$

one verifies by calculation that

$$\ell([F_1, F_2]E) \leq \ell([F_1, F_2]) - 1.$$

If $n < m$, the result follows by a similar calculation. □

PROPOSITION II.2. *Assume that* $\mathrm{tsr}(C(X)) = 1$. *Then* $\mathrm{tsr}(\mathfrak{U}) \leq 2$.

PROOF. Let $A_1, A_2 \in \mathfrak{U}$ and $\varepsilon > 0$ be given. If we can show that there exist $H_i, G_i \in \mathfrak{U}, ||H_i - A_i|| < \varepsilon, i = 1, 2$ with $H_1 G_1 + H_2 G_2 = 1$, this will imply that $\mathrm{rtsr}(\mathfrak{U})$ is at most two. Let $\mathcal{B}_\varepsilon = \{[H_1, H_2] \in \mathfrak{U}^2 : ||H_i - A_i|| < \varepsilon\}$, and set $\ell_0 = \min\{\ell([H_1, H_2]T) : [H_1, H_2] \in \mathcal{B}_\varepsilon,\ T \in \mathrm{GL}(2,\mathfrak{U})\}$. Observe that ℓ_0 is finite, since the elements of $\mathfrak{U}$ with finite Fourier series are dense. In fact, we claim $\ell_0 = 0$. For suppose $\ell_0 > 0$, and let $[H_1, H_2] \in \mathcal{B}_\varepsilon$ and $T \in \mathrm{GL}(2,\mathfrak{U})$ be such that $\ell([H_1, H_2]T) = \ell_0$. Set $[F_1, F_2] = [H_1, H_2]T$. Since the map

$$\mathfrak{U}^2 \to \mathfrak{U}^2, \quad [K_1, K_2] \to [K_1, K_2]T$$

is a homeomorphism, $\mathcal{B}_\varepsilon T$ is open. Hence there exists $\delta > 0$ such that if $||F_i' - F_i|| < \delta, i = 1, 2, [F_1', F_2'] \in \mathcal{B}_\varepsilon T$. If $F_i = \sum_{k=0}^{n_i} U^k f_k^{(i)}$, let $F'_i = \sum_{k=0}^{n_i-1} U^k f_k^{(i)} + U^{n_i} f^{(i)}$, where $f^{(i)}$ is invertible in $C(X)$ and $||f^{(i)} - f_{n_i}^{(i)}|| < \delta, i = 1, 2$. This is possible since the invertible elements of $C(X)$ are dense. Now $\ell([F'_1, F'_2]) = \ell([F_1, F_2]) > 0$. By the lemma, there is an $E \in \mathrm{GL}(2, \mathfrak{U})$ such that $\ell([F'_1, F'_2]E) < \ell([F'_1, F'_2])$. But as $TE \in \mathrm{GL}(2, \mathfrak{U})$,this contradicts the minimality of ℓ_0, and hence ℓ_0 must be zero.

Thus there exist $[H_1, H_2] \in \mathcal{B}_\varepsilon$ and $T \in \mathrm{GL}(2, \mathfrak{U})$ so that $[H_1, H_2] = [f^{(1)}, f^{(2)}] \in C(X)^2$. As above, we may further assume that $f^{(1)}$ and $f^{(2)}$ are invertible. Let $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = T \begin{bmatrix} (f^{(1)})^{-1} \\ 0 \end{bmatrix}$. Then

$$\begin{aligned} H_1G_1 + H_2G_2 &= [H_1, H_2] \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = [H_1, H_2] T \begin{bmatrix} (f^{(1)})^{-1} \\ 0 \end{bmatrix} \\ &= [f^{(1)}, f^{(2)}] \begin{bmatrix} (f^{(1)})^{-1} \\ 0 \end{bmatrix} = 1. \ \square \end{aligned}$$

COROLLARY II.3. *If* $\mathrm{tsr}(C(X)) = 1$, *then* $\mathrm{tsr}(\mathfrak{U}) = 2$.

PROOF. By the Proposition, $\mathrm{rtsr}(\mathfrak{U}) \leq 2$, whereas, by I.1 and I.2, $\mathrm{rtsr}(\mathfrak{U}) > 1$. On the other hand, by a left version of the above Proposition, $\mathrm{ltsr}(\mathfrak{U}) \leq 2$, and consequently, by I.1 and I.2, $\mathrm{ltsr}(\mathfrak{U}) = 2$. $\square$

REMARKS II.4. (i) As mentioned in the introduction, the disk algebra is a semi-crossed product $\mathbf{Z}^+ \times_\varphi C(X)$ where X is a one point space and φ the identity map. The fact that the disk algebra has topological stable rank 2, which follows from the theorem, was already well known [**6**].

(ii) Does $\mathrm{tsr}(\mathbf{Z}^+ \times_\varphi C(X)) = 2$ for an arbitrary continuous surjection φ on a compact Hausdorff space X, assuming $\mathrm{tsr}(C(X)) = 1$?

(iii) The topological stable rank of the C^*-algebra $C^*(S)$, where S is the unilateral shift of multiplicity one, is determined from the short exact sequence

$$0 \to K \to C^*(S) \to C(T) \to 0$$

(K the compact operators on $L^2(T)$) [**6**; Example 4.13]. Let φ be a minimal action—such as irrational translation—on the torus. Now the semi-crossed product $\mathbf{Z}^+ \times_\varphi C(T)$ can be represented on $H^2(T)$ such that U is mapped to a unilateral shift of multiplicity one. (Indeed, any representation $\Pi_t (t \in T)$ is the restriction to $\mathbf{Z}^+ \times_\varphi C(T)$ of a covariant representation of the C^*-crossed product $\mathbf{Z} \times_\varphi C(T)$, and such a representation is isometric. See [**4**, II.4]). There is also here a short exact sequence

$$0 \to U\mathfrak{U} \to \mathfrak{U} \to C(T) \to 0$$

($\mathfrak{U} = \mathbf{Z}^+ \times_\varphi C(T)$), and we know that

$$\max\{\operatorname{rtsr}(U\mathfrak{U}), \operatorname{rtsr}(C(T)) + 1\} \geq \operatorname{rtsr}(\mathfrak{U})$$

[**6**, 4.12]. However as we have no *a priori* information concerning $\operatorname{rtsr}(U\mathfrak{U})$, the computation of $\operatorname{tsr}(\mathfrak{U})$ cannot imitate that of $C^*(S)$.

III. Summability and K-theory. Let I denote either the natural numbers $\mathbf{N}$ or else the interval $[0, 1)$ with the usual ordering; by $\lim_r$ we will mean the limit as $r \uparrow \infty$ if $I = \mathbf{N}$ or else the limit as $r \uparrow 1$ if $I = [0, 1)$. Let $\{K_r\}_{r \in I}$ be a family of L^1-kernels on $[-\pi, \pi]$, and, for $0 < \delta < \pi$, set $\mu_r(\delta) = \sup_{\delta \leq |t| \leq \pi}\{|K_r(t)|\}$. Consider the conditions

(i) $\frac{1}{\pi}\int_{-\pi}^{\pi} K_r(t)dt = 1$;

(ii) $K_r(t) \geq 0,\ -\pi \leq t \leq \pi$;

(iii) $\lim_r \mu_r(\delta) = 0$ for each fixed $\delta,\ 0 < \delta < \pi$.

Let $\sigma_r(x; F) = \frac{1}{\pi}\int_{-\pi}^{\pi} \tau_{x+t}(F)K_r(t)dt$.

Proposition III.1. *Suppose the family of kernels $\{K_r\}_{r \in I}$ satisfies conditions* (i), (ii) *and* (iii). *Then*

$$\lim_r \sigma_r(x; F) = \tau_x(F)$$

for every $x \in [-\pi, \pi]$ and $F \in \mathbf{Z}^+ \times_\varphi C(X)$.

Proof. The proof is similar to the classical one for Fourier Series [**8**; Theorem 2.21]; for completeness, we reproduce it here. By (i),

$$\sigma_r(x; F) - \tau_x(F) = \frac{1}{\pi}\int_{-\pi}^{\pi} (\tau_{x+t}(F) - \tau_x(F))K_r(t)dt.$$

Thus, by (ii),

$$||\sigma_r(x; F) - \tau_x(F)|| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} ||\tau_{x+t}(F) - \tau_x(F)||K_r(t)dt.$$

Now

$$||\tau_{x+t}(F) - \tau_x(F)|| = ||\tau_x(\tau_t(F) - (F))|| \\ = ||\tau_t(F) - F||$$

since, by 0.1, τ_x is isometric. Thus,

$$||\sigma_r(x;F) - \tau_x(F) \leq \frac{1}{\pi}\int_{|t|\leq\delta} ||\tau_t(F) - F||K_r(t)\delta dt \\ + \frac{1}{\pi}\int_{\delta\leq t\leq\pi} ||\tau_t(F) - F||K_r(t)dt.$$

By Lemma 0.2, τ_t is pointwise-norm continuous, so, given $\varepsilon > 0$, there is a $\delta > 0$ such that, for $|t| < \delta, ||\tau_t(F) - F|| < \varepsilon$. So by (i), the first term on the right is less than ε. By (iii) there is an $r_0 \in I$ such that, for $r \geq r_0$ and $|t| \geq \delta, 0 \leq K_r(t) < \varepsilon/2$, from which the second term of the right is less than ε. □

REMARK III.2. For any dynamical system (X, φ) (X compact), the semi-crossed product contains the disk algebra as a closed subalgebra; this is the set of elements F whose Fourier series are constant functions. Since, for functions in the disk algebra, the partial sums of the Fourier series may not converge, it is necessary to consider questions of summability in $\mathbf{Z}^+ \times_\varphi C(X)$. Just as in the classical situation, taking $K_n(t) = \frac{2}{n+1}\left\{\frac{\sin(n+1)t/2}{2\sin t/2}\right\}^2$ (Fejér kernel) we obtain from III.1 that the Fourier series of $F \in \mathbf{Z}^+ \times_\varphi C(X)$ is Cesàro summable to F. Also, and more importantly for what follows, taking $K_r(t) = \frac{1}{2}\left(\frac{1-r^2}{1-2r\cos t+r^2}\right), r \in [0,1)$ (Poisson kernel), III.1 yields that the Fourier series of F is Abel summable to F.

Now if F has Fourier series $\sum_{n=0}^{\infty} U^n f^n$, then

$$\sigma_r(0;F) = \frac{1}{\pi}\int_{-\pi}^{\pi} \tau_t(F)K_r(t)\,dt$$

has Fourier series $\sum_{n=0}^{\infty} U^n r^n f_n$. Denote, by $F_r, \sigma_r(0;F)$, where K_r is the Poisson kernel, $0 \leq r < 1$, and set $F_1 = F$. We can express $F_r = \sum_{n=0}^{\infty} U^n r^n f_n, r < 1$, since the series is properly convergent in the sense that the partial sums converge.

DEFINITION III.3. [**2**; 4.2] Let $\mathfrak{U}$ be a Banach Algebra and $f, g \in \mathfrak{U}$. A *path* in $\mathfrak{U}$ joining f with g is a mapping $[0,1] \to \mathfrak{U}, r \mapsto f_r$, which is norm continuous and satisfies $f_0 = f, f_1 = g$.

Lemma III.4. *Let* $\mathfrak{U} = \mathbf{Z}^+ \times_\varphi C(X)$.

(i) *If* $F \in \mathfrak{U}, \{F_r\}_{0\le r\le 1}$ *is a path joining* $P_0(F) = f_0$ *with* F.

(ii) *If* $F \in \mathfrak{U}$ *is an idempotent, so is* $F_r, r \in [0,1]$.

(iii) *If* $F, G \in \mathfrak{U}, (FG)_r = F_r G_r$.

(iv) *If* $F \in \mathfrak{U}$ *is invertible with inverse* G*, then* F_r *is invertible with inverse* $G_r, r \in [0,1]$.

Proof. The map $r \mapsto F_r$ is clearly continuous at every $r, 0 \le r < 1$. It is continuous at $r = 1$ by III.1.

Next, $F \in \mathfrak{U}$ is an idempotent iff $P_n(F^2) = P_n(F), n = 0, 1, 2, \ldots$. If F has Fourier series $\sum_{n=0}^\infty U^n f_n$, this is expressed by the equation $\sum_{k=0}^n f_{n-k} \circ \varphi^k f_k = f_n, n = 0, 1, \ldots$. Replacing f_n by $r_n f_n$ for all n, the above set of equations are still satisfied. Thus, $P_n(F_r^2) = P_n(F_r)$, so F_r is idempotent.

That $(FG)_r = F_r G_r$ follows from comparing Fourier series, and the statement (iv) concerning inverses is an immediate consequence of this. $\square$

Let M_n denote the n by n complex matrices, and $M_n(\mathfrak{U}) = M_n \otimes \mathfrak{U}, \mathfrak{U}$ a Banach Algebra.

Lemma III.5. *Let* $\mathfrak{U} = \mathbf{Z}^+ \times_\varphi C(X)$. $\underline{F} = (F_{ij})$. *Let* $\underline{F}_r = ((F_r)_{ij})$.

(i) *If* $\underline{F} \in M_n(\mathfrak{U}), \underline{F}_r$ *is a path in* $M_n(\mathfrak{U})$ *joining* $\underline{f}$ *with* $\underline{F}$*, where* $\underline{f} = (P_0(F_{ij})) \in M_n(C(X))$.

(ii) *If* $\underline{F} \in M_n(\mathfrak{U})$ *is an idempotent, so is* $\underline{F}_r, 0 \le r < 1$.

(iii) *If* $\underline{F}, \underline{G} \in M_n(\mathfrak{U}), (\underline{F}\,\underline{G})_r = \underline{F}_r \underline{G}_r$.

(iv) *If* $\underline{F} \in M_n(\mathfrak{U})$ *is invertible with inverse* $\underline{G}$*, then* $\underline{F}_r$ *is invertible with inverse* $\underline{G}_r$.

The proof is similar to that of III.4.

If $m < n, M_m(\mathfrak{U})$ is imbedded in $M_n(\mathfrak{U})$ as the upper left hand corner, and $M_\infty(\mathfrak{U})$ is the inductive limit of $\{M_n(\mathfrak{U})\}$. Two idempotents (respectively, invertible elements) in a Banach algebra are said to be homotopic if there is a path consisting of idempotents (respectively, invertible elements) joining them. This is denoted by $e \sim_h f$.

LEMMA III.6. *Let $\mathfrak{U} = \mathbf{Z}^+ \times_\varphi C(X)$. Two idempotents (respectively, invertible elements) $\underline{f}, \underline{g} \in M_n(C(X))$ are homotopic in $M_n(C(X))$ iff they are homotopic in $M_n(\mathfrak{U}), 1 \leq n \leq \infty$.*

PROOF. Let $\underline{f}, \underline{g}$ be idempotents in $M_n(C(X))$. Clearly, if $\underline{f}, \underline{g}$ are homotopic in $M_n(C(X))$ they are homotopic in $M_n(\mathfrak{U})$. Suppose they are homotopic as elements of $M_n(\mathfrak{U})$, and let $\psi(t), 0 \leq t \leq 1$, be a path of idempotents in $M_n(\mathfrak{U})$ joining them. Then $\psi(t)_r$ is a path of idempotents in $M_n(\mathfrak{U})$ joining $\underline{f} = \psi(0)_r$ with $\underline{g} = \psi(1)_r$ for every $r, 0 \leq r \leq 1$. In particular, setting $r = 0$ we obtain a path in $M_n(C(X))$. If $\underline{f}, \underline{g}$ are invertible, the proof is analogous. □

PROPOSITION III.7. $K_0(\mathbf{Z}^+ \times_\varphi C(X)) \simeq K_0(C(X))$.

PROOF. By Lemmas III.5 and III.6, every equivalence class of idempotents in $M_\infty(C(X))$ is contained in a unique equivalence class of idempotents in $M_\infty(\mathfrak{U})$. The conclusion follows from the definition of K_0. □

If $\mathfrak{U}$ is a Banach algebra with identity, let $\mathfrak{U}^{-1}$ denote the group of invertible elements of $\mathfrak{U}$, and $\mathfrak{U}_0^{-1} = \exp \mathfrak{U}$ the connected component of the identity in $\mathfrak{U}^{-1}$ [**7**; Proposition 4.6]. Let $H^1(X, Z)$ be as in [**7**].

COROLLARY III.8. *Let $\mathfrak{U} = \mathbf{Z}^+ \times_\varphi C(X)$ and $F \in \mathfrak{U}^{-1}$. Then F has a factorization $F = Gf_0$, where $G \in \mathfrak{U}_0^{-1}$ and $f_0 \in C(X)^{-1}$. Hence $\mathfrak{U}^{-1} = \mathfrak{U}_0^{-1} C(X)^{-1}$. Consequently $H^1(X, \mathbf{Z})$ is isomorphic with $\mathfrak{U}^{-1}/\mathfrak{U}_0^{-1}$.*

PROOF. If $F \in \mathfrak{U}^{-1}$, then $f_0 = P_0(F) \in C(X)^{-1}$. Set $G = Ff_0^{-1}$; G has Fourier series $1 + \sum_{n=1}^\infty U^n g_n, g_n = f_n f_0^{-1}, n \geq 1$. By III.4, $\{G_r\}_{0 \leq r \leq 1}$ is a path in $\mathfrak{U}^{-1}$ connecting 1 with G, so $G \in \mathfrak{U}_0^{-1}$.

Next observe $C(X)_0^{-1} = C(X)^{-1} \cap \mathfrak{U}_0$; indeed, this follows from Lemma III.6, $n = 1$. By [**7**; Proposition 3.9], $H^1(X, \mathbf{Z}) \simeq C(X)^{-1}/C(X)_0^{-1}$. Finally,

$$\begin{aligned} C(X)^{-1}/C(X)_0^{-1} &= C(X)^{-1}/C(X)^{-1} \cap \mathfrak{U}_0^{-1} \\ &\simeq \mathfrak{U}_0^{-1} C(X)^{-1}/\mathfrak{U}_0^{-1} \\ &= \mathfrak{U}^{-1}/\mathfrak{U}_0^{-1}, \end{aligned}$$

the final equality being the factorization result, and the isomorphism preceding it the second isomorphism theorem for groups. □

COROLLARY III.9. $\mathrm{GL}(n,\mathfrak{U})/\mathrm{GL}(n,\mathfrak{U})_0 \simeq \mathrm{GL}(n, C(X))/\mathrm{GL}(n, C(X))_0$, $1 \leq n \leq \infty$. *Consequently,* $K_1(\mathfrak{U})$ *is isomorphic with* $K_1(C(X))$.

PROOF. Repeat the above argument, replacing $C(X)^{-1}, \mathfrak{U}^{-1}$ by $\mathrm{GL}(n, C(X)), \mathrm{GL}(n, \mathfrak{U})$ respectively. □

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