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RATIONAL C*-ALGEBRAS AND NONSTABLE K-THEORY

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1. Introduction. In this article, we will be concerned with various aspects of comparison theory for idempotents (or finitely generated projective modules) over unital rings, particularly over unital **C***-algebras.

The usual way equivalence and comparison is studied is via K-theory, which involves "stabilizing" the relations. There is a very extensive and powerful body of machinery allowing the computation of the K-groups of many rings; the problem is then to relate the K-theory data back to the actual structure of the ring and its projective modules. This process has become known as *nonstable* K*-theory*.

We will discuss several aspects of nonstable K-theory and develop some new relationships based on results in the theory of abelian semigroups. Then we will apply these results to solve nonstable K-theory problems for certain "rationalized" rings. Actually, a majority of the paper is devoted to developing the semigroup theory; in fact, a more appropriate title might be "applications of abelian semigroups in algebraic K-theory."

1.1 *Review of* K_0*-Theory.* Let us first give a very brief review of the construction of $K_0(A)$ for a unital ring A, in order to establish notation. A much more complete treatment of the subject can be found in [**3**].

DEFINITION 1.1.1. Let p and q be idempotents in A. $p \sim q$ if there are $x, y \in A$ with $xy = p, yx = q$. $p \prec q$ if p is equivalent to an idempotent r with $qr = rq = q$ and $r \neq q$. $p \preceq q$ if $p \prec q$ or $p \sim q$. A is *finite* if $1_A \not\prec 1_A$; A is *stably finite* if the $n \times n$ matrix algebra $M_n(A)$ is finite for all n. $M_\infty(A)$ is the (nonunital) ring $\varinjlim M_n(A)$, where $M_n(A)$ is embedded in $M_{n+1}(A)$ in the upper left-hand corner, extended by zeros.

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Recall that there is a one-to-one correspondence between equivalence classes of idempotents in $M_\infty(A)$ and isomorphism classes of finitely generated projective modules over A. If $A = C(X)$ for a compact Hausdorff space X, then these classes are also in one-one correspondence with the isomorphism classes of complex vector bundles on X. Each of these sets has a natural operation of "direct sum," making it into an abelian semigroup.

Definition 1.1.2. $V(A)$ is the semigroup of equivalence classes of idempotents in $M_\infty(A)$, or of isomorphism classes of finitely generated projective modules over A. $V_0(A)$ is the subsemigroup of nonzero elements in $V(A)$.

$V(A)$ has an identity [0], but $V_0(A)$ does not have an identity in general. (Note, however, that $V_0(A)$ can have an identity (1.1.3(b)).)

Examples 1.1.3. (a) Let A be the integers $\mathbf{Z}$ or any field (or, more generally, any PID.) Then $V(A)$ is isomorphic to the additive semigroup of nonnegative integers, and $V_0(A)$ is isomorphic to the natural numbers $\mathbf{N}$.

(b) Let A be the Cuntz algebra O_2 [**3**, 6.3.2]. A is not finite. $V(A)$ is isomorphic to the semigroup $\mathbf{J}$ with elements $\{0, 1\}$, with $0+0 = 0, 0+1 = 1+0 = 1+1 = 1$. $V_0(A)$ is the one-element semigroup, which is a group. (Actually, if A is any purely infinite simple unital $\mathbf{C}^*$-algebra, then $V_0(A)$ is a group [**7**].)

We define $K_0(A)$ to be the Grothendieck group of $V(A)$ (or of $V_0(A)$) [**3**, 1.3]. (Note: the subscript 0 in K_0 has a completely different meaning than the subscript in V_0.) The semigroups $V(A)$ and $V_0(A)$ need not have cancellation, even if A is commutative, so the natural homomorphism $\phi : V(A) \to K_0(A)$ is not generally injective. If p and q are idempotents in matrix algebras over A, then $[p] = [q]$ in $K_0(A)$ if and only if p and q are stably equivalent, i.e., if there is an idempotent r with $p \oplus r \sim q \oplus r$.

We can put a preordering on $K_0(A)$ by taking $K_0(A)_+$ to be $\phi(V(A))$. This is a partial ordering if A is stably finite. This preordering determines stable comparability of idempotents, i.e., $[p] \le [q]$ in $K_0(A)$ if and only if there is an r with $p \oplus r \preceq q \oplus r$.

1.2. *Nonstable K-Theory.* Probably the central question in nonstable K-theory is

QUESTION 1.2.1. Let p and q be idempotents in $M_\infty(A)$ which are stably equivalent [respectively stably comparable]. Under what conditions does it follow that p and q are equivalent [respectively comparable]?

One can more specifically ask whether a given ring A has cancellation or strict cancellation:

DEFINITION 1.2.2. A has *cancellation* if $p \oplus r \sim q \oplus r$ for some r implies $p \sim q$. A has *strict cancellation* if $p \oplus r \prec q \oplus r$ implies $p \prec q$.

Cancellation implies strict cancellation, but the converse is probably false (cf. 2.1.6.) A ring with strict cancellation must be stably finite; but even a commutative ring need not have strict cancellation.

A has cancellation if and only if the semigroup $V(A)$ has cancellation. If A is stably finite, then A has cancellation if and only if $V_0(A)$ has cancellation (but see 1.1.3(b).)

Another nonstable K-theory question for $\mathbf{C}^*$-algebras is

QUESTION 1.2.3. Let A be a unital $\mathbf{C}^*$-algebra, $U_1(A)$ its unitary group, $U_1(A)_0$ its connected component of the identity, and μ_1 the canonical homomorphism from $U_1(A)/U_1(A)_0$ to $K_1(A)$ [**3**, §9]. Under what conditions is μ_1 injective? When is μ_1 surjective?

In addition, there are some other important questions which are, strictly speaking, not nonstable K-theory questions, but which have a similar flavor. For example:

QUESTION 1.2.4. Let A be a finite ring. Is A stably finite?

QUESTION 1.2.5. Let p and q be idempotents in $M_\infty(A)$. Write $n \cdot p = p \oplus \cdots \oplus p$ (n times.) If $n \cdot p \sim n \cdot q$ [respectively $n \cdot p \prec n \cdot q$], under what conditions is $p \sim q$ [respectively $p \prec q$]?

DEFINITION 1.2.6. A has *n-power cancellation* if $n \cdot p \sim n \cdot q$ implies $p \sim q$.

A is *n-unperforated* if $n \cdot p \preceq n \cdot q$ implies $p \preceq q$.

A is *strictly n-unperforated* if $n \cdot p \prec n \cdot q$ implies $p \prec q$.

A is [*strictly*] *unperforated* if A is [strictly] n-unperforated for all n, and similarly for power cancellation.

The power cancellation or unperforated property seems at first glance to be closely related to cancellation; however, further reflection will reveal that there are fundamental conceptual differences. The main results of this article are that the two are nevertheless intimately related.

Finally, the concept of *stable rank* [**14**] is important in studying nonstable K-theory questions.

See [**4**] for a discussion of all of these questions and related matters for simple C^*-algebras.

1.3. *Results and final comments.* The main result is that if A is a stably finite simple ring with [strict] n-unperforation for some n, then A has [strict] cancellation. (Actually the results are somewhat more general.) As a consequence, we show that certain tensor products have cancellation. In particular, if A is any stably finite simple unital $\mathbf{C}^*$-algebra and B is any simple unital AF algebra, then $A \otimes B$ has cancellation. So far as I know, the results of this paper are the first cancellation theorems which do not use some kind of stable rank arguments.

Perhaps the most remarkable fact about the results is that essentially no $\mathbf{C}^*$-algebra theory or even ring theory is used in the proofs; the results are easy consequences of some rather simple facts in the theory of abelian semigroups. While the semigroup results were previously known to specialists in that field, their relevance to nonstable K-theory questions does not seem to have been noticed (although Cuntz and Pedersen have used very similar arguments in [**8**, 2.1 and 5.2].)

In §2 we do the semigroup theory relevant for the applications. Actually, we do somewhat more than is necessary; the extra material such as the theory of ordered semigroups is closely related to the work we will need to use, and I believe its inclusion is worthwhile because of its potential applications to other nonstable K-theory problems (cf. [**4,** §6]). Then in §3 we give the applications to nonstable K-theory. While the author and many potential readers are primarily interested in $\mathbf{C}^*$-algebras, all the results of this section work equally well in general unital rings, and are so expressed. §4 contains some consequences in $\mathbf{C}^*$-algebra theory.

2. Some semigroup theory. In this section, we will make a digression to develop some of the theory of abelian semigroups which will be relevant for K-theory (we will, of course, want to apply the results to the study of $V(A)$ and $V_0(A)$ for a unital ring A.) Although few, if any, of the results here are new, since most potential readers are somewhat unfamiliar with the subject we will give a concise but reasonably self-contained treatment of the parts of the theory we will need.

In what follows, all semigroups are always assumed to be *abelian*, and will be written using additive notation. We do not assume that our semigroups have identities. The identity of a semigroup, if it exists, will be denoted 0. If $n \in \mathbf{N}$, then nx will denote $x + \cdots + x$ (n times.)

2.1. *Basic theory of semigroups.*

Definition 2.1.1. If S is a semigroup, we define a preorder $<$ on S, called the *algebraic preordering,* by setting $y < x$ if there is a $z \in S$ with $x = y + z$.

The algebraic preordering is a translation-invariant preorder (transitive relation) on S, but it is not in general a partial order. For example, in the extreme case where S is a group, $x < y$ for all $x, y \in S$. It is convenient to think of the algebraic preordering as an "ordering" even in the case where it is not a true partial order.

As usual, we will write $y \leq x$ if $y < x$ or $y = x$.

Definition 2.1.2. A semigroup S is a *strict semigroup* if the algebraic preordering on S is a strict partial order.

S is a strict semigroup if and only if $x+y \neq x$ for all $x, y \in S$. A strict semigroup can never have an identity or even an idempotent. (Note that there are important examples of semigroups with idempotents, such as the power set of any set with union as the operation, on which the algebraic ordering is a nonstrict partial order.)

Example 2.1.3. If A is a stably finite unital ring, then $V_0(A)$ is a strict semigroup.

If S is a strict semigroup and ϕ is the canonical homomorphism from S into its Grothendieck group $G(S)$, then the cancellation semigroup $\phi(S)$ of S is also a strict semigroup. So if we set $G(S)_+ = \phi(S) \cup \{0\}$, then $G(S)$ becomes an ordered group in the sense of [**3,** §6]. This is called the *induced order* on $G(S)$.

Definition 2.1.4. A semigroup S is *archimedean* if, for any $x, y \in S$, there is an $n \in \mathbf{N}$ such that $y < nx$ (i.e., there is a $z \in S$ with $nx = y + z$.)

An archimedean semigroup is one in which no element is "infinitely large" or "infinitely small" with respect to any other. Any group is an archimedean semigroup. An archimedean semigroup with an identity (or even containing an idempotent) must be very close to being a group. See [**15**] for a description of the structure of archimedean semigroups.

If S is an archimedean strict semigroup, then its Grothendieck group $G(S)$ is a simple ordered group in its induced order. See [**3,** §6] for a development of the theory of ordered groups.

DEFINITION 2.1.5. Let $n > 1$. A semigroup S is *n-unperforated* if $nx \leq ny$ implies $x \leq y$.

S is *strictly n-unperforated* if $nx < ny$ implies $x < y$.

S has *n-power cancellation* if $nx = ny$ implies $x = y$.

S is *unperforated* (etc.) if it is n-unperforated (etc.) for all n.

If S is strictly n-unperforated and has n-power cancellation, then S is n-unperforated. The converse is false in general: $\mathbf{Z}_n$ is unperforated but does not have n-power cancellation, and if $S = \mathbf{N} \cup \{0, 1'\}$, with $0+n = n$ for $n \neq 1'$, $1'+1' = 2$, $n+1' = n+1$ for all other n, then S is unperforated but not strictly unperforated since $2\cdot 1' = 2 < 2$ $[2 = 2+0]$ but $1' \not< 1'$.

If S is a strict semigroup, then S is n-unperforated if and only if it is strictly n-unperforated and has n-power cancellation.

If S is n-unperforated (etc.), then S is k-unperforated (etc.) whenever k is a power of n.

Recall also that a semigroup S has *cancellation* if $x+z = y+z$ for some z implies $x = y$. We say S has *strict cancellation* if $x+z < y+z$ implies $x < y$. It is easy to see that cancellation implies strict cancellation. The converse is false:

EXAMPLE 2.1.6. Let S be the semigroup $\mathbf{N} \cup \{1'\}$, with $1' + 1' = 2$ and $1' + n = n + 1' = n + 1$ for $n \in \mathbf{N}$. Then S has strict cancellation but not cancellation.

It will be important for our main results to have the following notion of "stable cancellation":

DEFINITION 2.1.7. A semigroup S has *cancellation up to powers* if whenever $x + z = y + z$ for some z, then $kx = ky$ for all sufficiently large k. S has *strict cancellation up to powers* if $x + z < y + z$ implies $kx < ky$ for all sufficiently large k. (The minimal k may depend on x and y.)

PROPOSITION 2.1.8. Let *S be a semigroup.*

(a) *If S has cancellation up to powers, then S has strict cancellation up to powers.*

(b) *If S has n-power cancellation for some n and cancellation up to powers, then S has cancellation.*

(c) *If S is strictly n-unperforated for some n and has strict cancellation up to powers, then S has strict cancellation.*

PROOF. (a). Suppose $x+z < y+z$. Then there is a w with $x+z+w = y+z$. By cancellation up to powers, $k(x+w) = kx+kw = ky$ for all sufficiently large k, so $kx < ky$ for large k.

(b). If $x+z = y+z$, then, by cancellation up to powers, $kx = ky$ whenever k is a sufficiently large power of n. But then $x = y$ by k-power cancellation.

(c). Nearly identical to (b). □

We now come to the main result, due to Kimura and Tsai [**12**], which says that an archimedean semigroup "almost" has cancellation.

THEOREM 2.1.9. *Let S be an archimedean semigroup. Then S has cancellation up to powers.*

PROOF. Suppose $x+z = y+z$. We can find $a, b \in S$ and $m, n \in \mathbf{N}$ with $z+a = nx, z+b = my$. Then

$$x+nx = x+z+a = y+z+a = y+nx$$

and similarly $x+my = y+my$. So in the presence of at least $n\,x$'s or $m\,y$'s an x may be converted to a y or vice versa. Thus, if $k \geq m+n$, then

$$kx = nx+mx+rx = nx+my+ry = ny+my+ry = ky. \quad \square$$

COROLLARY 2.1.10. *If S is an archimedean semigroup with n-power cancellation for some n, then S has cancellation. If S is archimedean and strictly n-unperforated for some n, then S has strict cancellation.*

Proof. Follows immediately from 2.1.9 and 2.1.8. □

Finally, we have a result about unperforation which is closely related in spirit to 2.1.9. The proof is combination of 2.1.9 with the arguments in the proof of [**4**, III.2.6].

Theorem 2.1.11. *Let S be an archimedean semigroup, $x, y \in S$. If $nx < ny$ for some n, then $kx < ky$ for all sufficiently large k.*

Proof. Fix n with $nx < ny$. Then $nx + z = ny$ for some $z \in S$. There is a j such that $y < jz$; if $m = (n-1)j$, then, for $0 \le r \le n-1$,

$$\begin{aligned} &mnx + ry \le mnx + (n-1)y < mnx + (n-1)jz \\ &= mnx + mz = m(nx+z) = mny \end{aligned}$$

So, for $0 \le r \le n-1$, $(mn-r)x+ry \le mnx+ry < (mn-r)y+ry$. Thus, for fixed r, we have $s(mn-r)x < s(mn-r)y$ for all sufficiently large s by strict cancellation up to powers (2.1.9, 2.1.8(a)). Fix s relatively prime to n and large enough that $s(mn-r)x < s(mn-r)y$ for all r with $0 \le r \le n-1$.

Now suppose $k \ge smn$. Write $k = pn - q$ with $0 \le q \le n-1$; then there is an r with $0 \le r \le n-1$ with $rs \equiv q \mod n$. For this $r, k \equiv s(mn-r) \bmod n$, so $k = tn + s(mn-r)$; $t \ge 0$ since $k \ge smn \ge s(mn-r)$. Then $tnx < tny$ and $s(mn-r)x < s(mn-r)y$, so

$$kx = tnx + s(mn-r)x < tny + s(mn-r)y = ky. \quad \square$$

Corollary 2.1.12. *Let S be an archimedean semigroup. Suppose there is an $n > 1$ such that S is uniquely n-divisible, i.e., for any $x \in S$ there is a unique $y \in S$ with $ny = x$. Then S has cancellation and is strictly unperforated.*

Proof. S has n-power cancellation, so S has cancellation by 2.1.9. To show S is strictly unperforated, suppose $my < mx$. Then $ky < kx$ for some sufficiently large k which is a power of n by 2.1.11, so $ky + z = kx$

for some z. By unique n-divisibility (iterated) $z = kw$ for some w, and then by n-power cancellation $y + w = x, y < x$. □

2.2. *Ordered semigroups.* This section will not be used in the rest of the article, except in 2.5.5. We will extend the main result of the previous section to certain ordered semigroups where the ordering is not the algebraic ordering. In this section, we will work only with *strict* semigroups.

Definition 2.2.1. An *ordered semigroup* is a semigroup S with a strict partial order $<$ which is translation-invariant and which extends the algebraic ordering, i.e.:

(1) If $y < x$, then $y + z < x + z$ for all z.

(2) If $x = y + z$, then $y < x$.

The ordering is *well-behaved* if $x+z = y+z$ implies $x+w = y+w$ for all w with $z < w$. The ordering is *strictly well-behaved* if $x + z < y + z$ implies $x + w < y + w$ for all w with $z < w$.

An ordered semigroup is necessarily a strict semigroup. An ordered semigroup with cancellation is well-behaved; in fact, the well-behaved [respectively strictly well-behaved] property may be regarded as a weak form of cancellation [respectively strict cancellation].

We can also define the notions of strict cancellation, strict cancellation up to powers, and strict n-unperforation for ordered semigroups in the obvious way; these concepts depend of course on the order structure and not just on the algebraic structure of the semigroup. Unlike the case of the algebraic ordering, cancellation does not imply strict cancellation in a general ordered semigroup (2.2.2(e)). Similarly, a well-behaved ordered semigroup is not strictly well-behaved in general (2.2.2(f)). And, in analogy with 2.1.6, a strictly well-behaved semigroup is not necessarily well-behaved (2.2.2(d)), so the well-behaved and strictly well-behaved properties are independent.

Examples 2.2.2. (a) Any strict semigroup is a well-behaved, strictly well-behaved ordered semigroup with respect to the algebraic ordering.

(b) More generally, if S is a subsemigroup of a strict semigroup T, then the restriction of the algebraic ordering on T to S is well-behaved, strictly well-behaved partial ordering on S which is not in general the same as the algebraic ordering (for example, let $T = \mathbf{N}, S = \{n : n \geq 2\}$.)

(c) As a special case of (b), let T be the semigroup of all bounded functions from a topological space X to the positive real numbers, with pointwise addition, and S the subsemigroup of bounded lower semicontinuous functions. The induced ordering on S is the pointwise strict ordering, i.e., $f < g$ if $f(x) < g(x)$ for all $x \in X$. Since the difference of two lower semicontinuous functions is not generally lower semicontinuous, this ordering is not generally the algebraic ordering. S is denoted $LSC_{++}(X)$.

(d) Let $\mathbf{J}$ be the semigroup defined in 1.1.3(b). $\mathbf{J}$ is an *idempotent semigroup* (one in which every element is idempotent.) Let $S = \mathbf{J} \times \mathbf{N}$, and order S by $(a, m) < (b, n)$ if and only if $m < n$. S is not well-behaved because $(0, 1) + (1, 1) = (1, 1) + (1, 1)$ and $(1, 1) < (0, 2)$, but $(0, 1) + (0, 2) \neq (1, 1) + (0, 2)$. S is strictly well-behaved; in fact, it has strict cancellation.

(e) Let S be the subsemigroup $\{n \,|\, n \geq 2\}$ of $\mathbf{N}$. Order S with the usual ordering but with the exception that 2 and 3 are incomparable. Then S has cancellation, but not strict cancellation since $2+2 < 3+2$ but $2 \not< 3$.

(f) It is more difficult to give an example of an ordered semigroup which is well-behaved but not strictly well-behaved. Let $S = \{3\} \cup \{n \,|\, n \geq 5\} \subseteq \mathbf{N}$. S is a subsemigroup. Define the order on S to be the usual order but with the following pairs incomparable: (3,5), (6,7), (6,8), (9,10), (9,11). S has cancellation, hence is well-behaved; but it is not strictly well-behaved since $3+5 < 5+5$ and $5 < 6$, but $3+6 \not< 5+6$.

It is possible that example (b) is the most general well-behaved, strictly well-behaved ordered semigroup. To attempt to prove this, if S is an ordered semigroup, let S^+ be S with a zero adjoined, i.e., $S^+ = S \cup \{0\}, 0 + x = x + 0 = x$ for all $x, 0 < x$ for all $x \in S$. Let $\underline{S}$ be the semigroup with generators $\{[x, y] |\, x \in S, y \in S^+, y < x\}$ and relations

$$[x, 0] + [y, 0] = [x + y, 0] \quad \text{for all } x, y \in S,$$

$$[y,0]+[x,y]=[x,0] \text{ for all } x,y \in S, y<x.$$

Think of $[x,y]$ as $x-y$. $\underline{S}$ is called the *full semigroup* of S.

There is a homomorphism $\Psi : S \to \underline{S}$ given by $\Psi(x)=[x,0]$. $(\underline{S},\Psi)$ has the universal property that any order-preserving homomorphism from S into a semigroup T with the algebraic ordering factors through $\underline{S}$.

Definition 2.2.3. S is *subalgebraically ordered* if Ψ is injective and if the ordering on S is the restriction of the algebraic ordering on $\underline{S}$.

Because of the universal property of $\underline{S}$, S is subalgebraically ordered if and only if S arises as in 2.2.2(b).

It seems to be a difficult problem to intrinsically characterize subalgebraically ordered semigroups. An obvious necessary condition is that the ordering be well-behaved and strictly well-behaved; it is possible that these two conditions are also sufficient.

There is a homomorphism from $\underline{S}$ to the Grothendieck group $G(S)$ sending $[x,y]$ to $x-y$; the composition with Ψ gives the natural homomorphism from S to $G(S)$. Thus if $\Psi(x)=\Psi(y)$, then $x+z=y+z$ for some $z \in S$. In particular, if S has cancellation, then Ψ is injective. (Note, however, that injectivity of Ψ is not sufficient to make S subalgebraically ordered; see 2.2.2(f).)

Problem 2.2.4. Give necessary and sufficient intrinsic conditions for Ψ to be injective; give conditions for S to be subalgebraically ordered.

I have been unable to find a treatment of this problem in the literature, except for the special cases treated in the old papers [**1**] and [**13**].

Even if S is subalgebraically ordered, $\underline{S}$ seems to be too large to be very useful in studying S. In particular, $\underline{S}$ is very far from having cancellation. One would hope for a quotient of $\underline{S}$ which still contains S and algebraically induces the order on S, and which is roughly as close to having cancellation as S.

To understand the situation better, suppose S has cancellation. We want to order-embed S in an algebraically ordered cancellation semigroup. This can be done if and only if the cancellation semigroup $\overline{S}$ of $\underline{S}$ induces the right order on S. An obvious necessary condition on S is strict cancellation; this is also sufficient because one can then order the Grothendieck group of S by taking the image of $\underline{S}$ under the canonical homomorphism (which is isomorphic to $\overline{S}$) to be the nonzero positive cone, and the induced order on S will be the original one. This is the theorem proved in [**13**]. So example 2.2.2(e), which is easily seen to be subalgebraically ordered, cannot be order-embedded in an algebraically ordered cancellation semigroup, showing that a quotient of $\underline{S}$ can induce a different ordering on S than $\underline{S}$ itself. $\overline{S}$ is called the *full cancellation semigroup* of S.

The definition of an archimedean ordered semigroup is formally identical to the definition for the algebraic ordering: S is archimedean if, for any $x, y \in S$, there is an n such that $y < nx$. If S is archimedean, then $\overline{S}$ is not necessarily archimedean (2.2.5(c)).

We say S is *strongly archimedean if*, for any $x, z \in S$ and $y \in S^+$ with $y < x$, there is an n such that $ny + z < nx$. An archimedean algebraically ordered semigroup is always strongly archimedean, but a general archimedean ordered semigroup is not necessarily strongly archimedean (2.2.5(c)). If S is strongly archimedean, then $\underline{S}$ is archimedean; if S has cancellation, then $\overline{S}$ is archimedean if and only if S is strongly archimedean.

EXAMPLES 2.2.5. (a) If S is archimedean in the algebraic ordering, then S is archimedean in any other ordering. More generally, if S is archimedean in one ordering, then it is archimedean in any stronger ordering.

(b) The semigroup S of 2.2.2(d) is strongly archimedean.

(c) Let $S = LSC_{++}(X)$ as in 2.2.2(c), with X compact. Then S is archimedean since any lower semicontinuous positive function on X is bounded away from 0. But S is not strongly archimedean, since if f and g are lower semicontinuous and $f < g$, then $g - f$ is not necessarily bounded away from 0. $\overline{S}$ is isomorphic to the semigroup $DSC_{++}(X)$ of all functions from X to $(0, \infty)$ which can be written as the difference of two bounded lower semicontinuous functions. $DSC_{++}(X)$ is not archimedean since functions in $DSC_{++}(X)$ are not necessarily bounded away from 0.

(d) Let S be the same semigroup as in (c), but with a different ordering: $f < g$ if and only if $g - f$ is positive everywhere and bounded away from 0. This ordered semigroup is called $LSC_{\varepsilon}(X)$; it is strongly archimedean. $\overline{S}$ is isomorphic to the subsemigroup $DSC_{\varepsilon}(X)$ of $DSC_{++}(X)$ consisting of all elements which are bounded away from 0. $DSC_{\varepsilon}(X)$ is archimedean (with the algebraic ordering.)

Note that examples (b), (c) and (d) are not archimedean with respect to the algebraic ordering (in $LSC_{++}(X)$, if f is continuous and g is discontinuous no multiple of f dominates g algebraically.)

See [**4**, 6.3] for an example of how semigroups similar to the ones in 2.2.5(c) and (d) arise naturally in the study of simple $\mathbf{C}^*$-algebras.

We now come to the analog of the main result of §2.1 for ordered semigroups:

Theorem 2.2.6. *Let S be an archimedean ordered semigroup.*

(a) *If S is well-behaved, then S has cancellation up to powers.*

(b) *If S is strictly well-behaved, then S has strict cancellation up to powers (with respect to the given ordering).*

Proof. The proof of both statements is essentially identical to the proof of 2.1.9. We prove (b). Suppose $x + z < y + z$. Find n and m so that $z < nx$ and $z < my$. Then, by the strictly well-behaved property, $x + nx < y + nx, x + my < y + my$. So, if $k \geq n + m$,

$$kx = nx + mx + rx < nx + my + ry < ny + my + ry = ky. \quad \square$$

Corollary 2.2.7. *Let S be an archimedean ordered semigroup.*

(a) *If S is well-behaved and has n-power cancellation for some n, then S has cancellation.*

(b) *If S is strictly well-behaved and is strictly n-unperforated for some n, then S has strict cancellation.*

Theorem 2.2.6(a) says that if S is a (strict) semigroup, and S admits an archimedean ordering under which it is well-behaved, then S must have cancellation up to powers. (2.2.6(b) cannot be rephrased in the same way because the notion of strict cancellation up to powers depends on the ordering.)

The hypothesis that S be well-behaved in 2.2.6 (or something like it) is necessary: the semigroup of 2.2.5(b) is strongly archimedean but does not have either cancellation up to powers or strict cancellation up to powers.

We also have an analog of 2.1.11 with a nearly identical proof:

THEOREM 2.2.8. *Let S be an ordered semigroup which is strictly well-behaved and strongly archimedean, and let $x, y \in S$. If $nx < ny$ for some n, then $kx < ky$ for all sufficiently large k.*

2.3 *Tensor products of semigroups.* This section will also not be used in an essential way in the sequel.

The theory of tensor products of abelian semigroups works out in much the same formal way as the theory of tensor products of abelian groups. For a complete treatment see [**10**].

DEFINITION 2.3.1. Let S and T be semigroups. The *tensor product* of S and T, denoted $S \otimes T$, is the universal (abelian) semigroup with generators $\{s \otimes t : s \in S, t \in T\}$ and relations $\{s_1 \otimes t + s_2 \otimes t = (s_1 + s_2) \otimes t, s \otimes t_1 + s \otimes t_2 = s \otimes (t_1 + t_2) : s_1, s_2 \in S, t, t_1, t_2 \in T\}$.

If t_0 is fixed, then $s \to s \otimes t_0$ is a homomorphism from S into $S \otimes T$. A fixed element of S likewise gives a homomorphism from T into $S \otimes T$. These homomorphisms are not injective in general.

$S \otimes T$ has the usual universal property: any "bilinear" map from $S \times T$ into an abelian semigroup R induces a unique homomorphism from $S \otimes T$ to R.

PROPOSITION 2.3.2. *Let S and T be semigroups.*

(a) *If $S = \varinjlim S_i$, then $S \otimes T$ is naturally isomorphic to $\varinjlim S_i \otimes T$.*

(b) *If* S *and* T *are archimedean, then* $S \otimes T$ *is archimedean.*

(c) *If* S *and* T *have cancellation, then* $S \otimes T$ *has cancellation.*

(d) *If* S *has no nontrivial idempotent quotients and* T *is a group, then* $S \otimes T$ *is a group.*

Proof. (a). This is a straightforward exercise left to the reader.

(b). A typical element of $S \otimes T$ can be written in the form $\sum s_i \otimes t_i$. Since $\sum s_i \otimes t_i > s_1 \otimes t_1$, it suffices to show that, for any elementary tensor $x \otimes y$, there is a k with $x \otimes y < k(s_1 \otimes t_1)$. Choose m and n with $x < ms_1, y < nt_1$. Then $x \otimes y < ms_1 \otimes y < ms_1 \otimes nt_1 = mn(s_1 \otimes t_1)$.

(c) See [**11**].

(d) Since $\{s \otimes 0 : s \in S\}$ is a subsemigroup of $S \otimes T$ in which every element is idempotent, and this subsemigroup is a quotient of S, the hypothesis implies that this subsemigroup consists of only one element, i.e., $s_1 \otimes 0 = s_2 \otimes 0$ for all $s_1, s_2 \in S$. This element is clearly an identity for $S \otimes T$. The inverse for the element $\sum s_i \otimes t_i$ is $\sum s_i \otimes (-t_i)$. □

Corollary 2.3.3. *If* S *is archimedean and* T *is a group, then* $S \otimes T$ *is a group.*

Proof. A quotient of an archimedean semigroup is archimedean, and an archimedean semigroup can have at most one idempotent. □

Examples 2.3.4. (a) If S is any semigroup, then $S \otimes \mathbf{N}$ is naturally isomorphic to S.

(b) If S is a semigroup with no nontrivial idempotent quotients (e.g., S is archimedean), then $S \otimes \mathbf{Z}$ is naturally isomorphic to the Grothendieck group of S.

(c) If S is an idempotent semigroup (e.g., $S = \mathbf{J}$ of 2.2.2(d)), then $S \otimes \mathbf{Z}$ is isomorphic to S. Thus the hypothesis is 2.3.2(d) is necessary.

2.4 *Simple dimension semigroups.* In this section, we give a description of the theory of simple dimension semigroups. This theory is nothing but the theory of simple dimension groups from a slightly different point of view. We treat only simple dimension groups because the theory of general (non-simple) dimension groups cannot be cleanly expressed in semigroup language in the same way.

We will examine certain semigroups which are inductive limits of ones isomorphic to $\mathbf{N}^r$. Not all inductive limits will fit well in our theory; to specify the right ones, we must examine the possible connecting homomorphisms.

Let σ be a homomorphism from $\mathbf{N}^r$ to $\mathbf{N}^s$. Then σ extends uniquely to a homomorphism between the Grothendieck groups $\mathbf{Z}^r$ and $\mathbf{Z}^s$, hence is given by multiplication by an $s \times r$ matrix R_σ with integer entries. R_σ maps $\mathbf{N}^r$ into $\mathbf{N}^s$ if and only if all entries of R_σ are nonnegative and R_σ has no zero rows (to see this, regard R_σ as giving a linear map from $\mathbf{R}^r$ to $\mathbf{R}^s$; this map must send $\mathbf{Q}_+^r$ into $\mathbf{Q}_+^s$ and therefore must send points with nonnegative coordinates into points with nonnegative coordinates).

Definition 2.4.1. σ is *simple* if:

(1) Either $r = s = 1$ or r and s are both greater than 1;

(2) All entries in R_σ are positive;

(3) σ is not the identity map on $\mathbf{N}^1$.

A simple map cannot be surjective; in fact, $(1, 1, \ldots, 1)$ is never in the image. A simple map can also fail to be injective.

Proposition 2.4.2. *Let $\sigma : \mathbf{N}^r \to \mathbf{N}^s$ be a composition of n simple maps. Then σ is simple, and every entry in the matrix R_σ is at least 2^{n-1}.*

Proof. This is an easy proof by induction, since R_σ is a product of n matrices with positive entries. Treat the cases $r = s = 1$ and $r, s > 1$ separately. □

Definition 2.4.3. A *simple dimension semigroup* is a semigroup S which is isomorphic to $\varinjlim(\mathbf{N}^{r_i}, \rho_i)$, where each $\rho_i : \mathbf{N}^{r_i} \to \mathbf{N}^{r_{i+1}}$ is simple.

We have excluded $\mathbf{N}$ as a simple dimension semigroup by our insistence that the identity map on $\mathbf{N}$ is not simple. This is an arbitrary choice which is in slight conflict with conventions in dimension group theory; however, excluding $\mathbf{N}$ makes many statements in succeeding sections considerably cleaner.

Every simple dimension semigroup is strict, countable, archimedean, unperforated, and has cancellation, since $\mathbf{N}^r$ has these properties. Simple dimension semigroups are characterized by these properties plus one additional property which is less obvious, due to the Effros-Handelman-Shen theorem [**9**]:

Theorem 2.4.4. *A semigroup is a simple dimension semigroup if and only if it is strict, countable, archimedean, unperforated, has cancellation and the* Riesz interpolation property (*if* $x_1, x_2 \leq y_1, y_2$, *then there is a* z *with* $x_1, x_2 \leq z \leq y_1, y_2$), *and is not isomorphic to* $\mathbf{N}$ (*equivalently, is not finitely generated*). [The cancellation hypothesis is redundant by 2.1.10.]

Proof. The Grothendieck group of a simple dimension semigroup is a simple dimension group with the induced order. Conversely, the nonzero positive cone in any simple dimension group except $\mathbf{Z}$ is a simple dimension semigroup (this is not entirely obvious, but is not difficult to show.) Apply the Effros-Handelman-Shen theorem. □

A representation of a simple dimension semigroup as an inductive limit as in Definition 2.4.3 is called a *Bratteli diagram* for the semigroup. An abstract simple dimension semigroup could *a priori* have completely unrelated Bratteli diagrams; however, it turns out that any two Bratteli diagrams must be equivalent in the usual sense of equivalence of inductive systems:

THEOREM 2.4.5. *Let S be a simple dimension semigroup, and let $(\mathbf{N}^{r_i}, \rho_i)$ and $(\mathbf{N}^{s_i}, \sigma_i)$ be Bratteli diagrams for S. Then there are integers $n_1 < k_1 < n_2 < k_2 < \cdots$ and simple homomorphisms $\alpha_i : \mathbf{N}^{r_{n_i}} \to \mathbf{N}^{s_{k_i}}$ and $\beta_i : \mathbf{N}^{s_{k_i}} \to \mathbf{N}^{r_{n_{i+1}}}$ with $\beta_i \circ \alpha_i = \rho_{n_i, n_{i+1}}$* and $\alpha_{i+1} \circ \beta_i = \sigma_{k_i, k_{i+1}}$.

PROOF. The proof is contained in the proof of [**3**, 7.3.2]. □

The dimension semigroups in which $r_i = 1$ for all i are particularly important. These are classified by the *generalized integers,* formal products $g = 2^{m_2} 3^{m_3} 5^{m_5} \cdots$, where an infinite number of factors and infinite exponents are allowed. (We will not regard ordinary integers as generalized integers.) The semigroup D_g corresponding to a generalized integer g consists of all positive rational numbers whose denominators "divide" g. A Bratteli diagram for D_g is obtained by taking any sequence (n_i) of integers ≥ 2 with $g = \prod n_i$, and letting $\sigma_i : \mathbf{N} \to \mathbf{N}$ be multiplication by n_i. Clearly every simple dimension semigroup with all $r_i = 1$ is obtained in this way. If g is the "universal" generalized integer (with $m_p = \infty$ for all primes p), then $D_g = \mathbf{Q}_+$.

2.5. *Rationalization of semigroups.* In this section, we will discuss the "rationalization" of a semigroup with respect to a simple dimension semigroup, and some applications of the results of the previous sections.

DEFINITION 2.5.1. Let S be a semigroup, D a simple dimension semigroup. The *rationalization of* S by D, denoted S_D, is the tensor product $S \otimes D$. If $D = D_g$ for a generalized integer g, then S_{D_g} is denoted S_g, called the rationalization of S by g. The rationalization of S by $\mathbf{Q}_+$ is called the *rational semigroup of* S, denoted $S_{\mathbf{Q}}$.

The definition of S_D can be rephrased without any explicit mention of tensor products. Let $(\mathbf{N}^{r_i}, \rho_i)$ be a Bratteli diagram for D. Then S_D is isomorphic to $\varinjlim (S \otimes \mathbf{N}^{r_i}, 1 \otimes \rho_i) \cong \varinjlim (S^{r_i}, \sigma_i)$, where σ_i is the obvious map from S^{r_i} to $S^{r_{i+1}}$. (If S_D is defined this way, then 2.4.5 must be used to show that the definition is independent of the Bratteli diagram chosen.)

If u is a fixed element in D, then there is a homomorphism ω_u from S into S_D which sends x to $x \otimes u$.

If $D = D_g$ and $u = 1$, then S_g is a semigroup containing (a homomorphic image of) S, in which every element of S is divisible by any integer "dividing" g. This is the reason for the term "rationalization." S_D can also be regarded as a type of "localization" of S, particularly in the case where $D = D_g$: then the Grothendieck group $G(S_g)$ is isomorphic to the localization of $G(S)$ at g in the usual sense, regarding $G(S)$ as a $\mathbf{Z}$-module and $G(S_g)$ as a $\mathbf{Z}_{(g)}$-module ($\mathbf{Z}_{(g)}$ is the Grothendieck group of D_g, or the subgroup of $\mathbf{Q}$ generated by D_g.)

If S is archimedean, then S_D is archimedean for any D. This follows immediately from 2.3.2(b), or can be easily proved directly by noting that the class of archimedean semigroups is closed under finite products and under direct limits. A similar proof shows that any rationalization of a strict semigroup is also strict.

We have the following corollary of 2.1.9 and 2.1.11, which is the main result of this section:

THEOREM 2.5.2. *Let S be an archimedean semigroup, D a simple dimension semigroup. Then S_D has cancellation and is strictly unperforated.*

PROOF. Suppose $\overline{x}, \overline{y}, \overline{z} \in S_D$ with $\overline{x} + \overline{z} = \overline{y} + \overline{z}$. Choose a Bratteli diagram $(\mathbf{N}^{r_i}, \rho_i)$ for D, and let (S^{r_i}, σ_i) be the corresponding diagram for S_D; then, for some i, there are preimages $x, y, z \in S^{r_i}$ of $\overline{x}, \overline{y}, \overline{z}$ with $x + z = y + z$. Let $x = (x_1, \dots, x_{r_i})$, and similarly for y and z. Then $x_j + z_j = y_j + z_j$ for $1 \le j \le r_i$. Since S is archimedean, there is a k_0 such that $kx_j = ky_j$ for all $k \ge k_0$ and all j, $1 \le j \le r_i$ by 2.1.9.

Let n be such that $2^{n-1} \ge k_0$, and let $\sigma_{i,i+n} = \sigma_{i+n-1} \circ \cdots \circ \sigma_i : S^{r_i} \to S^{r_{i+n}}$. If the matrix of $\sigma_{i,i+n}$ is $[m_{pq}]$, $1 \le p \le r_{i+n}, 1 \le q \le r_i$, then $\sigma_{i,i+n}(x) = (\sum_q m_{1q} x_q, \dots, \sum_q m_{r_{i+n}q} x_q)$, and similarly for $\sigma_{i,i+n}(y)$. Since all $m_{pq} \ge k_0$ by 2.4.2, we have $\sigma_{i,i+n}(x) = \sigma_{i,i+n}(y)$ and hence $\overline{x} = \overline{y}$, so S_D has cancellation.

The proof that S_D is strictly unperforated is virtually identical, using 2.1.11 in place of 2.1.9. ◻

It is not true that a rationalized semigroup is always unperforated or has power cancellation:

EXAMPLE 2.5.3. Let $S = \mathbf{N} \times \mathbf{Z}_3, g = 2^\infty$. Then $S_g \cong D_g \times \mathbf{Z}_3$ does not have 3-power cancellation and is not 3-unperforated. ($\mathbf{Z}_3 = \mathbf{Z}/3\mathbf{Z}$).

However, we do have

COROLLARY 2.5.4. *Let S be an archimedean semigroup. Then its rational semigroup $S_{\mathbf{Q}}$ has cancellation and is unperforated.*

PROOF. It is easy to check that $S \otimes \mathbf{Q}_+$ has power cancellation for any semigroup S. □

We also have an analog of 2.5.2 for ordered semigroups. If S is an ordered semigroup, then we can order S^r by $(x_1, \dots, x_r) < (y_1, \dots, y_r)$ if and only if $x_i < y_i$ for all i. Then S^r is archimedean [respectively strongly archimedean, well-behaved, strictly well-behaved] if and only if S is archimedean [respectively strongly archimedean, well-behaved, strictly well-behaved]. So, for any D, the rationalized semigroup has a natural ordering, which inherits all of the properties of the ordering on S.

THEOREM 2.5.5. *Let S be an archimedean ordered semigroup, and D a simple dimension semigroup.*

(a) *If S is well-behaved, then S_D has cancellation.*

(b) *If S is strictly well-behaved and strongly archimedean, then S_D is strictly unperforated.*

So if S is a (strict) semigroup, and S admits an ordering under which it is archimedean and well-behaved, then any rationalization of S has cancellation.

3. Applications to K-Theory. In this section, all rings will be unital unless otherwise specified.

3.1. *K-Simple Rings.* Although our original motivation for developing the semigroup theory was to apply it to simple $\mathbf{C}^*$-algebras, the results apply equally well to all simple rings and even to a somewhat larger class of rings:

Definition 3.1.1. Let A be a unital ring. An idempotent in $M_\infty(A)$ is *full* if it is not contained in any proper two-sided ideal. Let $V_f(A)$ be the subsemigroup of $V_0(A)$ consisting of equivalence classes of full idempotents. A is *K-simple* if every nonzero idempotent in $M_\infty(A)$ is full.

An idempotent is full if and only if the corresponding projective module is a generator. A is K-simple if and only if $V_f(A) = V_0(A)$. The term "K-simple" is used because such a ring might as well be simple for the purposes of nonstable K-theory (at least for the part of the theory we consider here).

Examples 3.1.2. (a) Any simple ring is K-simple.

(b) If A is a unital $\mathbf{C}^*$ algebra and $\mathrm{Prim}(A)$ contains no nontrivial compact open subsets, then A is K-simple. In particular, if $\mathrm{Prim}(A)$ is Hausdorff and connected (e.g., $A = C(X)$ for X connected), then A is K-simple [**3**, 6.3.6].

Proposition 3.1.3. [**3**, 6.3.5] *Let A be a ring. Then the semigroup $V_f(A)$ is archimedean.*

The main result of this section is the following immediate corollary of 3.1.3 and 2.1.9:

Theorem 3.1.4. *Let A be a ring, p, q, r idempotents in $M_\infty(A)$ with p, q full. If $p \oplus r \sim q \oplus r$, then $k \cdot p \sim k \cdot q$ for all sufficiently large k.*

COROLLARY 3.1.5. *Let A be a ring. If A has n-power cancellation for some n, then $V_f(A)$ has cancellation. If A is strictly n-unperforated for some n, then $V_f(A)$ has strict cancellation.*

COROLLARY 3.1.6. *If A is stably finite and K-simple, and has n-power cancellation (respectively is strictly) n-unperforated for some n, then A has cancellation (respectively strict cancellation).*

PROOF OF SECOND STATEMENT. If $p \oplus r \prec q \oplus r$, then, for some s, we have $p \oplus r \oplus s \sim q \oplus r$. Then, by the theorem, $k \cdot p \oplus k \cdot s \sim k \cdot q$ (i.e. $k \cdot p \prec k \cdot q$) for some k which is a power of n. So by strict k-unperforation $p \prec q$. (One could also simply apply 2.1.10.) □

DEFINITION 3.1.7. A stably finite ring A is *weakly n-unperforated* if the ordered group $(K_0(A), K_0(A)_+)$ has the property that $nx > 0$ implies $x > 0$ [**3**, 6.7.1].

Weak n-unperforation can be rephrased in the following awkward way: A is weakly n-unperforated if, for any idempotents $p, q, r \in M_\infty(A)$ with $n \cdot p \oplus r \prec n \cdot q \oplus r$, there is an idempotent s with $p \oplus s \prec q \oplus s$.

Weak unperforation is quite important in the study of the ordered group $K_0(A)$ via traces on A; see [**3**, §6] and [**4**].

COROLLARY 3.1.8. *Let A be a stably finite ring. Then A is strictly n-unperforated if and only if A is weakly n-unperforated and A has strict cancellation.*

PROOF. It is obvious from the above rephrasing that weak n-unperforation plus strict cancellation implies strict n-unperforation. Conversely, strict n-unperforation obviously implies weak n-unperforation, and also implies strict cancellation by 3.1.6. □

3.2. *Rationalization of rings.* In this section, we will discuss applications to the K-theory of certain tensor products of rings, which may be regarded as "rationalized" rings in the same sense as the tensor products of 2.5 are rationalized semigroups.

We will denote by $\mathbf{M}_n$ the ring $M_n(\mathbf{Z})$ of $n \times n$ matrices over $\mathbf{Z}$. (This notation is inconsistent with the notation of [**3**], where $\mathbf{M}_n$ denotes $M_n(\mathbf{C})$, but this will cause no confusion.)

We first describe the analog for rings of the construction of a simple AF algebra from a scaled dimension group. Let (D, u) be a scaled simple dimension semigroup, i.e., D is a simple dimension semigroup and u is a fixed element of D. Let $(\mathbf{N}^{r_i}, \rho_i)$ be a Bratteli diagram for D. Assume without loss of generality that there is an element $\nu = (n_{11}, \ldots, n_{r_1 1})$ in $\mathbf{N}^{r_1}$ with image u. Set $\rho_{1,k}(\nu) = (n_{1k}, \ldots, n_{r_k k})$. Let $M_k = \mathbf{M}_{n_{1k}} \oplus \cdots \oplus \mathbf{M}_{n_{r_k k}}$. ρ_k defines a unital embedding of M_k into M_{k+1} in the obvious way, using the entries of R_{ρ_k} as the multiplicities of the partial embeddings. Let $\mathbf{M}_{(D,u)}$ be the direct limit ring $\varinjlim M_k$. 2.4.5 shows that $\mathbf{M}_{(D,u)}$ depends only on (D, u) and not on the choice of the Bratteli diagram.

Definition 3.2.1. $M_{(D,u)}$ is called the *simple AF ring* of (D, u). If $(D, u) = (D_g, 1)$ for a generalized integer g, write $\mathbf{M}_g$ for $\mathbf{M}_{(D,u)}$. Write $\mathbf{M}_{\mathbf{Q}}$ for $\mathbf{M}_{(\mathbf{Q}_+,1)}$.

Proposition 3.2.2. (a) $\mathbf{M}_{(D,u)}$ *is a K-simple ring.*

(b) $(V_0(\mathbf{M}_{(D,u)}), [1])$ *is isomorphic to* (D, u), so $\mathbf{M}_{(D_1,u_1)} \cong \mathbf{M}_{(D_2,u_2)}$ *if and only if* $(D_1, u_1) \cong (D_2, u_2)$.

Proof. (a). Since $\mathbf{Z}$ is K-simple, any idempotent in a matrix algebra over M_k generates an ideal which is a direct summand. But since the matrix R_{ρ_k} has all positive entries, the image in M_{k+1} of any summand of M_k generates all of M_{k+1} as an ideal.

(b). $(V_0(M_k), [1])$ is isomorphic to $([\mathbf{N} \cup \{0\}]^{r_k} \setminus (0, \ldots, 0), (n_{1k}, \ldots, n_{r_k k}))$, and ρ_k embeds this semigroup into $(\mathbf{N}^{r_{k+1}}, (n_{1,k+1}, \ldots, n_{r_{k+1},k+1}))$. $\square$

Now we can define the rationalization of a general ring with respect to a simple dimension semigroup:

DEFINITION 3.2.3. Let A be a (unital) ring, (D,u) a scaled simple dimension semigroup. The *rationalization of* A *by* (D,u), denoted $M_{(D,u)}(A)$, is the ring tensor product $A \otimes \mathbf{M}_{(D,u)}$. Denote $A \otimes \mathbf{M}_g$ by $M_g(A)$ and $A \otimes \mathbf{M_Q}$ by $M_{\mathbf{Q}}(A)$. $M_{\mathbf{Q}}(A)$ is called the *rational ring* of A. A is *rational* if $A \cong M_{\mathbf{Q}}(A)$.

The notation is consistent with the notation $M_n(A) = A \otimes \mathbf{M}_n$ for the $n \times n$ matrix algebra over A, if we regard n as corresponding to the scaled semigroup $(\mathbf{N}, n) = (\frac{1}{n}\mathbf{N}, 1)$. $M_{(D,u)}(A)$ may be regarded as a sort of infinite matrix algebra over A.

EXAMPLES 3.2.4. (a) $M_{(D,u)}(\mathbf{Z}) = \mathbf{M}_{(D,u)}$.

(b) $M_{(D,u)}(\mathbf{C})$ is the dense locally finite *-subalgebra of the simple unital AF algebra with scaled dimension group $(G(D), D \cup \{0\}, u)$.

Just as in the case of semigroups, the rationalization of A by (D,u) can be defined without explicit mention of tensor products. We have $M_{(D,u)}(A) = \varinjlim A \otimes M_k$, and $A \otimes M_k$ is isomorphic to $M_{n_{1k}}(A) \oplus \cdots \oplus M_{n_{r_k k}}(A)$. The connecting maps are defined in the obvious way by the matrices R_{ρ_k}.

In particular, the rational ring of A can be defined to be $\varinjlim M_{n!}(A)$, where the embeddings are as "diagonal blocks" (i.e., $a \to \text{diag}(a, a, \dots, a)$).

PROPOSITION 3.2.5. (a) *If A is simple, then $M_{(D,u)}(A)$ is simple.*

(b) *If A is K-simple, then $M_{(D,u)}(A)$ is K-simple.*

(c) *If A is stably finite, then $M_{(D,u)}(A)$ is stably finite.*

PROOF. (a) and (b) are almost identical to 3.2.2(a), and (c) is obvious. □

Proposition 3.2.6. *$V_f(M_{(D,u)}(A))$ is isomorphic to the rationalization $V_f(A)_D$, and similarly for V_0.*

Proof. Similar to 3.2.2(b). □

Now the main result of this section is an immediate corollary of 2.5.2 and 2.5.4:

Theorem 3.2.7. *Let A be a ring, and (D,u) a scaled simple dimension semigroup.*

(a) *$V_f(M_{(D,u)}(A))$ has cancellation and is strictly unperforated. $V_f(M_{\mathbf{Q}}(A))$ has cancellation and is unperforated.*

(b) *If A is stably finite and K-simple, then $M_{(D,u)}(A)$ has cancellation and is strictly unperforated; $M_{\mathbf{Q}}(A)$ has cancellation and is unperforated.*

We can extend these results somewhat. The following is not the most general statement possible, but is the best that can be easily stated:

Theorem 3.2.8. *Let A be a (unital) ring which is K-simple and stably finite. Suppose there is an n such that, for any idempotent p in $M_\infty(A)$, there is an idempotent q, unique up to equivalence, for which $n \cdot q \sim p$. Then A has cancellation and is strictly unperforated.*

Proof. Follows immediately from 2.1.12. □

4. Applications to C^*-algebras. In this section, we will give some consequences of the preceding results which are specific to the K-theory of $\mathbf{C}^*$-algebras. We will continue to assume that all $\mathbf{C}^*$-algebras are *unital.*

4.1. *Rationalized $\mathbf{C}^*$-algebras.* Suppose A is a $\mathbf{C}^*$-algebra and (D,u) is a scaled simple dimension semigroup. Then $M_{(D,u)}(A)$ is a local $\mathbf{C}^*$-

algebra [**3**, §3] which is not complete. Its completion is the $\mathbf{C}^*$-tensor product $A \otimes M$, where M is the simple unital AF algebra with scaled dimension semigroup (D, u).

DEFINITION 4.1.1. $A \otimes M$ is called the *rationalized* $\mathbf{C}^*$*-algebra of* A *by* (D, u). If $(D, u) = (\mathbf{Q}_+, 1)$, $A \otimes M$ is called the *rational* $\mathbf{C}^*$*-algebra of* A, denoted $A_{\mathbf{Q}}$. A $\mathbf{C}^*$-algebra A is *rational* if $A \cong A_{\mathbf{Q}}$. More generally, a $\mathbf{C}^*$-algebra is a *rationalized* $\mathbf{C}^*$*-algebra* if it is isomorphic to the rationalization of some $\mathbf{C}^*$-algebra by a simple dimension semigroup.

Since the embedding of a local $\mathbf{C}^*$-algebra A into its completion $\overline{A}$ induces an isomorphism $V(A) \cong V(\overline{A})$, we have

THEOREM 4.1.2. *Let* A *be a* $\mathbf{C}^*$*-algebra,* M *an infinite-dimensional simple AF algebra. Then* $V_f(A \otimes M)$ *has cancellation and is strictly unperforated. If* A *is stably finite and* K*-simple, then* $A \otimes M$ *has cancellation and is strictly unperforated,* i.e., *a stably finite rationalized* K*-simple* $\mathbf{C}^*$*-algebra has cancellation and is strictly unperforated. A stably finite rational* K*-simple* $\mathbf{C}^*$*-algebra has cancellation and is unperforated.*

Similar results were previously obtained in [**2**] under a stable rank hypothesis. 4.1.2 is a great improvement for two reasons: first, the stable rank hypotheses may very well not be satisfied in general, or even in cases of interest, and secondly, techniques for calculating or estimating stable rank are presently so rudimentary that it is usually impossible to check the hypotheses of [**2**] except in very special cases.

If A is a rationalized $\mathbf{C}^*$-algebra, say $A = M_{(D,u)}(B)$, then $A = \varinjlim A_k$, where A_k is direct sum of matrix algebras over B. The sizes of the matrix algebras become uniformly large as $k \to \infty$ by 2.4.2. So the following other nonstable K-theory results are an immediate consequence:

PROPOSITION 4.1.3. *If* A *is a finite rationalized* $\mathbf{C}^*$*-algebra, then* A *is stably finite.*

PROOF. A finite $\Rightarrow$ each A_k finite $\Rightarrow$ B stably finite $\Rightarrow$ A stably finite. □

PROPOSITION 4.1.4. *If A is a rationalized $\mathbf{C}^*$-algebra, then $\mu_1 : U_1(A)/U_1(A)_0 \to K_1(A)$ is an isomorphism.*

PROOF. Let $A_k = M_{n_{1k}}(B) \oplus \cdots \oplus M_{n_{r_k k}}(B)$. Then $U_1(A_k)/U_1(A_k)_0 \cong [U_{n_{1k}}(B)/U_{n_{1k}}(B)_0] \oplus \cdots \oplus [U_{n_{r_k k}}(B)/U_{n_{r_k k}}(B)_0]$, and the map $\mu_1 : U_1(A_k)/U_1(A_k)_0 \to K_1(A_k) \cong K_1(B)^{r_k}$ is of the form $(\mu_{n_{1k}}, \dots, \mu_{n_{r_k k}})$. Thus the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \dfrac{U_1(A_k)}{U_1(A_k)_0} & \longrightarrow & \dfrac{U_1(A_{k+1})}{U_1(A_{k+1})_0} & \longrightarrow & \cdots & \longrightarrow & \dfrac{U_1(A)}{U_1(A)_0} \\
 & & \Big\downarrow \mu_1 & & \Big\downarrow \mu_1 & & & & \Big\downarrow \mu_1 \\
\cdots & \longrightarrow & K_1(A_k) & \longrightarrow & K_1(A_{k+1}) & \longrightarrow & \cdots & \longrightarrow & K_1(A)
\end{array}$$

is actually of the form

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \dfrac{U_{n_{1k}}(B)}{U_{n_{1k}}(B)_0} \oplus \cdots \oplus \dfrac{U_{n_{r_k k}}(B)}{U_{n_{r_k k}}(B)_0} & \longrightarrow & \cdots & \longrightarrow & \dfrac{U_1(A)}{U_1(A)_0} \\
 & & \Big\downarrow (\mu_{n_{1k}}, \cdots, \mu_{n_{r_k k}}) & & & & \Big\downarrow \mu_1 \\
\cdots & \longrightarrow & K_1(B) \oplus \cdots \oplus K_1(B) & \longrightarrow & \cdots & \longrightarrow & K_1(A)
\end{array}$$

Since $n_{1k}, \dots, n_{r_k k} \to \infty$ as $k \to \infty$, it is easy to see that the direct limits in the rows are isomorphic. □

PROPOSITION 4.1.5. *If A is a rational $\mathbf{C}^*$-algebra, then the stable rank of A is* 1,2, *or* ∞.

PROOF. $A \cong M_n(A)$ for all n, so the result follows from [**14**, 6.1]. □

Of course, most of the results of §3 such as 3.1.4, 3.1.5, 3.1.6, 3.1.8, and 3.2.8 also apply verbatim to $\mathbf{C}^*$-algebras.

4.2. *K-Theory with Localized Coefficients.* If B is any $\mathbf{C}^*$-algebra, then the map $A \to A \otimes B$ is a functor from the category of $\mathbf{C}^*$-algebras to itself (if B is not nuclear, then the maximal or minimal cross norm must be specified.) One can then define a B-isomorphism as $A_1 \sim_B A_2$ if $A_1 \otimes B \cong A_2 \otimes B$, and similarly for stable B isomorphisms. In particular, the map $A \to A_{\mathbf{Q}}$ is a functor from the category of $\mathbf{C}^*$-algebras to the category of rational $\mathbf{C}^*$-algebras, which can be used to define rational isomorphisms and stable rational isomorphisms for $\mathbf{C}^*$-algebras.

The notion of rationalization can be used to define K-theory and KK-theory with coefficients (cf., [**3**, 23.15.6]). Let g be a generalized integer; suppose for simplicity that $\mathbf{Z}_{(g)}$ is a subring of $\mathbf{Q}$ (this is equivalent to requiring that the exponent m_p is 0 or ∞ for all p.) Let M be the UHF algebra corresponding to g; then $M \cong M \otimes M$.

Definition 4.2.1. If A is a $\mathbf{C}^*$-algebra, then the *K-theory of A with coefficients in* $\mathbf{Z}_{(g)}$, denoted $K_*(A; \mathbf{Z}_{(g)})$, is $K_*(A \otimes M)$. If A and B are $\mathbf{C}^*$-algebras, define $KK(A, B; \mathbf{Z}_{(g)})$ to be $KK(A \otimes M, B \otimes M) \cong KK(A, B \otimes M)$. $KK(A, B; \mathbf{Q}) = KK(A_{\mathbf{Q}}, B_{\mathbf{Q}})$.

$KK(A, B; \mathbf{Z}_{(g)})$ is a $\mathbf{Z}_{(g)}$-module. $K_*(A; \mathbf{Z}_{(g)}) \cong K_*(A) \otimes_{\mathbf{Z}} \mathbf{Z}_{(g)}$, but $KK(A, B; \mathbf{Z}_{(g)}) \ncong KK(A, B) \otimes_{\mathbf{Z}} \mathbf{Z}_{(g)}$ in general (take $A = M, B = \mathbf{C}$.) The usual properties of KK-theory (homotopy invariance, stability, Bott periodicity, exact sequences for semisplit extensions) carry over to these KK-theories with coefficients. In addition, there are versions of the Universal Coefficient Theorem and the Künneth Theorem for Tensor Products [**3**, §23]. All of these results follow easily from the corresponding results in ordinary KK-theory (they may be regarded as special cases.)

In the case of KK-theory with rational coefficients, the theory can be developed from scratch with some technical simplifications. In particular, the rational versions of the UCT and KTP are very easy to state and prove (cf. [**3**, 23.2–23.4]):

Theorem 4.2.2. (Universal Coefficient Theorem.) *Let A and B be separable* $\mathbf{C}^*$*-algebras, with* $A \in N$ [**3**, 22.3.4]. *Then*

$$KK(A,B;\mathbf{Q}) \cong \mathrm{Hom}(K_0(A;\mathbf{Q}), K_0(B;\mathbf{Q})) \oplus \mathrm{Hom}(K_1(A;\mathbf{Q}), K_1(B;\mathbf{Q})).$$

Theorem 4.2.3. (Künneth Theorem for Tensor Products.) *Let A and B be separable* $\mathbf{C}^*$*-algebras, with* $A \in N$*. Then*

$$K_0(A \otimes B;\mathbf{Q}) \cong [K_0(A;\mathbf{Q}) \otimes_{\mathbf{Q}} K_0(B;\mathbf{Q})] \oplus [K_1(A;\mathbf{Q}) \otimes_{\mathbf{Q}} K_1(B;\mathbf{Q})].$$

It is less clear that there is a rational version of the Künneth Theorem, since the ordinary Künneth Theorem [**3**, 23.1.2] has a finite generation hypothesis.

KK-Theory with rational coefficients, although not as rich a theory as ordinary KK-theory, is of interest in certain contexts. For example, it seems to be natural to consider rational KK-theory in connection with the Chern Character.

4.3 *Rational* $\mathbf{C}^*$*-algebras with many projections.* Sharper results than the ones in §4.1 can be obtained in the case of $\mathbf{C}^*$-algebras with "many" projections. Recall that A has the (SP) property if every nonzero hereditary $\mathbf{C}^*$-subalgebra of A contains a nonzero projection, and A has (HP) if every hereditary $\mathbf{C}^*$-subalgebra of A has an approximate identity of projections. A has stable (HP) if all matrix algebras over A have (HP).

Proposition 4.3.1. [**6**, 1.9] *Every rationalized* unital $\mathbf{C}^*$*-algebra has* (SP).

There are, however, rational $\mathbf{C}^*$-algebras which do not have (HP) [**6**, 1.6]. In the presence of (HP), the following result is a corollary of 4.1.2 and [**3**, 6.5.2] (cf. [**4**, 4.3.7]):

Theorem 4.3.2. *Let A be rationalized* unital *K-simple* $\mathbf{C}^*$*-algebra with* (HP). *Then the following are equivalent:*

(1) *A is finite.*

(2) *A has cancellation.*

(3) *A has stable rank* 1, i.e., *the invertible elements in A are dense.*

DEFINITION 4.3.3. A stably finite unital $\mathbf{C}^*$-algebra A has *stable strict comparability* if, for all projections $p, q \in M_\infty(A)$ with $\tau(p) < \tau(q)$ for all normalized quasitraces τ on A, it follows that $p \prec q$.

The *Fundamental Comparability Question is* whether every stably finite simple unital $\mathbf{C}^*$-algebra has stable strict comparability. See [**4**] for a complete discussion of strict comparability and the Fundamental Comparability Question.

THEOREM 4.3.4. *Let A be a finite rationalized K-simple unital* $\mathbf{C}^*$-*algebra with* (HP). *Then A has stable strict comparability.*

PROOF. By [**4**, 3.4.9], A has stable strict comparability if and only if A has strict cancellation and enough quasitraces. A has strict cancellation by 4.1.2 and enough quasitraces by [**4**, 4.3.8] (cf. [**5**, III.1.3].) □

REFERENCES

1. N.G. Alimov, *On ordered semigroup*, Izv. Akad. Nauk SSSR Ser. Mat. **14** (1950), 569–576.

2. B. Blackadar, *A stable cancellation theorem for simple* $\mathbf{C}^*$*-algebras*, Proc. London Math. Soc. **47** (1983), 303–305.

3. ———, *K-Theory for Operator Algebras.* Mathematical Sciences Research Institute Publication vol. 5, Springer-Verlag, New York/Heidel-berg/Berlin/Tokyo, 1986.

4. ———, *Comparison theory for simple* $\mathbf{C}^*$*-algebras*, in *Operator Algebras and Applications* (ed. D. Evans and M. Takesaki), London Math. Soc. Lecture Note Ser. **135** (1988), 21–54.

5. B. Blackadar and D. Handelman, *Dimension functions and traces on* $\mathbf{C}^*$*-algebras*, J. Funct. Anal. **45** (1982), 297–340.

6. B. Blackadar and A. Kumjian, *Skew products of relations and the structure of simple* $\mathbf{C}^*$*-algebras*, Math. Z. **189** (1985), 55–63.

7. J. Cuntz, *K-Theory for certain* $\mathbf{C}^*$*-algebras*, Ann. of Math. **113** (1981), 181–197.

8. J. Cuntz and G. Pedersen, *Equivalence and traces on* $\mathbf{C}^*$*-algebras*, J. Funct. Anal. **33** (1979), 135–164.

9. E. Effros, D. Handelman, and C.-L. Shen, *Dimension groups and their affine representations*, Amer. J. Math. **102** (1980), 385–407.

10. P. Grillet, *The tensor product of commutative semigroups*, Trans. Amer. Math. Soc. **138** (1969), 281–293.

11. ———, *The tensor product of cancellative semigroups*, J. Natur. Sci. Math. **10** (1970), 199–208.

12. N. Kimura and Y.-S. Tsai, *On power cancellative archimedean semigroups.* Proc. Japan Acad. Ser. A Math. Sci. **48** (1972), 553–554.

13. O. Nakada, *Partially ordered abelian semigroups* I. *On the extension of the strong partial order defined on abelian semigroups*, J. Fac. Sci. Hokkaido Univ. **11** (1951), 181–189.

14. M. Rieffel, *Dimension and stable rank in the* K*-theory of* $\mathbf{C}^*$*-algebras*, Proc. London Math. Soc. (3) **46** (1983), 301–333.

15. T. Tamura, *Construction of trees and commutative archimedean semigroups*, Math. Nachr. **36** (1968), 255–287.

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