ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 3, Summer 1990

ON MINIMAL UPPER SEMICONTINUOUS COMPACT-VALUED MAPS

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1. Introduction. In what follows, X and Y are Hausdorff topological spaces, and the term *map* is reserved for set-valued mappings. Also, for $x \in X$ and $y \in Y$, $\mathcal{U}(x)$ and $\mathcal{V}(y)$ are always used to denote a base of neighborhoods of x in X and y in Y, respectively. If $F : X \to Y$ is a (set-valued!) map, then

$$\mathrm{Gr}\,(F) = \{(x, y) \in X \times Y : y \in F(x)\}$$

is the *graph* of F.

Given two maps $F, G : X \to Y$, we write $G \subset F$ and say that G *is contained in* F if $G(x) \subset F(x)$ for every x in X; equivalently, if $\mathrm{Gr}\,(G) \subset \mathrm{Gr}\,(F)$. The relation of containment being a partial order in the family of all maps (with domain X and range Y), if a set $\mathcal{F}$ of maps is specified, we can look for maps which are minimal elements of $(\mathcal{F}, \subset)$.

A map $F : X \to Y$ is *upper semincontinuous at* a point $x \in X$ (*usc at* x) if, for every open set V containing $F(x)$, there exists $U \in \mathcal{U}(x)$ such that

$$F(U) = \cup\{F(u) : u \in U\} \subset V.$$

F is *upper semicontinuous* (*usc*) if it is usc at each point of X. We say, shortly, that a map F is *usco* if it is usc and takes nonempty compact values. Finally, a map F is said to be *minimal usco* if it is a minimal element in the family of all usco maps (with domain X and range Y); that is, if it is usco and does not contain properly any other usco map from X into Y. (See [**5**] for references.)

Historically, minimal usco maps seem to have appeared first in complex analysis (in the second half of the 19th century), in the form of a bounded holomorphic function and its "cluster sets," see, e.g., [**3**]. Starting with a 1982 paper of Christensen [**2**], a series of "multi-valued Namioka theorems" has been discovered (see [**9, 4**]). These theorems tell us that, under unexpectedly general assumptions about X and Y, a minimal usco map $F : X \to Y$ reduces to a (point-valued) function f on a

Received by the editors on February 9, 1987.

dense subset D of X, i.e., $F(x)$ is a singleton, $\{f(x)\}$, for all $x \in D$. It turns out that, then F is determined uniquely by its restriction $f = F|D$: for $x \in X \setminus D$, $F(x)$ is nothing else but the cluster set $\tilde{f}_D(x)$ of f at x; thus F is the cluster-set extension of f (see Corollary 4.9 below). This simple result was the origin of our research presented here; it is now a consequence of a more general (but still quite simple) result, Theorem 4.7, which characterizes a minimal usco map F in terms of its restriction $F|D$ to a dense subset D of X, and the cluster set extension of $F|D$ to the whole of X. In some cases, for instance when X is metrizable and Y is a Banach space in its weak topology, these cluster sets, $\tilde{F}_D(x)$, admit a "sequential" representation. We state this explicitly in Corollary 4.9 when $F|D = f$, but a similar result holds also without the assumption that $F|D$ is single-valued. This result requires some work, and we found it convenient to start, in §2, with two results about cluster sets of filterbases. In particular, we give some denumerability conditions under which such a cluster set has a sequential description (Proposition 2.2). Then, in §3, we use this last result to show that, under the same conditions, the graph of a compact-valued usc map has a very nice topological property: The sequential (or countable) closure of any of its subsets is a closed set (Proposition 3.5). From this our "sequential representation result," Corollary 4.9, follows directly.

Finally, although the existence of a minimal usco map contained in a given usco map follows by an application of the Kuratowski-Zorn Principle, something can be said about its size; this is done in the last part of the paper, §5.

2. Cluster sets of filterbases. Throughout this section Y stands for a Hausdorff space.

Let $\mathcal{B}$ be a filtering (= downward directed) family of subsets of Y. Then the *cluster set* of $\mathcal{B}$ is defined by

$$\mathrm{Cs}\,(\mathcal{B}) = \bigcap\{\overline{B} : B \in \mathcal{B}\}.$$

If $(y_\alpha) = (y_\alpha)_{\alpha\in A}$ is a net in Y, its set of cluster points is

$$(y_\alpha) = \bigcap_{\beta\in A} \overline{\{y_\alpha : \beta \le \alpha \in A\}}.$$

Writing $(y_\alpha) \prec \mathcal{B}$ to denote that, for every $B \in \mathcal{B}$, there is β such that $y_\alpha \in B$ for all $\alpha \ge \beta$, it is easily seen that

$$\mathrm{Cs}\,(\mathcal{B}) = \{y \in Y : y \text{ is a cluster point of a net } (y_\alpha) \prec \mathcal{B}\}.$$

Finally, we will say that $\mathcal{B}$ is *aimed at* a set $S \subset Y$ and write

$$B \rightsquigarrow S,$$

provided that every open set $V \supset S$ contains some $B \in \mathcal{B}$.

PROPOSITION 2.1. *Let $\mathcal{B}$ be a filtering family of subsets of Y aimed at a compact set. Then $L_0 = \mathrm{Cs}\,(\mathcal{B})$ is the smallest compact set at which $\mathcal{B}$ is aimed; moreover, $L_0 \neq \varnothing$ when $\mathcal{B}$ is a filterbase, and $L_0 = \varnothing$ otherwise.*

PROOF. We consider only the nontrivial case when $\mathcal{B}$ is a filterbase. Let $L \subset Y$ be compact and suppose $\mathcal{B} \rightsquigarrow L$. If $y \in Y \setminus L$, then there exist disjoint open sets $V \supset L$ and $W \in y$. Since $B \rightsquigarrow L$, $B \subset V$ for some $B \in \mathcal{B}$; then $y \notin \overline{B}$ and so $y \notin L_0$. Thus $L_0 \subset L$, and L_0 is obviously compact.

Suppose $L_0 = \varnothing$. Then, for every $y \in L$, there is $B_y \in \mathcal{B}$ with $y \notin \overline{B}_y$, so $V_y \cap B_y = \varnothing$ for some $V_y \in \mathcal{V}(y)$. As L is compact, there exist $y_1, \dots, y_n \in L$ such that $L \subset V = V_{y_1} \cup \cdots \cup V_{y_n}$, and then $V \cap B = \varnothing$ for all $B \in \mathcal{B}$ contained in $B_{y_1} \cap \cdots \cap B_{y_n}$. However, $\mathcal{B}$ is a filterbase and $B \rightsquigarrow L$, and we quickly arrive at a contradiction.

Thus it remains to show that $B \rightsquigarrow L_0$. Suppose it is not so; then there is an open set $W \supset L_0$ such that $B \setminus W \neq \varnothing$ for all $B \in \mathcal{B}$. It follows that $\mathcal{B}' = \{B \setminus W : B \in \mathcal{B}\}$ is a filterbase and, clearly, $\mathcal{B}' \rightsquigarrow L$. Applying to $\mathcal{B}'$ what we have proved above for $\mathcal{B}$, we now have

$$\varnothing \neq \bigcap\{\overline{B \setminus W} : B \in \mathcal{B}\} \subset \bigcap\{\overline{B} \setminus W : B \in \mathcal{B}\} = L_0 \setminus W = \varnothing;$$

a contradiction.

In the proposition below we show that when $\mathcal{B}$ is countable and Y satisfies some appropriate topological countability requirements, $\mathrm{Cs}\,(\mathcal{B})$ admits also a sequential description.

Given a set $B \subset Y$, let ${}^s\overline{B}$ (respectively ${}^c\overline{B}$) denote the *sequential* (*countable*) *closure* of B, consisting of all points y in Y such that y is the limit (a cluster point) of a sequence (y_n) in B; B is *sequentially* (*countably*) *closed* if ${}^s\overline{B} = B({}^c\overline{B} = B)$. We say that the space Y has *sequentially* (*countably*) *determined closure for relatively compact sets* if the following condition (A_1) $((\mathrm{A}_2))$ is satisfied.

(A$_1$) ${}^s\overline{K} = \overline{K}$ for every relatively compact set K in Y.

(A$_2$) ${}^c\overline{K} = \overline{K}$ for every relatively compact set K in Y.

Evidently, if the space Y satisfies (A$_1$) ((A$_2$)), then its sequentially (countably) closed relatively compact subsets are compact.

PROPOSITION 2.2. *Let $\mathcal{B} = (B_n)$ be a decreasing sequence of nonempty subsets of Y aimed at a compact set, and define*

$$\begin{aligned} L\ &= \mathrm{Cs}\,(\mathcal{B}) = \cap_n \overline{B}_n;\\ L_c &= \{y : y \textit{ is a cluster point of a sequence } (y_n) \prec \mathcal{B}\};\\ L_s &= \{y : y \textit{ is the limit of a sequence } (y_n) \textit{ such that } y_n \in B_n \textit{ for all } n\}\\ &= \{y : y = \lim y_n \textit{ for some sequence } (y_n) \prec \mathcal{B}\}. \end{aligned}$$

Then $L_s \subset L_c \subset L_0$ and

(a) *$L_0 \neq \varnothing$ and L_0 is the smallest compact set with $\mathcal{B} \rightsquigarrow L_0$;*

(b) *for every sequence $(y_n) \prec \mathcal{B}$, the set (y_n) of its cluster points is a nonempty compact subset of L_0, and the set $\{y_n : n \in N\}$ is relatively compact;*

(c) *L_c is a countably closed, dense subset of L_0 and $\mathcal{B} \rightsquigarrow L_c$;*

(d) *if Y satisfies* (A$_2$), *then $L_c = L_0$;*

(e) *if Y satisfies* (A$_1$), *then $L_s = L_0$.*

PROOF. (a) is a particular case of Proposition 2.1.

(b) Let $(y_n) \prec \mathcal{B}$ and set $H_n = \{y_m : m \geq n\}$, $n = 1, 2, \ldots$. Then $B_n \rightsquigarrow L_0$ implies $H_n \rightsquigarrow L_0$ and, applying (a) to (H_n), we get that $C = (y_n)$ is a nonempty compact subset of L_0 and $H_n \rightsquigarrow C$. To verify that $H = \{y_n : n \in N\}$ is relatively compact, first observe that $\overline{H} = H \cup C$. Now, if $\mathcal{V}$ is an open covering of $\overline{H}$, first choose a finite cover $\mathcal{V}' \subset \mathcal{V}$ of C, next an n so that $H_n \subset \cup\mathcal{V}'$ (which is possible as $H_n \rightsquigarrow C$), and finally a finite cover $\mathcal{V}'' \subset \mathcal{V}$ of $\{y_1, \ldots, y_n\}$. Then $\mathcal{V}' \cup \mathcal{V}''$ is a finite cover of $\overline{H}$.

(c) Suppose it is not true that $B_n \rightsquigarrow L_c$. Then there is an open set $V \supset L_c$ and a sequence (y_n) such that $y_n \in B_n \setminus V$ for each n. Hence $(y_n) \subset Y \setminus V$ and $(y_n) \subset L_c$, which is impossible because $(y_n) \neq \varnothing$ by (b).

Now, $B_n \rightsquigarrow L_c$ implies $B_n \rightsquigarrow \overline{L}_c \subset L_0$, so $\overline{L}_c = L_0$ by (a).

Finally, in order to show that L_c is countably closed, let $Z = \{z^k : k \in N\}$ be a countable subset of L_c. For each k choose a sequence $(y_n^k) \prec \mathcal{B}$ so that $z^k \in (y_n^k)$ and, as we may, $(y_n^k) \subset B_k$. Arrange the elements $y_n^k (k, n \in N)$ in a single sequence (y_n). Then $(y_n) \prec \mathcal{B}$ and

$$(y_n) \supset \bigcup_k (y_n^k) \supset Z.$$

Hence $L_c \supset (y_n) \supset \overline{Z}$, which proves L_c is countably closed.

(d) is an obvious consequence of (c) and (A_2).

(e) Since (A_1)$\Rightarrow$(A_2) so that $L_c = L_0$ by (d), and, since $L_s \subset L_c$, it remains to be shown that $L_c \subset L_s$. Let $y \in (y_n)$ where $(y_n) \prec \mathcal{B}$. In view of (b) the set $\{y_n : n \in N\}$ is relatively compact, hence, by (A_1), y is the limit of a subsequence (y_{k_n}) of (y_n). Since $(y_{k_n}) \prec \mathcal{B}$, $y \in L_s$.

3. Compact-valued upper semicontinuous maps. In this section we apply the results of the preceding section to use maps. (As before, X and Y are two Hausdorff spaces.) This approach is quite natural because a map $F : X \to Y$ is usc at a point x of X if and only if $F(\mathcal{U}(x)) \rightsquigarrow F(x)$, where

$$F(\mathcal{U}(x)) = \{F(U) : U \in \mathcal{U}(x)\}.$$

We will also need the following lemma whose straightforward proof is omitted.

LEMMA 3.1. *Given a map $F : X \to Y$ and a set $D \subset X$, let $\tilde{F}_D : X \to Y$ be the map defined by*

$$\tilde{F}_D(x) = \bigcap \{\overline{F(U \cup D)} : U \in \mathcal{U}(x)\}.$$

Then

$$\mathrm{Gr}\,(\tilde{F}_D) = \overline{\mathrm{Gr}\,(F|D)},$$

where $F|D$ is the restriction of F to D and the closure is taken in $X \times Y$.

If $D = X$, we write $\tilde{F} = \tilde{F}_X$. Note that F has a closed graph if and only if $\tilde{F} = F$, that is, $\tilde{F}(x) = F(x)$ for all x in X.

The following useful result appears explicitly in Christensen [**2**].

PROPOSITION 3.2. *Let a map $F : X \to Y$ be usc and compact-valued. Then*:

(a) $\mathrm{Gr}\,(F)$ *is closed in* $X \times Y$.

(b) *If* $G : X \to Y$ *is a map such that* $G \subset F$ *and* $\mathrm{Gr}\,(G)$ *is closed, then* G *is usc and compact-valued.*

PROOF. (a) For any $x \in X$, $F(\mathcal{U}(x)) \rightsquigarrow F(x)$ because F is usc at x. Since $F(x)$, moreover, is compact, $\tilde{F}(x) = \mathrm{Cs}\,(F(\mathcal{U}(x))) \subset F(x)$ by Proposition 2.1; in fact $\tilde{F}(x) = F(x)$. Thus the graph of F is closed.

(b) Fix an $x \in X$. Then $G(\mathcal{U}(x)) \rightsquigarrow F(x)$ because $G \subset F$ and F is usc at x. By Proposition 2.1, $G(\mathcal{U}(x)) \rightsquigarrow \mathrm{Cs}\,(G(\mathcal{U}(x))) = \tilde{G}(x)$, and $\tilde{G}(x) = G(x)$ because G has a closed graph. Hence G is usc at x and $G(x)$ is compact.

COROLLARY 3.3. *Let a map $F : X \to Y$ be usc and compact-valued. If a map $G : X \to Y$ is contained in F and is usc and compact-valued when Y is equipped with a weaker Hausdorff topology, then it is also usc and compact-valued for the original topology of Y.*

COROLLARY 3.4. *If $F : X \to Y$ is a compact-valued usc map, then the topology of* $\mathrm{Gr}\,(F)$ *(induced from $X \times Y$) does not change when the topology of Y is replaced by any weaker Hausdorff topology.*

PROOF. Let ρ be the original topology of Y, and τ any weaker Hausdorff topology on Y. Let C be a closed subset of $\mathrm{Gr}\,(F) \subset X \times (Y, \rho)$. Then $C = \mathrm{Gr}\,(G)$ for some map $G : X \to Y$, and $G : X \to (Y, \rho)$ is usc and compact-valued by Proposition 3.1(b). Clearly, also $G : X \to (Y, \tau)$ is usc and compact-valued, hence $C = \mathrm{Gr}\,(G)$ is closed in $\mathrm{Gr}\,(F) \subset X \times (Y, \tau)$ by Proposition 3.1(a).

PROPOSITION 3.5. *Let a map $F : X \to Y$ be usc and compact-valued, and assume that X satisfied the first axiom of countability. If Y has sequentially* (*countably*) *determined closure for relatively compact sets, then the sequential* (*countable*) *closure of any subset of* $\mathrm{Gr}\,(F)$ *is closed. Equivalently, given any map $G : X \to Y$ such that $G \subset F$, then, for every*

$x \in X$,

$$(*) \qquad \tilde{G}(x) = \{y : \textit{there exist sequences } x_n \to x \textit{ and } y_n \to y \textit{ such that } y_n \in G(x_n), \forall\}$$

(*respectively*

$$\left(\begin{smallmatrix}*\\ *\end{smallmatrix}\right) \qquad \tilde{G}(x) = \{y : \textit{there exist sequences } x_n \to x \textit{ and } y_n \in G(x_n) \textit{ such that } y_n \in (y_n)\cdot\}).$$

PROOF. We give a proof of the "sequential" part of the proposition. Fix an x in X and a decreasing countable base (U_n) of neighborhoods of x. Then $G(U_n) \rightsquigarrow F(x)$ and $F(x)$ is compact. From Proposition 2.2 and Lemma 3.1 it now follows that $\tilde{G}(x) = \{y : y = \lim y_n$ for some sequence $(y_n) \prec (G(U_n))\}$, and this last set is easily seen to coincide with the set on the right-hand side of $(*)$.

The following result (needed in Example 4.1) is probably well-known; anyway, for finite products, it can be found in [**1**, p. 114].

PROPOSITION 3.6. *For each* $t \in T$ (*an index set*), *let* $F_t : X \to Y_t$ (*a Hausdorff space*) *be a compact-valued usc map. Then the product map*

$$F : X \to Y = \prod_{t \in T} Y_t$$

defined by

$$F(x) = \prod_{t \in T} F_t(x)$$

is also compact-valued and usc.

PROOF. F is compact-valued by Tychonoff's theorem. To prove that F is usc, fix an $x \in X$ and denote $K_t = F_t(x)$, $K = F(x)$. Let V be an open set in Y containing K. Since K is compact, an easy argument shows that there is an open set W such that $K \subset W \subset V$ and W is a finite union of open sets of the form $\prod W_t$, where $W_t = Y_t$ except for a finite number of indices t. It follows that, for some finite set $S \subset T$ we have $W = (\prod_{t \in T \setminus S} Y_t) \times W'$, where W' is an open set in $\prod_{t \in S} Y_t$ containing

$K' = \prod_{t \in S} K_t$. Finally, by [**6**, 3.2.10], there exist open sets $W'_t \subset Y_t$ for $t \in S$ such that $K' \subset \prod_{t \in S} W'_t \subset W'$. Now, since the maps $F_t (t \in S)$ are usc at x, we can find $U \in \mathcal{U}(x)$ for which $F_t(U) \subset W'_t$ for $t \in S$, and we conclude easily that $F(U) \subset W$.

Given a subset D of X, we make the following definitions.

A map $F : X \to Y$ is *D-usc at* a point $x \in X$ if, for every open set $V \supset F(x)$, there exists $U \in \mathcal{U}(x)$ such that $F(U \cap D) \subset V$. F is *D-usc* if it is D-usc at each point of X. F is *D-usco* if it is D-usc and assumes nonempty compact values.

Note that F is D-usc if and only if all its restrictions $F|D \cup \{x\} (x \in X)$ are usc.

PROPOSITION 3.7. *Let $F : X \to Y$ be a map, and let $D \subset X$.*

(a) *If F is D-usc and compact-valued, then so is $\tilde{F}_D$; moreover, $\tilde{F}_D \subset F$ and $\tilde{F}_D|D = F|D$.*

(b) *If D is dense in X and F is D-usco, so is $\tilde{F}_D$.*

PROOF. (a) Since $F|D$ is usc, its graph is a closed subset of $D \times Y$ by Proposition 3.2(a); hence $\tilde{F}_D|D = F|D$. Now fix an $x \in X$. Since $F|D \cup \{x\}$ is usc and compact-valued, it has a closed graph and hence $\tilde{F}_D|D \cup \{x\} \subset F|D \cup \{x\}$; in particular, $\tilde{F}_D(x)$ is a (compact) subset of $F(x)$. Moreover, as the graph of $\tilde{F}_D|D \cup \{x\}$ is obviously closed, Proposition 3.2(b) applies to show that this map is usc. In consequence, $\tilde{F}_D$ is D-usc and compact-valued.

(b) In view of (a), we only have to check that $\tilde{F}_D$ assumes nonempty values. If, for some $x \in X$, $\tilde{F}_D(x) = \varnothing$, then, since $\tilde{F}_D$ is usc, there is $U \in \mathcal{U}(x)$ such that $F_D(U \cap D) = \varnothing$, which is impossible.

4. Characterizations of minimal usco maps. Recall that a map $F : X \to Y$ is said to be *minimal usco* if it is usco and does not contain properly any other usco map defined on X and with values in Y. Of course, any (point-valued) continuous function $f : X \to Y$, more precisely the map $F(x) = \{f(x)\}$, is minimal usco; but it is also easy to find examples of genuine minimal usco maps. A standard one is the map $F : [0, 1] \to \mathbf{R}$, where $F(x) = \{\sin(1/x)\}$ for $x \neq 0$ and $F(0) = [-1, 1]$. Or take any function $f : \mathbf{R} \to \mathbf{R}$ that has finite left- and right-sided limits $f(x-)$ and $f(x+)$ at each point x, and set $F(x) = \{f(x-), f(x+)\}$. Here $F(x)$ is a singleton if f is continuous at x, and it is well known that this happens everywhere except for a countable number of points. The Namioka type theorems mentioned in the Introduction provide us with surprisingly general situations where a minimal usco map is single-valued on a large subset of its domain. In general, however, a minimal usco map need not be single-valued even at one point of its domain. An example to this effect can be found in [**2**]; the following example is slightly simpler.

EXAMPLE 4.1. A nowhere single-valued minimal usco map. For each $t \in T = [0,1]$, the map $F_t : [0,1] \to [0,1]$ defined by

$$F_t(x) = \begin{cases} \{0\} & \text{for } 0 \le x < t \\ \{0,1\} & \text{for } x = t \\ \{1\} & \text{for } t \le x \le 1 \end{cases}$$

is evidently minimal usco. By Proposition 3.6 the corresponding product map

$$F : [0,1] \to [0,1]^T$$

is usco. Suppose $G : [0,1] \to [0,1]^T$ is usco, $G \subset F$, and $G(s) \neq F(s)$ for some s in $[0,1]$. Let p_s be the projection of $[0,1]^T$ onto the sth copy of $[0,1]$ in this product. Then the map $G_s : [0,1] \to [0,1]$, defined by $G_s(x) = p_s[G(x)]$, is easily seen to be usco. Moreover, $G_s \subset F_s$ and $G_s(s) \neq F_s(s)$, where the latter follows from the fact that $F_t(s)$ is a singleton for $t \neq s$. Since F_s is minimal usco, we must have $G_s = F_s$; a contradiction.

REMARK 4.2. Despite the product form of the minimal usco map F in the above example, even the product of two minimal usco maps need not be minimal (though it *is* usco by Proposition 3.6). A suitable counterexample can be obtained by a slight modification of Example 4.4 in [**4**]. In fact, define $F : [0,\infty] \to \mathbf{R}^2$ as therein, but with $F(\infty) = L = [-1,1] \times [-1,1]$. Since the functionals $y_1^* = (1,0)$ and $y_2^* = (0,1)$ can be identified with the projections p_1 and p_2 of $\mathbf{R}^2$ onto its first and second axis, respectively, the argument used in that example shows that the maps $F_i = p_i F$ from $[0,\infty]$ into $\mathbf{R}$, $i = 1,2$, are minimal usco. However, the map $F = F_1 \times F_2$ is not minimal usco because $F(\infty) = L \supset C$.

We now turn to the general theory of minimal usco maps. The following basic result (cf. [**2, 5, 8**]) follows from Proposition 3.2 by an easy application of the Kuratowski-Zorn Principle.

Proposition 4.3. *Every usco map $F : X \to Y$ contains a minimal usco map (defined on X).*

The next result can be readily deduced from Corollary 3.3.

Proposition 4.4. *Let $F : X \to Y$ be a usco map. Then F is minimal usco iff it is minimal usco when Y is equipped with a weaker Hausdorff topology.*

We now give some characterizations (and representations) of minimal usco maps.

We start with a result which is essentially known: The fact that (b), (c) and (d) below are implied by (a) has been observed and employed by various authors.

Proposition 4.5. *For any usco map $F : X \to Y$, the following are equivalent.*

(a) *F is minimal usco.*

(b) *$F|U$ is minimal usco for every open subset U of X.*

(c) *Whenever U is an open subset of X and C is a closed subset of Y such that $F(x) \cap C \neq \varnothing$ for all x in U, then $F(U) \subset C$.*

(d) *Given $x \in X$, $U \in \mathcal{U}(x)$ and an open subset V of Y, if $F(x) \cap V \neq \varnothing$ then $F(u) \subset V$ for some u in U.*

(e) *For each x in X, the map F is minimal usco at x; that is, for every usc at x map $G : X \to Y$ such that G assumes nonempty values in a neighborhood of x and $G(x)$ is compact, if $G \subset F$, then $G(x) = F(x)$.*

Proof. We omit the proofs of implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and (e) $\Rightarrow$ (a) (cf. [**4**, Proposition 4.1]), and show only how (d) implies (e): For a map G as specified in (e), suppose that $G(x) \neq F(x)$, and let $y \in F(x) \setminus G(x)$. Then there exist disjoint open sets V and W in Y such that $G(x) \subset W$ and $y \in V$. Since G is usc at x, we can find $U \in \mathcal{U}(x)$ so that $\varnothing \neq G(u) \subset W$ for all $u \in U$. But $G \subset F$, so there is no u in U for which $F(u) \subset V$, contradicting (d).

REMARK 4.6. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and (e) $\Rightarrow$ (a) and, for Y regular, (d) $\Rightarrow$ (e), hold also for the class of usc maps assuming nonempty closed values. (Note that minimal maps in this case need not be minimal usco.)

We say that a map $F : X \to Y$ is *minimal D-usco*, where $D \subset X$, if it is D-usco (see §3) and does not contain properly any other D-usco map defined on X; we say F is *D-regular* if $F = \tilde{F}_D$. Note that for every map F, $\tilde{F}_D$ is always D-regular.

THEOREM 4.7. *For any usco map $F : X \to Y$ and any dense subset D of X, the following are equivalent*:

(a) *F is minimal usco.*

(b) *F is minimal D-usco.*

(c) *$F|D$ is minimal usco and F is D-regular.*

PROOF. (a) $\Rightarrow$ (b) Suppose $G : X \to Y$ is D-usco and $G \subset F$. Then, by Proposition 3.7, $\tilde{G}_D \subset G$ and $\tilde{G}_D$ is D-usco. On the other hand, $\tilde{G}_D$ has a closed graph and $\tilde{G}_D \subset G \subset F$, so $\tilde{G}_D$ is usco by Proposition 3.2(b). It follows that $\tilde{G}_D = G = F$ and thus F is minimal D-usco.
(b) $\Rightarrow$ (c) If $H \subset F|D$ is usco, then the map G such that $G|D = H$ and $G|(X \setminus D) = F|(X \setminus D)$ is D-usco. Since $G \subset F$, we must have $G = F$ and, in consequence, $H = F|D$. Thus $F|D$ is minimal usco. By Proposition 3.7, $\tilde{F}_D$ is D-usco and $\tilde{F}_D \subset F$, hence $\tilde{F}_D = F$, i.e., F is D-regular.

(c) $\Rightarrow$ (a) Suppose $G \subset F$ and G is usco. Then $G|D = F|D$ so that $\tilde{G}_D = \tilde{F}_D$, and $\tilde{F}_D = F$ because F is D-regular. Now $F = \tilde{G}_D \subset G \subset F$, so $G = F$, which proves that F is minimal usco.

COROLLARY 4.8. *If $F : X \to Y$ is usco and $F|D$ is minimal usco for some dense subset D of X, then $\tilde{F}_D$ is a unique minimal usco map contained in F. In particular, if F is minimal usco, then $F = \tilde{F}_D$.*

PROOF. $\tilde{F}_D$ is of course usco and D-regular, and $\tilde{F}_D|D = F|D$ is minimal usco. Hence $\tilde{F}_D$ is minimal usco, by the implication (c) $\Rightarrow$ (a) of the above theorem. If G is a minimal usco map contained in F, then we must have $G|D = F|D$. Hence $\tilde{G}_D = \tilde{F}_D$, $\tilde{G}_D$ is usco and $\tilde{G}_D \subset G$, so $G = \tilde{F}_D$.

Corollary 4.9. *Let $F : X \to Y$ be a minimal usco map and suppose F is single-valued on a dense subset D of X; that is, there is a function $f : D \to Y$ such that $F(x) = \{f(x)\}$ for x in X. Then $F = \tilde{f}_D$, i.e., for every $x \in X$,*

$$F(x) = \cap\{\overline{f(U \cap D)} : U \in \mathcal{U}(x)\}.$$

Moreover, if X is 1st countable (in particular, metrizable) and Y has sequentially determined closured for relatively compact sets, then

$$F(x) = \{y : y = \lim f(x_n) \\ \text{for some sequence } (x_n) \text{ in } D \text{ converging to } x\}.$$

Proof. The first assertion is immediate from the preceding corollary; the second follows from the first by an easy application of Proposition 3.5.

Here is a typical situation in which the last result can be applied. Let X be a metric Baire space, Y a Banach space, and $F : X \to Y$ a minimal weakly usco map (i.e., F is usco into (Y, weak)). Then, thanks to the results of Christensen [**2**] and Saint-Raymond [**11**], F reduces to a point-valued function f on a dense G_δ-set D in X. Further, (Y, weak) is an angelic space (see [**7**]); thus, in particular, it has sequentially determined closure for relatively weakly compact sets. Hence, by the above corollary, for each x in X, $F(x) = \{y \in Y : y = \text{weak-lim} f(x_n)$ for some sequence (x_n) in D converging to $x\}$.

Remark 4.10. We may treat the equality $F = \tilde{F}_D$ in 4.8 (and $F = \tilde{f}_D$ in 4.9) as sort of a representation result for a minimal usco map F in terms of its restriction $F|D$. In general, however, given a dense subset D of X and a continuous function $f : D \to Y$, the map F defined by $F = \tilde{f}_D$ need not be usc or compact-valued; see Levi [**10**] for more information about such extensions.

5. Some estimates of minimal usco maps. In what follows we intend to give some (lower and upper) "estimates" of the size of the minimal usco maps contained in a given usco map F; recall that the existence of such minimal maps is established, *ineffectively*, in Proposition 4.3.

A usco map F may contain many minimal usco maps and so, in general, will not have the smallest usco map contained in it. However, the infimum of all usco maps contained in F always exist as a compact-valued (the value $\varnothing$ not excluded) usc map from X into Y, and its value at a point $x \in X$ is given as $\cap\{G(x) : G \subset F, G \text{ usco}\}$.

Given a map $F : X \to Y$, we define, for every $x \in X$, $F_*(x)$ to be the set of all points $y \in Y$ which are "minimal for F at x" (cf. Proposition 4.5(d)), that is, satisfy the following condition:

(μ) For every $V \in \mathcal{V}(y)$ and $U \in \mathcal{U}(x)$, there exists $u \in U$ such that $F(u) \subset V$.

Proposition 5.1. *If $F : X \to Y$ is a usco map, then, for every $x \in X$,*

$$F_*(x) = \bigcap\{G(x) : G \subset F, G \textit{ usco}\}.$$

Proof. Given a usco map $G : X \to Y$ contained in F, we first prove that $F_* \subset G$. Suppose that, for some x in X, there is $y \in F_*(x) \setminus G(x)$. Then we can find disjoint open sets $W \supset G(x)$ and $V \in Y$. Next, since G is usc at x, we find $U \in \mathcal{U}(x)$ such that $G(U) \subset W$. Now (μ) implies that, for some $u \in U$, $F(u) \subset V$. But $G(u) \subset F(u)$, so we have $G(u) \subset W$ and $G(u) \subset V$, which is impossible because $G(u) \neq \varnothing$ and $V \cap W = \varnothing$. Thus the inclusion "$\subset$" holds.

Now, let $y \notin F_*(x)$ so that there exist $U \in \mathcal{U}(x)$ and $V \in \mathcal{V}(y)$ such that $F(u) \not\subset V$ for all $u \in U$. Consider the map $G : X \to Y$ such that $G(z) = F(z)$ for $z \in X \setminus U$ and $G(z) = F(z) \cap (Y \setminus V)$ for $z \in U$. Then G is usco, $G \subset F$, and $y \notin G(x)$. We have thus shown that the inclusion "$\subset$" cannot be proper.

Proposition 5.2. *Let $F : X \to Y$ be a usco map, where Y is a uniform space with a base $\mathcal{V}$ for its uniformity. Then the following are equivalent for every x in X.*

(a) $F_*(x) \neq \varnothing$.

(b) *For every $U \in \mathcal{U}(x)$ and $V \in \mathcal{V}$, there exists $u \in U$ such that $F(u)$ is V-small (i.e., $F(u) \times F(u) \subset V$).*

Proof. We can assume that $\mathcal{V}$ consists of sets open in $Y \times Y$. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a) Suppose $F_*(x) = \varnothing$. Then, for every $y \in F(x)$ there exist $u_y \in \mathcal{U}(x)$ and $V_y \in \mathcal{V}$ such that $F(u) \not\subset V_y(y)$ for all $U \in U_y$. By compactness, there exist a finite number of points $y_1, \ldots, y_n$ in $F(x)$ such that $F(x) \subset \cup_{i=1}^n V_i(y_i)$, where $V_i = V_{y_i}$. Then $U_1 = \cap_{i=1}^n U_{y_i}$ is a neighborhood of x, and $F(u) \not\subset V_i(y_i)$ for $I = 1, \ldots, n$ and all $u \in U_1$. By the compactness of $F(x)$, there is $V \in \mathcal{V}$ such that, for every $y \in F(x)$, we can find i for which $V(y) \subset V_i(y_i)$; clearly, we then have $F(u) \not\subset V(y)$ for all $u \in U_1$ and $y \in F(x)$. Since F is usc at x, there is $U \in \mathcal{U}(x), U \subset U_1$ such that $F(U) \subset W(F(x))$, where $W \in \mathcal{V}$ and $W \cdot W \subset V$. From the above,

$$(*) \qquad F(u) \not\subset V(y) \quad \text{for all } u \in U \quad \text{and} \quad y \in F(x).$$

Now we claim that

$$\forall u \in U : F(u) \text{ is not } W\text{-small}.$$

Suppose that $F(u) \times F(u) \subset W$ for some $u \in U$. Since $F(u) \subset W(F(x))$, taking any $z \in F(u)$ we can find $y \in F(x)$ such that $(z, y) \in W$. But $F(u)$ is W-small, hence $F(u) \subset V(y)$, which contradicts $(*)$.

We have thus found $U \in \mathcal{U}(x)$ and $W \in \mathcal{V}$ such that $F(u)$ is not W-small for any $u \in U$, contrary to our assumption (b).

5.3. Let $F : X \to Y$ be usco. From Corollary 4.8 it follows easily that if we define

$$(R_1F)(x) = \cap_D \tilde{F}_D(x) \text{ for } x \in X,$$

where the intersection is taken over all dense subsets D of X, then, for every minimal usco map $G \subset F$, one has

$$G \subset R_1F \subset F.$$

Clearly, $R_1F : X \to Y$ is a usco map.

Suppose $R_\alpha F$ has already been defined for all ordinals $\alpha < \beta$. Then we define $R_\beta F$ as follows:

If β has a predecessor α, then $R_\beta F = R_1(R_\alpha F)$.

If β is a limit ordinal, then $R_\beta F(x) = \cap_{\alpha<\beta} R_\alpha F(x)$ for all $x \in X$. We also set $R_0 F = F$.

Let γ be the first ordinal for which $R_{\gamma+1} = R_\gamma F$, and set

$$F^* = R_\gamma F.$$

The map $F^* : X \to Y$ thus obtained may be called "*the dense regularization*" of F: it is D-regular for every dense subset D of X.

The following facts are easily verified.

(a) If G is a usco map contained in F, then $G^* \subset F^*$.

(b) F^* is the largest usco map contained in F that is regular with respect to all dense subsets of X.

(c) If G is any minimal usco map contained in F, then $G = G^* \subset F^*$.

Thus, if F is a usco map and G is a minimal usco map contained in F, then

$$F_* \subset G \subset F^*;$$

these are the "estimates" we alluded to at the beginning of this section.

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