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INNER MULTIPLIERS OF THE BESOV SPACE, $0 < p \leq 1$

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0. For $\alpha > 0$ let k be the integer so that $k-1 \leq \alpha < k$. Then, for $p > 0$, the Besov space B_α^p is the set of functions f, holomorphic in the unit disc U such that

$$||f||_{p,\alpha}^p = \int |f^{(k)}(z)|^p (1-|z|)^{p(k-\alpha)-1} dm(z) < \infty.$$

Here dm denotes area measure in U. We will assume from now on that $1 - p\alpha > 0$. (When $1 - p\alpha < 0$ the functions in B_α^p are continuous out to the boundary of U.) In [**9**], I. Verbitsky characterized those inner functions $B \in MB_\alpha^p$, i.e., for which $Bf \in B_\alpha^p$ for all $f \in B_\alpha^p, p \geq 1$. See [**5**, Chapter 17], for a discussion of inner functions. In this paper we consider the case $0 < p \leq 1$.

The first step is to show that any such inner function is a Blaschke product whose zero set is a finite union of interpolating sequences. The proof of this for $p \leq 1$ is similar to Verbitsky's proof for $p \geq 1$. Indeed, after some preliminaries we appeal directly to his argument. So the question becomes: Which such Blaschke products are in MB_α^p?

For $p > 1$, the Carleson measures for B_α^p were determined by D. Stegenga [**6**]. Using this result one immediately gets a necessary and sufficient condition on B in order that $B \in MB_\alpha^p$. However, this condition does not involve the distribution of zeros of B in any direct way. The whole point of Verbitsky's paper is to find a necessary and sufficient condition on the zeros of B in order that $B \in MB_\alpha^p$. We take the same point of view.

In the first section we find the Carleson measures for $B_\alpha^p, 0 < p \leq 1$. For the case $p > 1$, Stegenga used the ideas involved in E. Stein's proof [**7**] of the original Carleson measure theorem together with the strong capacitary estimates of D. Adams [**1**]. Our proof is the same except we must use the recently proved "strong Hausdorff capacity" estimates

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of Adams [**2**]. Then we find that (at least in the case $0 < \alpha < 1$), $B \in MB_\alpha^p$ if and only if

$$\int_{S(I)} |B'|^p (1-|z|)^{p(1-\alpha)-1} dm(z) \le C|I|^{1-\alpha p}, \tag{0.1}$$

for all arcs I. Here $S(I) = \{re^{i\theta} : e^{i\theta} \in I, 1-|I| \le r < 1\}$, $|I|$ = length of I. Our main result is that if $p > 1/(1+\alpha)$, then the above condition is equivalent to

$$\sum_{a_n \in S(I)} (1-|a_k|)^{1-\alpha p} \le C|I|^{1-\alpha p}, \tag{0.2}$$

for all arcs I. Here $\{a_k\}$ are the zeros of B. Indeed we show that, for any $\alpha > 0$ and $p \le 1$ such that $1/(1+\alpha) < p < 1/\alpha$, the condition 0.2 is equivalent to $B \in MB_\alpha^p$. We also show that there is no theorem for $p \le 1/(1+\alpha)$, i.e., in this case 0.2 need not imply that $B \in MB_\alpha^p$.

We end the introduction by introducing some more notation. For $f(z) = \sum_{k=0}^\infty a_k z^k$ and $\alpha > 0$, let

$$D^\alpha f(z) = \sum_{k=0}^\infty (k+1)^\alpha a_k z^k.$$

We will often use

$$\int |D^\alpha f(z)|^p (1-|z|)^\gamma dm \doteq \int |D^\beta f(z)|^p (1-|z|)^{p(\beta-\alpha)+\gamma} dm \tag{0.3}$$

as long as γ and $p(\beta-\alpha)+\gamma$ are greater than -1. Here $A \doteq B$ means that $A/c \le B \le CA$ for some universal constant C.

It follows from 0.3 that $f \in B_\alpha^p$ if and only if

$$\int |D^{1+\alpha} f(z)|^p (1-|z|)^{p-1} dm < \infty.$$

And from this it follows, by the Littlewood-Paley inequality that

$$B_\alpha^p \subseteq \{f : D^\alpha f \in H^p\}, \qquad 0 < p \le 2.$$

Finally we will use the estimate $|D^\alpha f(z)| \leq C(1-|z|)^{-\alpha}||f||_\infty$. For proofs of this and 0.3 see the paper of T. Flett [**4**].

Our main result is that if $p \leq 1, \alpha > 0$ and $1/(1+\alpha) < p < 1/\alpha$, then $B \in MB_\alpha^p$ if and only if 0.2 holds. We will give detailed proof only in case $0 < \alpha < 2$. The general case is technically rather complicated but requires no new ideas. Our proof in the case $1 \leq \alpha < 2$ will be rather sketchy.

1. To determine the Carleson measure for B_α^p we need the strong Hausdorff capacity estimates of D. Adams [**2**].

DEFINITION. For $0 < m < 1$ define, for $E \subseteq T, H^m(E) = \inf\{\sum |I_n|^m : E \subseteq \cup I_n, I_n \text{ open arc}\}$.

THEOREM A. *Suppose* $0 < m = 1 - \alpha p < 1, 0 < p \leq 1$. *Then there is a constant* C *such that*

$$\int_0^\infty H^m(\{Nf > t\}) t^{p-1} dt \leq C||D^\alpha f||_{H^p}.$$

Here Nf denotes the usual non-tangential maximal function of f.

LEMMA. *Suppose* $0 < m < 1$, $\{I_n\}$ *is a sequence of open intervals and* $\cup I_n = \cup J_k$, *where* $\{J_k\}$ *are disjoint open intervals. Then*

$$\sum |J_k|^m \leq \sum |I_n|^m.$$

PROOF. Since $\{J_k\}$ are pairwise disjoint $J_k = \cup\{I_n : I_n \subseteq J_k\}$ and hence $|J_k| \leq \sum_{I_n \subseteq J_k} |I_n|$. Since $0 < m < 1$ we have

$$|J_k|^m \leq \sum_{I_n \subseteq J_k} |I_n|^m,$$

so

$$\sum_k |J_k|^m \leq \sum_k \sum_{I_n \subseteq J_k} |I_n|^m = \sum_n |I_n|^m. \ \square$$

DEFINITION. A positive Borel measure on U is called a Carleson measure for B^p_α if there is a constant C so that

$$\int |f|^p d\mu \le C||f||_{p,\alpha}, \text{ all } f \in B^p_\alpha.$$

THEOREM 1. *If* $0 < p \le 1$, *then* μ *is a Carleson measure for* B^p_α *if and only if there is a constant* C *so that*

$$(*) \qquad \mu(S(I)) \le C|I|^{1-\alpha p}.$$

PROOF. To prove sufficiency of $(*)$, take $f \in B^p_\alpha$, then

$$\int |f|^p d\mu = \int_0^\infty \mu(\{|f| > t\}) t^{p-1} dt.$$

Fix $0 < t < \infty$ and suppose $\{I_n\}$ is a sequence of open intervals such that $\{Nf > t\} \subseteq \cup I_n$ and $\cup I_n = \cup J_k$, where $\{J_k\}$ are pairwise disjoint open intervals. Now if $|f(z)| > t$, then $Nf > t$ on some interval of length greater than $1 - |z|$ and hence $z \in S(J_k)$ for some k. That is $\{|f(z)| > t\} \subseteq \cup S(J_k)$, so

$$\begin{aligned}\mu(\{|f| > t\}) &\le \sum \mu(S(J_k)) \\ &\le C \sum |J_k|^{1-\alpha p} \le C \sum |I_n|^{1-\alpha p},\end{aligned}$$

by the lemma. It follows from the definition of $H^{1-\alpha p}$ that $\mu(\{|f| > t\}) \le CH^{1-\alpha p}(\{Nf > t\})$. From this it follows that

$$\int |f|^p d\mu \le C||D^\alpha f||_{H^p} \le C||f||^p_{p,\alpha}.$$

The necessity of condition $(*)$ follows in a standard way by testing μ against functions of the form $f(z) = (1 - \overline{w}z)^{-\beta}$. We omit the details. □

2. Our first step is to show that if B is an inner function that multiplies B^p_α, then B is a Blaschke product whose zero set is a finite union of interpolating sequences.

LEMMA 2.1. *If $0 < p \le 1$ and $B \in MB^p_\alpha$ is inner, then*

$$\int_{S(I)} (1-|B(z)|)(1-|z|)^{-1-\alpha p} dm(z) \le C|I|^{1-\alpha p}.$$

PROOF. We assume that I has its center at $\zeta = 1$. Let $f(z) = (1-rz)^{-1}$, where $r = 1-\delta$, $\delta = |I|$. Now $|f(z)| \doteq \delta^{-1}$ in $S(I)$, so we have

$$\begin{aligned}\int_{S(I)} &(1-|B(z)|)(1-|z|)^{-1-\alpha p} dm(z)\\ &\le C\delta \int_{S(I)} |f(z)|(1-|B(z)|)(1-|z|)^{-1-\alpha p} dm(z)\\ &\le C\delta \int_U |f(z)|(1-|B(z)|)(1-|z|)^{-1-\alpha p} dm(z).\end{aligned}$$

Since B is inner,

$$1-|B(re^{i\theta})| \le \int_r^1 |B'(\rho e^{i\theta})| d\rho \quad \text{a.e. } d\theta.$$

Hence we have

$$\begin{aligned}\int_{S(I)} &(1-|B(z)|)(1-|z|)^{-1-\alpha p} dm(z)\\ &\le C\delta \int_0^{2\pi}\int_0^1 |f(re^{i\theta})|(1-r)^{-1-\alpha p} \int_r^1 |B'(\rho e^{i\theta}) d\rho dr d\theta\\ &\le C\delta \int_0^{2\pi}\int_0^1 |B'(\rho e^{i\theta})| \int_0^\rho |f(re^{i\theta})|(1-r)^{-1-\alpha p} dr d\rho d\theta\\ &\le C\delta \int_0^{2\pi}\int_0^1 |B'(\rho e^{i\theta})|\, |f(\rho e^{i\theta})| \int_0^\rho (1-r)^{-1-\alpha p} dr d\rho d\theta\\ &\le C\delta \int_0^{2\pi}\int_0^1 |B'(\rho e^{i\theta})|\, |f(\rho e^{i\theta})|(1-\rho)^{-\alpha p} d\rho d\theta.\\ &\le C\delta \bigg[\int |(Bf)'(z)|(1-|z|)^{-\alpha p} dm\\ &\qquad + \int |B(z)|\, |f'(z)|(1-|z|)^{-\alpha p} dm(z) \bigg]\\ &\le C\delta \int |(Bf)'(z)|(1-|z|)^{-\alpha p} dm + C\delta \int |f'(z)|(1-|z|)^{-\alpha p} dm.\end{aligned}$$

An elementary calculation shows that $\delta\int|f'(z)|(1-|z|)^{-\alpha p}dm \leq C\delta^{1-\alpha p}$, so we turn our attention to

$$\begin{aligned}\delta\int&|(Bf)'(z)|(1-|z|)^{-\alpha p}dm\\ &\leq C\delta\int|D^{1+\alpha}(Bf)|(1-|z|)^{\alpha-\alpha p}dm\\ &= C\delta\int|D^{1+\alpha}(Bf)|^p\,|D^{1+\alpha}(Bf)|^{1-p}(1-|z|)^{\alpha-\alpha p}dm\\ &\leq c\delta||Bf||_\infty^{1-p}\int|D^{1+\alpha}(Bf)|^p(1-|z|)^{(1+\alpha)(p-1)+\alpha-\alpha p}dm\\ &\leq C\delta||f||_\infty^{1-p}\int|D^{1+\alpha}(Bf)|^p(1-|z|)^{p-1}dm(z)\\ &= C\delta||f||_\infty^{1-p}||Bf||_{p,\alpha}^p \leq C\delta||f||_\infty^{1-p}||f||_{p,\alpha}^p,\end{aligned}$$

because $B \in MB_\alpha^p$. Now $||f||_\infty \doteq \delta^{-1}$, and we may calculate that $||f||_{p,\alpha}^p \doteq \delta^{1-\alpha p-p}$. This completes the proof of the lemma. □

We now give our main result.

THEOREM 2.1. *Suppose* $0 < p \leq 1, 1/(1+\alpha) < p < 1/\alpha$, *and* B *is an inner function. Then* $B \in MB_\alpha^p$ *if and only if* B *is a Blaschke product whose zeros* $\{a_k\}$ *satisfy*

$$(*)\qquad \sum_{a_k\in S(I)}(1-|a_k|)^{1-\alpha p} \leq C|I|^{1-\alpha p}, \quad \text{all } I.$$

PROOF. Suppose $B \in MB_\alpha^p$. Then, by Lemma 2.1, we see that

$$\int_{S(I)}(1-|B(z)|)(1-|z|)^{-1-\alpha p}dm(z) \leq C|I|^{1-\alpha p}, \quad \text{all } I.$$

In [**9**] Verbitsky shows that B is a Blaschke product whose zero set $\{a_k\}$ is a finite union of interpolating sequences. This in turn implies that

$$\frac{1-|B(z)|}{1-|z|} \geq C\sum\frac{1-|a_k|}{|1-\overline{a}_n z|^2},$$

see [**8**]. If we use this and Lemma 2.1 again we see that

$$|I|^{1-\alpha p} \geq C \int_{S(I)} \sum \frac{1-|a_k|}{|1-\overline{a}_k z|^2}(1-|z|)^{-\alpha p} dm$$
$$\geq C \sum_{a_k \in S(I)} (1-|a_k|) \int_{S(I)} \frac{(1-|z|)^{-\alpha p}}{|1-\overline{a}_k z|^2} dm.$$

We need to show that if $a_k \in S(I)$ then

$$\int_{S(I)} \frac{(1-z)^{-\alpha p}}{|1-\overline{a}_k z|^2} dm(z) \geq C(1-|a_k|)^{-\alpha p}.$$

Fix such an a_k; then there is an arc $J \subseteq I$ such that $a_k \in S(J)$ and $|J|/2 \leq 1-|a_k| \leq |J|$. It follows that, for $z \in S(J)$, $|1-\overline{a}_k z| \doteq |J| \doteq (1-|a_k|)$. So,

$$\int_{S(I)} \frac{(1-|z|)^{-\alpha p}}{|1-\overline{a}_k z|^2} dm \geq \int_{S(J)} \frac{(1-|z|)^{-\alpha p}}{|1-\overline{a}_k z|^2} dm$$
$$\geq C(1-|a_k|)^{-2} \int_{S(J)} (1-|z|)^{-\alpha p} dm$$
$$= C(1-|a_k|)^{-\alpha p}.$$

We turn to the proof of the sufficiency. We will let $d_k = 1-|a_k|$. Note, if $(*)$ holds, that

$$\sum_{a_k \in S(I)} d_k = \sum_{a_k \in S(I)} d_k^{\alpha p} d_k^{1-\alpha p} \leq C|I|^{\alpha p}|I|^{1-\alpha p} = C|I|,$$

and hence $(*)$ implies that

$$\sum \frac{d_k}{|1-\overline{a}_k z|^2} \doteq \frac{1-|B(z)|}{1-|z|},$$

as we have seen. We will use this fact later. First we assume that $0 < \alpha < 1$. Since $(Bf)' = fB' + f'B$ it follows that $B \in MB_\alpha^p$ if and only if $|B'|^p(1-|z|)^{p(1-\alpha)-1}dm(z)$ is a Carleson measure for B_α^p. By Theorem 1.1 this is equivalent to

$$(**) \qquad \int_{S(I)} |B'|^p(1-|z|)^{p(1-\alpha)-1} dm \leq C|I|^{1-\alpha p}.$$

We need to show that $(*)$ implies $(**)$. Suppose that I is centered at ζ and let $\delta = |I|$. Then $S(I) \subseteq \{z : |z-\zeta| < 2\delta\}$. Now

$$\begin{aligned}|B'(z)| &\le \sum \frac{d_k}{|1-\overline{a}_k z|^2} = \sum_{|\zeta - a_k|\le 3\delta} \frac{d_k}{|1-\overline{a}_k z|^2} + \sum_{j=0}^{\infty} \sum_{a_k \in A_j} \frac{d_k}{|1-\overline{a}_k z|^2} \\ &= I + II,\end{aligned}$$

where $A_j = \{z : 2^j 3\delta < |\zeta - z| \le 2^{j+1}\cdot 3\delta\}$. Notice that, if $z \in S(I)$ and $a_k \in A_j$, we have

$$\begin{aligned}|1-\overline{a}_k z| &= |\overline{\zeta} - \overline{a}_k \overline{\zeta} z| = |\overline{\zeta} - \overline{a}_k + \overline{a}_k \overline{\zeta}(\zeta - z)| \\ &\ge |\zeta - a_k| - |\zeta - z| \ge 2^j \cdot 3\delta - 2\delta \ge 2^j \delta.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{a_k\in A_j} \frac{d_k}{|1-\overline{a}_k z|^2} &\le \frac{1}{2^{2j}\delta^2} \sum_{a_k \in A_j} d_k \\ &= 2^{-2j}\delta^{-2} \sum_{a_k\in A_j} d_j^{1-\alpha p} d_k^{\alpha p} \\ &\le 2^{-2j}\delta^{-2}(2^{j+1}\cdot 3\delta)^{\alpha p} \sum_{a_k \in A_j} d_k^{1-\alpha p}.\end{aligned}$$

Now $A_j \subseteq S(I_j)$, where $|I_j| = 2^{j+3}\cdot 3\delta$, and so

$$\begin{aligned}\sum_{a_k\in A_j} d_k^{1-\alpha p} &\le \sum_{a_k \in S(I_j)} d_k^{1-\alpha p} \le C|I_j|^{1-\alpha p} \\ &\le C(2^j\delta)^{1-\alpha p}.\end{aligned}$$

As a consequence

$$\sum_{a_k\in A_j} \frac{d_k}{|1-\overline{a}_k z|^2} \le C 2^{-j}\delta^{-1},$$

and hence

$$II \le C\sum_{j=0}^{\infty} 2^{-j}\delta^{-1} \le C\delta^{-1}, \qquad \text{for all } z \in S(I).$$

We now have

$$\begin{aligned}\int_{S(I)} &|B'|^p(1-|z|)^{p(1-\alpha)-1}dm(z)\\ &\le \int_{S(I)}(I^p + C\delta^{-p})(1-|z|)^{p(1-\alpha)-1}dm(z)\\ &\le C\sum_{|\zeta-a_k|\le 3\delta} d_k^p \int_{S(I)} \frac{1}{|1-\overline{a}_k z|^{2p}}(1-|z|)^{p(1-\alpha)-1}dm(z)\\ &\quad + C\delta^{-p}\int_{S(I)}(1-r)^{p(1-\alpha)-1}dm(z)\\ &\le C\sum_{|\zeta-a_k|\le 3\delta} d_k^p \int_U \frac{(1-|z|)^{p(1-\alpha)-1}}{|1-\overline{a}_k z|^{2p}}dm(z) + C\delta^{1-\alpha p}\\ &\le C\delta^{1-\alpha p}.\end{aligned}$$

3. We suppose that $1<\alpha<2,\ p\le 1, p\alpha<1$. Now we want to find a condition on a measure μ so that

$$\int^{|f'|^p} d\mu \le C||f||_{p,\alpha}, \quad \text{all } f\in B^p_\alpha. \tag{3.1}$$

Since $\alpha>1, f\in B^p_\alpha$ if and only if $f'\in B^p_{\alpha-1}$, and hence 3.1 holds if and only if

$$\mu(S(I))\le C|I|^{1-p(\alpha-1)} = C|I|^{1-p\alpha+p}.$$

From this it follows from Leibnitz's rule and Theorem 1.1 that $B\in MB^p_\alpha$ if and only if

$$\int_{S(I)}|B''|^p(1-|z|)^{p(2-\alpha)-1}\le C|I|^{1-p\alpha}, \tag{3.2}$$

and

$$\int_{S(I)}|B'|^p(1-|z|)^{p(2-\alpha)-1}\le C|I|^{1-p\alpha+p}, \tag{3.3}$$

for all arcs I.

We will show that condition $(*)$ of Theorem 2.1 implies 3.2 and 3.3.

First we discuss 3.2. An easy calculation shows that

$$|B''(z)| \le C\left[\left(\sum \frac{d_k}{|1-\overline{a}_k z|^2}\right)^2 + \sum \frac{d_k}{|1-\overline{a}_k z|^3}\right].$$

We divide up each sum dyadically, just as in the case $0 < \alpha < 1$. The terms corresponding to the dyadic annuli A_j are handled exactly as in the case $0 < \alpha < 1$. The remaining terms must be handled slightly differently. They are

$$\left(\sum_{|\zeta - a_k| \le 3\delta} \frac{d_k}{|1-\overline{a}_k z|^2}\right)^2 + \sum_{|\zeta - a_k| \le 3\delta} \frac{d_k}{|1-\overline{a}_k z|^3}.$$

Hence we must estimate

$$\int_{S(I)} \left(\sum_{|\zeta - a_k| \le 3\delta} \frac{d_k}{|1-\overline{a}_k z|^3}\right)^{2p} (1-r)^{p(2-\alpha)-1} dm \tag{3.4}$$

and

$$\int_{S(I)} \left(\sum_{|\zeta - a_k| \le 3\delta} \frac{d_k}{|1-\overline{a}_k z|^3}\right)^{p} (1-r)^{p(2-\alpha)-1} dm. \tag{3.5}$$

The estimations of 3.5 offer no difficulty, just replace the p^{th} power of the sum by the sum of the p^{th} powers and integrate over all of U, not just $S(I)$ to get the right result. If $2p < 1$, then 3.4 can be handled the same way. Suppose $2p > 1$. As we have noted, in our situation (the sum extended over $|\zeta - a_k| \le 3\delta$),

$$\sum \frac{d_k}{|1-\overline{a}_k z|^2} \le C \frac{1-|B(z)|}{1-|z|} \le \frac{C}{1-|z|};$$

hence, 3.4 is at most a constant times

$$\int_U \left(\sum \frac{d_k}{|1-\overline{a}_k z|^2}\right)(1-r)^{1-2p+p(2-\alpha)-1} dm$$
$$\le \sum d_k \int_U \frac{(1-r)^{-p\alpha}}{|1-\overline{a}_k z|^2} dm \le C \sum d_k^{1-\alpha p}.$$

Recalling that the sum is extended only over those a_k such that $|\zeta - a_k| \le 2\delta$, we have our result.

This leaves the case $p = 1/2$ which is similarly treated.

We turn to 3.3. We estimate $|B'(z)| \le \sum d_k/|1 - \overline{a}_k z|^2$, and break up the sum dyadically. The only term that offers any difficulty is

$$\sum_{|\zeta - a_k| \le 3\delta} \frac{d_k}{|1 - \overline{a}_k z|^2}.$$

The part of 3.3 corresponding to this term is at most

$$\sum d_k^p \int_{S(I)} \frac{(1-r)^{p(2-\alpha)-1}}{|1 - \overline{a}_k z|^{2p}} dm, \tag{3.6}$$

the sum extended over $|\zeta - a_k| \le 3\delta$. If $2p > 1$, replace the integral over $S(I)$ by the integral over U and the result follows. Suppose $2p < 1$, then note that

$$\int_I \frac{d\theta}{|1 - \overline{a}_k r e^{i\theta}|^{2p}} \le C|I|^{1-2p}.$$

Using this we see that 3.6 is at most

$$\begin{aligned} \sum d_k^p |I|^{1-2p} |I|^{p(2-\alpha)} &= \sum d_k^{1-\alpha p + \alpha p - 1 + p} |I|^{1-\alpha p} \\ &\le C \sum d_k^{1-\alpha p} |I|^{\alpha p - 1 + p} |I|^{1-\alpha p} \end{aligned}$$

(here we have used the fact that $p > 1/(1+\alpha)$ and that the sum is extended over $|\zeta - a_k| \le 3\delta$). Now the result follows.

The case $\alpha = 1$ follows in a similar way.

To show that there is no theorem when $p \le 1/(1+\alpha)$, we show that if $p = 1/2$ and $\alpha = 1$ then there is a Blaschke product B whose zeros satisfy condition $*$ of Theorem 2.1 but $B \notin MB_\alpha^p$, in fact $B \notin B_\alpha^p \supseteq MB_\alpha^p$. In [**3**], in the proof of Lemma 2 on page 112, there is constructed a Blaschke product $\sum d_k^{1/2} < 2\pi$, but $B' \notin H^{1/2}$. The zeros are given as

$$\theta_n = \sum_{k=n}^{\infty} d_k^{1/2}$$

and

$$a_n = (1 - d_n)e^{i\theta_n}.$$

Now, it is clear from this construction that

$$\sum_{a_k \in S(I)} d_k^{1/2} \le C|I| \le C|I|^{1/2}.$$

But $B \in B_1^{1/2}$ implies $B' \in H^{1/2}$, so $B \notin B_1^{1/2}$.

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