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LIFTING HANKEL OPERATORS FROM THE HARDY SPACE TO THE BERGMAN SPACE

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This article is about lifting bounded Hankel operators on the Hardy space of the disk to (bounded) Bergman-Hankel operators on the Bergman space of the disk. Though Hankel operators on the Bergman space have been studied extensively [**2, 4**], the natural map that produces this lifting generates some questions about its range, leading to interesting function-theory.

Let $\mathcal{L}_a^2$ be the Bergman space of analytic functions on the open unit disk $\mathbf{D}$ that belong to $\mathcal{L}^2(\mathbf{D})$ with respect to the normalized area measure. Let B denote the Bloch space, consisting of functions f analytic on the disk which satisfy the condition $\sup_{|z|<1}(1-|z|)|f'(z)| < \infty$. We write $||f||_B = \sup_{|z|<1}(1-|z|^2)|f'(z)|$. The little Bloch space B_0 is the subspace of the Bloch space consisting of functions for which $\lim_{|z|\to 1}(1-|z|)|f'(z)| = 0$. It is well-known that B_0 is the closure of polynomials in the Bloch norm [**1**]. A natural way to define a Hankel operator is as follows: If $\phi \in \mathcal{L}_a^2(\mathbf{D})$, let $S_\phi h = PJ(\phi h)$, where J is the self-adjoint unitary operator given by $(JF)(z) = f(\overline{z})$ and P is the orthogonal projection of $\mathcal{L}^2(\mathbf{D})$ onto $\mathcal{L}_a^2(\mathbf{D})$. For $\overline{\phi} \in \mathcal{L}_a^2(\mathbf{D})$ it is well-known that S_ϕ is a bounded operator on $\mathcal{L}_a^2$ if and only if $\overline{\phi}$ belongs to the Bloch space, in which case $||\overline{\phi}||_B \approx ||S_\phi||$ [**4**]. If $\overline{\phi}$ induces a bounded Hankel operator on the Hardy space (or, equivalently, if $\overline{\phi} \in \mathrm{BMOA}$, in which case $||\overline{\phi}||_{BMO} \sim ||S_\phi||$) [**10**], then with respect to the basis $\{z^n\sqrt{n+1}, n \geq 0\}$, the matrix of $S_\phi \in B(\mathcal{L}_a^2)$ is $[a_{i+j}m_{ij}]$ where $[a_{i+j}]$ is the matrix of $S_\phi \in B(H^2)$with respect to the basis $\{e^{in\theta}, n \geq 0\}$ and $m_{ij} = \sqrt{(i+1)(j+1)}/(i+j+1)$.

This brings us to the map Φ which sends bounded Hankel operators on H^2 into bounded Bergman-Hankel operators on $\mathcal{L}_a^2$ via Schur multiplication defined by $\Phi(A) = [a_{ij}m_{ij}]$ where $A = [a_{ij}]$ with respect to the standard basis $\{e^{in\theta}, n \geq 0\}$ of H^2 and m_{ij} is the multiplier defined above.

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PROPOSITION 1. *Φ is a completely positive and injective contraction.*

PROOF. If $A = [a_{ij}]$, then $\Phi(A) = [a_{ij}m_{ij}]$ with $m_{ij} > 0$ for all i, j. As Φ is obviously linear, this clearly implies that Φ is injective. If A is positive semi-definite then $\left[a_{ij}\frac{1}{i+j+1}\right]$ is the Schur product of A with the Hilbert matrix (which is known to be positive semi-definite). The Schur product of two positive semi-definite matrices is known to be positive semi-definite, and it follows that $\Phi(A)$ is positive semi-definite. A simple extension of this argument shows that Φ is in fact completely positive. $\Phi(I)$ is clearly a contraction and hence Φ is contractive. □

PROPOSITION 2. *Φ is not bounded below; the range of Φ is not closed.*

The proof hinges upon the fact that BMOA is not closed in the Bloch norm. The easiest direct proof consists of looking at function $f(z) = \sum_{k=1}^{\infty} z^{2^k}/\sqrt{k}$. If the coefficients of a gap series f are not square-summable, Payley's inequality implies that $f \notin$ BMOA [**5**]. However, $f \in B_0$ since $1/\sqrt{k} \to 0$ [**1**]. B_0 being the closure of polynomials in the Bloch norm, it is clearly contained in the closure of BMOA in the Bloch space.

A more indirect proof uses an example of a Blaschke product b given by Sarason [**11**]. This proof may not be as explicit but it also gives something more. (Note that if BMOA were closed in the Bloch norm, then two norms would be equivalent on BMOA.) If b is an infinite Blaschke product that belongs to B_0 and p_n is a sequence of polynomials that converge to b in the Bloch norm, clearly $p_n \not\to b$ in BMOA norm. For if $p_n \to b$ in BMOA then b would be in VMOA. However, no infinite Blaschke product belongs to VMOA. In particular, this provides an example of b that induces a compact Hankel operator on the Bergman space but a non-compact one on the Hardy space.

So the problem of characterizing the norm-closure of the range of Φ is equivalent to characterizing the closure of BMOA in the Bloch norm. These functions do have one property that is easy to write down: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f \in \overline{\text{BMOA}}^{\|\cdot\|_B}$, then $a_n \to 0$. Clearly $\overline{\text{BMOA}}^{\|\cdot\|_B}$ is a Möbius invariant closed subspace of the Bloch space. This prompts us to ask the following question: Let $f \in B$ and $(f \circ \phi_\lambda)(z) = \sum_{n=0}^{\infty} a_n(\lambda) z^n$, where $\phi_\lambda(z) = (\lambda - z)/(1 - \overline{\lambda} z)$ for $|\lambda| < 1$. If $a_n(\lambda) \to 0$ as $n \to \infty$ for all $\lambda \in \mathbf{D}$, does f belong to $\overline{\mathrm{BMOA}}^{||\cdot||_B}$?

Generalizing the example given in the proof of Proposition 2, we note that if $f(z) = \sum_{k=0}^{\infty} b_k z^{q^k}$, $q > 1$, is a gap-series belonging to B, then $b_k \to 0$ implies $f \in \overline{\mathrm{BMOA}}^{||\cdot||_B}$. Is there a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $a_n \to 0$ but $f \notin \overline{\mathrm{BMOA}}^{||\cdot||_B}$? One possible candidate is the function given by Holland and Twomey [**7**]. They write down the Taylor series for a function in B that doesn't belong to BMOA but belongs to $\cap_{p>0} H^p$. Obviously $a_n \to 0$, in fact $\Sigma |a_n|^2 < \infty$. We have been unable to determine if $f \in \overline{\mathrm{BMOA}}^{||\cdot||_B}$.

The closure of the range of Φ in the weak-operator topology is a little easier to characterize. Note that $B_0^{**} = B$ implies that B_0 is wk*-dense in B. As a consequence of the atomic decomposition of B, it follows that B_0 is in fact sequentially wk*-dense in B. However polynomials are norm-dense and hence wk*-dense in B_0 and hence BMOA is wk*-dense in B. As a consequence of the next proposition, the wk*-closure of range Φ is completely characterized.

PROPOSITION 3. *$\{B, \text{wk}^*\text{-}\}$ is homeomorphic to $\{S_\phi, \phi \in B, WOT\}$.*

PROOF. In one direction, the proof is trivial. Assume that $\phi_\alpha \overset{\text{wk}^*-}{\to} \phi$. In order to prove that $S_{\phi_\alpha} \overset{WOT}{\to} S_\phi$, let $g, h \in \mathcal{L}_a^2$, $\tilde{g}(z) = \overline{g}(\overline{z})$, $f = \tilde{g} h \in \mathcal{L}_a^1$ and note that $\langle S_{\phi_\alpha} g, h \rangle = \int \phi_\alpha(z) \overline{f}(z) dA(z) \to \int \phi(z) \overline{f}(z) dA(z) = \langle S_\phi g, h \rangle$. In order to make the proof work in the other direction, we need a result of Horowitz [**8**] which says that any $f \in \mathcal{L}_a^1$ can be expressed as $f = gh$ where $g, h \in \mathcal{L}_a^2$. Now $\int \phi_\alpha(z) \overline{f}(z) dA(z) = \langle S_{\phi_\alpha} \tilde{g}, h \rangle \to \langle S_\phi \tilde{g}, h \rangle = \int \phi(z) \overline{f}(z) dA(z)$ as before, i.e., if $S_{\phi_\alpha} \overset{WOT}{\to} S_\phi$ then we have $\phi_\alpha \overset{\text{wk}^*-}{\to} \phi$. □

COROLLARY. *Range Φ is dense in $\{S_\phi, \phi \in B\}$ in SOT as well.*

FOLK LEMMA. *If $\{A_n\}$ is a sequence of operators on a Hilbert space H and $A_n \overset{SOT}{\rightarrow} A$, then there exists $\{B_n\}, B_n = \sum_{k=1}^{m_n} c_k A_k$ such that $c_k \geq 0, \sum c_k = 1, B_n \overset{SOT}{\rightarrow} A$, and $B_n^* \overset{SOT}{\rightarrow} A^*$.*

PROPOSITION 4. *Given $\phi \in B$, there exists $\phi_0 \in B_0$ such that $||S_{\overline{\phi}}||_e = ||S_{\overline{\phi}} - S_{\overline{\phi_0}}||$.*

Thus a bounded Bergman-Hankel operator always has a best compact approximant that is also a Bergman-Hankel operator.

PROOF. By the previous corollary and folk lemma we may choose $\phi_n \in B_0$ such that $S_{\overline{\phi_n}} \overset{SOT}{\rightarrow} S_{\overline{\phi}}$ and $S_{\overline{\phi_n}}^* \overset{SOT}{\rightarrow} S_{\overline{\phi}}$. Then, by Theorems 1 and 2 of [**3**] there exists $\phi_0 \in B_0$ such that $||S_{\overline{\phi}} - S_{\overline{\phi_0}}|| = ||S_{\overline{\phi}}||_e$ (= the essential norm of $S_{\overline{\phi}}$). □

As no exposé about Hankel operators is ever complete without establishing some kind of a connection with Toeplitz operators, we state the connection that interests us.

As usual, $H^\infty(\mathbf{D})$ denotes the set of bounded analytic functions on $\mathbf{D}$. For $\phi \in H^\infty(\mathbf{D})$, the multiplication operator $T_\phi : \mathcal{L}_a^2(\mathbf{D}) \rightarrow \mathcal{L}_a^2(\mathbf{D})$ is defined by $T_\phi(g) = \phi g$. Luecking has given necessary and sufficient conditions for the Toeplitz operator T_ϕ to be bounded below. A careful look at the proof [**9**, p. 5] and the statement of the main theorems shows that Luecking has actually proved

PROPOSITION 5. *Let $\phi \in H^\infty(\mathbf{D})$. Then T_ϕ is bounded below if and only if $||\phi k_\lambda|| \geq c||k_\lambda||$ for all $\lambda \in \mathbf{D}$, where k_λ is the reproducing kernel for the Bergman space.*

QUESTION (A). If b is an infinite Blaschke product with zeros $\{\lambda_n, n \geq 1\}$ and $||bk_{\lambda_n}|| \geq c||k_{\lambda_n}||$ for all n, is T_ϕ bounded below? Equivalently: is b a finite product of interpolating Blaschke products?

This brings us to the definition of Hankel operators on the Bergman space given in [**2**]. The criterion for compactness remains the same but the connection with Toeplitz operators is a closer and a more natural one. A close look at Theorem 7 of [**2**] shows that a Blaschke product $b \in B_0$ if and only if $\lim_{|\lambda|\to 1} ||H_{\bar{b}} k_\lambda||/||k_\lambda|| = 0$.

QUESTION (B). Is this true if the set of λ's is restricted to the zeros of b?

If the answers to Questions (A) and (B) are affirmative, then using the simple equation $||H_{\bar{b}} k_{\lambda_n}||/||k_{\lambda_n}|| = ||bk_{\lambda_n}||/||k_{\lambda_n}||$, we have the following sets of characterizations: If b is an infinite Blaschke product,with zeros $\{\lambda_n\}$, then: (i) b is a finite product of interpolating Blaschke products if and only if $\inf_n ||bk_{\lambda_n}||/||k_{\lambda_n}|| > 0$ and (ii) b is not in B_0 if and only if $\underline{\lim}_n ||bk_{\lambda_n}||/||k_{\lambda_n}|| > 0$.

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