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A BI-MEASURABLE TRANSFORMATION GENERATED BY A NON-MEASURE PRESERVING TRANSFORMATION

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0. Introduction. In [**2**] V.A. Rohlin constructs an automorphism S on a probability space associated with a measure preserving transformation T on a given probability space. Questions concerning ergodicity, etc. about T may be examined in terms of S. In this note we show that with some extra restrictions on T (notably that T takes measurable sets to measurable sets), a similar construction of a bi-measurable bijection is possible, without having T being measure preserving. Moreover the state space need only be σ-finite. It is then shown that the composition operators on the various L^p spaces constructed in terms of S are extensions of the corresponding operators defined in terms of T. Moreover the extension does not increase the operator norm. The state space for the constructed measure is the inverse limit space as given in [**1**, Chapter 10]. Since the setting below is considerably different, several details of its construction and properties are included. It is noted that if T is measure preserving, this procedure reduces to the standard case.

1. Let $(X, \sum, m)$ be a σ-finite measure space and let T be a mapping of X onto X such that $T^{-1}\sum \subset \sum$ and $T\sum \subset \sum$. We assume that $m\circ T$ and $m \circ T^{-1}$ are mutually absolutely continuous with respect to m, where $m\circ T$ is viewed only as a function. Define

$$Y = \{y = \langle y_i \rangle | \text{ for each } i \geq 0, y_i \in X \text{ and } Ty_{i+1} = y_i\}.$$

Since $TX = X$ it follows from the countable axiom of choice that, for each $x \in X$ and each $n \geq 0$, there is a point y in Y with $y_n = x$. For each $A \in \sum$ and $n \geq 0$ let

$$(A)_n = \{y \in Y | y_n \in A\}$$

and define

$$F = \{(A)_n | A \in \sum, n \geq 0\}.$$

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The following properties of F will prove useful.

PROPOSITION 1. *Let A and B be $\sum$-sets and let $n, k \geq 0$. Then*
(a) $(A)_n = (T^{-k}A)_{n+k}$,
(b) $(A)_n = (B)_n$ *if and only if* $A = B$,
(c) $(A)_n = (B)_{n+k}$ *if and only if* $B = T^{-k}A$,
(d) $(A)_{n+1} \subset (TA)_n = (T^{-1}TA)_{n+1}$.

PROOF. (a). Let $y \in Y$. Then $y \in (T^{-k}A)_{n+k}$ if and only if $y_{n+k} \in T^{-k}A$, that is if and only if $T^k y_{n+k} \in A$. But $T^k y_{n+k} = y_n$, so $(T^{-k}A)_{n+k} = (A)_n$.

(b). This follows from the fact that for each x in X there is a y in Y with $y_n = x$.

(c). This follows from (a) and (b).

(d). From (a) we have $(TA)_n = (T^{-1}TA)_{n+1}$.
Let $y \in (A)_{n+1}$. Then $y_{n+1} \in A$ and so

$$y_n = Ty_{n+1} \in TA, \text{ hence } (A)_{n+1} \subset (TA)_n. \ \square$$

REMARK. These statements above are set-theoretic and do not rely on measurability restrictions. It follows readily from Proposition 1 that F is a Borel field. In fact

$$Y - (A)_n = (X - A)_n, \quad (\varnothing)_0 = \varnothing,$$

and

$$\bigcup_{k=0}^{N} (A_k)_k = \bigcup_{k=0}^{N} (T^{-(N-K)}A_k)_N = \left(\bigcup_{k=0}^{N} T^{-(N-K)}A_k \right)_N .$$

For notational convenience, whenever a finite union of F-sets is indicated, we will block them by "Y-coordinate" so that, for example, $(A)_1 \cup (B)_2 \cup (C)_1 = (D)_1 \cup (B)_2$ where $D = A \cup C$. Also, gaps in "Y-coordinates" will be filled with empty sets. Thus every finite union of F-sets will be expressed as $\cup_{k=0}^{N}(A_k)_k$. This convention applies to arbitrary unions with care taken when measurability questions arise.

Define h as $dm \circ T^{-1}/dm$. Since m and $m \circ T^{-1}$ are mutually absolutely continuous and m is σ-finite, $0 < h < \infty$ a.e. dm. Let the sequence of functions $\{H_n\}$ be defined by $H_0 = 1$ and $H_n = 1/[h \circ T \cdots h \circ T^n]$. Note that, for each $n, H_{n+1} = H_n \circ T/h \circ T$.

LEMMA 2. *For $(A)_n$ in F and $k \geq 0$,*

$$\int_{T^{-k}A} H_{n+k} dm = \int_A H_n dm.$$

PROOF. It suffices to establish the result for $k = 1$. Indeed,

$$\begin{aligned}\int_{T^{-1}A} H_{n+1} dm &= \int_{T^{-1}A} (H_n \circ T/h \circ T) dm \\ &= \int_A h(H_n/h) dm = \int_A H_n dm. \ \square\end{aligned}$$

Lemma 2 in conjunction with Proposition 1a shows that the mapping λ on F given by $\lambda(A)_n = \int_A H_n dm$ is a well defined non-negative set function with $\lambda\varnothing = 0$. Note that $\lambda(A)_n = 0$ if and only if $m(A) = 0$. Our primary goal in this article is to show that λ extends to a σ-finite measure over Y. Let Γ be the smallest σ-field containing F.

LEMMA 3. *λ is finitely additive.*

PROOF. Let $\{(A_k)_k : 0 \leq k \leq N\}$be a collection of mutually disjoint sets in F. Let R be their union. Then $R = (\cup_{k=0}^{N} T^{-(N-K)} A_k)_N$. One easily verifies that $\{T^{-(N-K)} A_k : 0 \leq k \leq N\}$ is a disjoint collection in $\sum$. We then see that

$$\begin{aligned}\lambda(R) &= \int_{\cup_{k=0}^{N} T^{-(N-k)}A_k} H_N dm = \sum_{k=0}^{N} \int_{T^{-(N-k)}A_k} H_N dm \\ &= \sum_{k=0}^{N} \int_{T^{-(N-k)}A_k} H_{k+N-k} dm = \sum_{k=0}^{N} \int_{A_k} H_k dm \\ &= \sum_{k=0}^{N} \lambda(A_k)_k. \ \square\end{aligned}$$

LEMMA 4. *λ is countably additive on λ-null sets.*

PROOF. Suppose $\lambda(A_k)_k = 0$ for each k, and $(B)_N \subseteq \cup_{k=0}^{\infty}(A_k)$. By repeated application of Proposition 1d we have $(A_k)_k \subset (T^k A_k)_0$. Since $m \circ T^k$ is absolutely continuous with respect to m, λ is finitely additive and $(B)_N \subset (\cup_{k=0}^{\infty} T^k A_k)_0$,

$$\lambda(B)_N \leq \lambda\Big(\bigcup_{k=0}^{\infty} T^k A_k\Big)_0 = m\Big(\bigcup_{k=0}^{\infty} T^k A_k\Big) = 0. \ \square$$

THEOREM 5. *λ extends to a measure on Γ.*

PROOF. It suffices to show that, whenever $\{(A_k)_k\} : 0 \leq k < \infty\}$ is a disjoint collection in F whose union is also in F, then λ sums in the appropriate fashion. Let $\{(A_k)_k\}$ be such a collection, with union $(B)_N$. Let C be a measurable subset of B such that $\int_C H_N dm < \infty$. Then

$$\begin{aligned}(C)_N = (B)_N \cap (C)_N &= \bigcup_{k=0}^{\infty} [(A_k)_k \cap (C)_N] \\ &= \bigcup_{k=0}^{\infty} [(T^{-N} A_k)_{N+k} \cap (T^{-k} C)_{N+k}] = \bigcup_{k=0}^{\infty} [T^{-N} A_k \cap T^{-k} C]_{N+k}.\end{aligned}$$

Let $D_k = T^{-N}A_k \cap T^{-k}C$. Since $D_k \subset T^{-N}A_k, \{(D_k)_{N+k} : 0 \le k < \infty\}$ is a disjoint collection in F whose union is $(C)_N$. Then, via 1d,

$$(C)_n = \bigcup_{k=0}^{\infty}(D_k)_{N+k} \subseteq \Big(\bigcup_{k=0}^{\infty} T^k D_k\Big)_N = \bigcup_{k=0}^{\infty}(T^{-k}T^kD_k)_{N+k}.$$

In particular $C \subset \cup_{k=0}^{\infty} T^k D_k$. Fix k and let $x \in T^k D_k$, say $x = T^k z$ for some $z \in D_k$. Let $y \in Y$ with $y_{N+k} = z$. Then $y_N = T^k y_{N+k} = T^k z = x$. Since $z \in D_k$ and $y_{N+k} = z$, $y \in (D_k)_{N+k}$ and hence $y \in (C)_N$. But $x = y_N$, so $x \in C$. Thus $C = \cup_{k=0}^{\infty} T^k D_k$. It then follows that $(C)_N = \cup_{k=0}^{\infty}(D_k)_{N+k} = \big(\cup_{k=0}^{\infty} T^kD_k\big)_N = \cup_{k=0}^{\infty}(T^{-k}T^kD_k)_{N+k}$. Suppose that $x \in D_k \cap T^{\ell}(D_{k+\ell})$. Let $x = T^{\ell}d$ for $d \in D_{k+\ell}$ and let $y \in Y$ with $y_{N+k+\ell} = d$. Then $y \in (D_{k+\ell})_{N+k+\ell}$. But

$$x = T^{\ell}d = T^{\ell}(Y_{N+k+\ell}) = y_{N+k} \quad \text{and} \quad x \in D_k.$$

Thus $y \in (D_k)_{N+k} \cap (D_{k+\ell})_{N+k+\ell}$, whence $\ell = 0$. We then have, for all k and all $\ell > 0, m(T^kD_k \cap T^{k+\ell}D_{k+\ell}) = m(T^k(D_k \cap T^{\ell}D_{k+\ell})) = 0$, and so

$$m(C) = \sum_{k=0}^{\infty} m(T^kD_k)$$

and

$$\lambda(C)_N = \sum_{k=0}^{\infty}\int_{T^kD_k} H_N dm.$$

Also, if $y \in (T^{-k}T^kD_k)_{N+k} \cap (T^{-\ell}T^{\ell}D_{\ell})_{N+\ell}$, then $y_N = T^ky_{N+k} = T^{\ell}y_{N+\ell} \in T^kD_k \cap T^{\ell}D_{\ell}$ and consequently $k = \ell$. It then follows from the finite additivity of λ that $\lambda(C)_N \ge \sum_{k=0}^{\infty}\lambda(T^{-k}T^kD_k)_{N+k}$. In particular, since $\lambda(C)_N = \int_C H_n dm < \infty$, each $\lambda(T^{-k}T^kD_k)_{N+k} < \infty$. Moreover, $(T^{-k}T^kD_k - D_k)_{N+k} = ((T^{-k}T^kD_k)_{N+k} - D_k)_{N+k}$ is a λ-null set, since $T^{-k}T^kD_k \supset D_k$, $\{(T^{-k}T^kD_k)_{N+k}\}$ and $\{(D_k)_{N+k}\}$ are disjoint collections and their unions are both $(C)_N$. We then have $\lambda(D_k)_{N+k} = \lambda(T^{-k}T^kD_k)_{N+k}$, and so

$$\begin{aligned}\sum_{k=0}^{\infty}\lambda(D_k)_{N+k} &= \sum_{k=0}^{\infty}\lambda(T^{-k}T^kD_k)_{N+k} = \sum_{k=0}^{\infty}\int_{T^{-k}T^kD_k} H_{N+k}dm \\ &= \sum_{k=0}^{\infty}\int_{T^kD_k} H_N dm = \int_{\cup_{k=0}^{\infty}T^kD_k} H_N dm = \lambda(C)_N.\end{aligned}$$

We have shown that, for any $\sum$-set $C \subset B$ with $\lambda(C)_N < \infty$, it is true that

$$\lambda(C)_N = \sum_{k=0}^{\infty} \lambda(T^{-N} A_k \cap T^{-k} C)_{N+k}.$$

Let λ_N be the measure on $\sum$ given by $\lambda_N(A) = \lambda(A)_N = \int_A H_N dm$, that is $H_N = d\lambda_N/dm$. Thus $d\lambda_N/dm > 0$ a.e. dm and so there is an increasing sequence of sets $\{C_i\}$ with $\lambda_N(C_i) < \infty$ for each i and $\cup C_i = B$. It follows that

$$\lambda(B)_N = \lambda_N(B) = \operatorname*{limit}_{i\to\infty} \lambda_N(C_i) = \operatorname*{limit}_{i\to\infty} \lambda(C_i)_N.$$

But, for each i,

$$\begin{aligned}\lambda(C_i)_N &= \sum_{k=0}^{\infty} \lambda(T^{-N} A_k \cap T^{-k} C_i)_{N+k} \\ &\leq \sum_{k=0}^{\infty} \lambda(T^{-N} A_k)_{N+k} = \sum_{k=0}^{\infty} \lambda(A_k)_k,\end{aligned}$$

so that $\lambda(B_N) \leq \sum_{k=0}^{\infty} \lambda(A_k)_k$. The reverse inequality follows from the finite additivity of λ. Thus λ extends to a measure on Γ. □

As in [**1**, p. 240] for measure preserving transformations we define S on Y by $S\langle y_0, y_1, \dots\rangle = \langle y_1, y_2, \dots\rangle$. Moreover $SF \subset F$ and $S^{-1}F \subset F$ $(S^{-1}(A)_n = (T^{-1}A)_n$ and $S(A)_n = (A)_{n+1})$.

Example. Let X be the set of non-negative integers and let m be the counting measure. Let $T(0) = 0$ and $T(k) = k - 1$ for $k \geq 1$. Every point in Y is of the form $\langle j, j+1, \dots\rangle$ or $\langle 0, 0, \dots, 0, 1, 2, \dots\rangle$, or $\langle 0, 0, 0, \dots\rangle = z_\infty$. For $j \geq 0$ let $z_{-j} = \langle j, j+1, \dots\rangle$, and for $j \geq 0$ let $z_j = \langle 0, \dots, 0, 1, 2, \dots\rangle$, the right-most zero being in position j. Thus $Y = \{z_j : -\infty < j \leq \infty\}$ may be identified with the set of all integers and a point at ∞. Now the bijection S on Y is given by $S\langle k_0, k_1, \dots\rangle = \langle Tk_0, Tk_1, \dots\rangle$. Thus, for $j > 0, Sz_j = z_{j+1}$, and, for $j > 0, Sz_{-j} = S\langle j, j+1, \dots\rangle = \langle j-1, j, j+1, \dots\rangle = z_{-j+1}$, and so S may be regarded as the shift $j \to j+1$. The measure λ, however, is not the counting measure. Indeed, $h(0) = (dm \circ T^{-1}/dm)(0) = 2$

and $h(k) = 1$ for $k \geq 1$. In general $T^k(j) = \max\{0, j-k\}$, so $h \circ T^k(j) = 2$ if $j \leq k$, and 1 otherwise. It then follows that $H_n(j) = 1/[h \circ Tj) \dots (h \circ T^n j)]$ is given by

$$H_n(j) = \begin{cases} 1 & \text{if } j > n, \\ 1/2^{n-j+1}, & \text{if } 0 \leq j \leq n. \end{cases}$$

Now, for $k > 0$ $(\{k\})_0 = \{z_{-k}\}$, and for $k \geq 0, (\{1\})_{k+1} = \{z_k\}$. From these set equalities we conclude that, for $k < 0, \lambda\{z_k\} = H_0(-k) = 1$ and, for $k \geq 0, \lambda\{z_k\} = H_{k+1}(1) = 1/2^{k+1}$. Finally, the point $z_\infty = \langle 0, 0, \dots \rangle$ satisfies $\{z_\infty\} = \cap_{k=0}^\infty (\{0\})_k$ so $\lambda\{z_\infty\} \leq \lim_{k\to\infty} \inf 1/2^{k+1} = 0$.

For each $p \geq 1$, we may identify $L^p_m = L^p(X, \sum, m)$ as the set of functions on Y depending only on y_0, i.e., those F on Y such that there exists a function f on X related to F by $F(y) = f(y_0)$. Using approximation by simple function it is easily verified that $\int_Y |F|^P d\lambda = \int_X |f|^P dm$. Suppose that $h = dm \circ T^{-1}/dm \in L^\infty(X, \sum)$. Then the composition operator given by $Cf = f \circ T$ is bounded on L^p_m with $||C|| = ||h||_\infty^{1/p}$. Let W be defined on L^p_λ by $WF = F \circ S$ where $S\langle y_0, y_1, \dots \rangle = \langle Ty_0, Ty, \dots \rangle$. Then S is a bi-measurable bijection. For f in L^2_m and $F(y) = f(y_0)$ we have $WF(y) = F(Sy) = f \circ (Ty_0) = f \circ T(y_0)$. Thus $WL^p_m \subseteq L^p_m$ and $W|L^p_m = C$. We are now ready to state a composition operator extension theorem analogous to Rohlin's result for measure preserving transformations.

THEOREM 1.6. *Let $(X, \sum, m)$ be a σ-finite measure space and let T be a mapping of X onto X such that $T^{-1} \sum \subset \sum, T\sum \subset \sum$, and such that $m\circ T^{-1}$ and $m \circ T$ are mutually absolutely continuous with respect to m. Then there is a σ-finite measure space (Y, Γ, λ) and a bi-measurable bijection S on Y such that, for $1 \leq p < \infty$, the composition operator $Cf = f \circ T$ on $L^p(X, \sum, m)$ is extended by $WF = F \circ S$ on $L^p(Y, \Gamma, \lambda)$. Moreover, for each $y \in Y, (d\lambda \circ s^{-1}/d\lambda)(y) = (dm \circ T^{-1}/dm)(y_0)$, and in particular $||W|| = ||C||$ for every L^p norm.*

PROOF. Since $||W|| = ||d\lambda \circ s^{-1}/d\lambda||^{1/p}$, we need only verify the characterization of $d\lambda \circ s^{-1}/d\lambda$ in terms of h. Note that, for any A and

B in $\sum$ and $n \geq 0$, we have

$$\begin{aligned}\int_{(A)_n} \chi(B)_0 d\lambda &= \lambda((T^{-n}B)_n \cap (A)_n) \\ &= \int_{A\cap T^{-n}B} H_n dm \\ &= \int_A H_n \cdot \chi_B \circ T^n dm.\end{aligned}$$

It then follows from the usual linearity-density argument that, for any F on Y of the form $F(y) = f(y_0)$,

$$\int_{(A)_n} F d\lambda = \int_A H_n \cdot f \circ T^n dm.$$

Now let $G = d\lambda \circ S^{-1}/d\lambda$. Then $\int_{(A)_n} G d\lambda = \lambda S^{-1}(A)_n = \lambda(T^{-1}A)_n = \int_{T^{-1}A} H_n dm$. For $n = 0$, $\int_{T^{-1}A} H_n dm = \int_{T^{-1}A} dm = \int_A h dm = \int_A h H_0 dm$, and, for $n \geq 1$,

$$\begin{aligned}\int_{T^{-1}A} H_n dm &= \int_{T^{-1}A} (H_{n-1} \circ T)/h \circ T dm \\ &= \int_A H_{n-1} dm = \int_A (H_{n-1}/H_n) H_n dm \\ &= \int_A h \circ T^n H_n dm.\end{aligned}$$

Thus, if we define $G_0(y) = h(y_0)$ then, for any set A in $\sum$,

$$\int_{(A)_n} G d\lambda = \int_A h \circ T^n H_n dm = \int_{(A)_n} G_0 d\lambda,$$

and so $G = G_0$, as stated. □

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