ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 3, Summer 1990

THE AVERAGE ERROR OF QUADRATURE FORMULAS FOR FUNCTIONS OF BOUNDED VARIATION

S. GRAF AND E. NOVAK

1. Introduction. The purpose of this paper is to discuss the average case error of quadrature formulas for functions of bounded variation. If one wants to consider average case errors then the main problem is to define a natural probability measure on the function set in question. One natural class of such measures is that of the Gaussian measures which have been considered by several authors (see for instance [**6, 2, 7, 3**]).

We take an alternative approach. The probability measures we are interested in should reflect "uniform distribution" in some sense (see also [**4**]). On bounded sets of finite dimensional spaces the normalized Lebesgue measure is the canonical candidate. On infinite dimensional spaces a translation invariant measure which is finite on bounded sets does not exist. Therefore, we construct a probability measure Q on the set $BV = \{f : [0,1] \to \mathbf{R} \,|\, f \text{ continuous}, \ f(0) = 0, \operatorname{Var}(f) \le 1\}$ in a different way using the "natural" measure on the homeomorphisms of $[0,1]$ introduced in [**1**].

Let e_n^Q denote the infimum of the average errors of quadrature formulas with n knots. We show that e_n^Q converges to 0 like $n^{-\log 6/(2\log 2)}$, where $\log 6/(2\log 2) = 1.29248\ldots$. This contrasts with the result for the worst case analysis. Much as in [**8**] one can show that, among all quadrature formulas with n knots, the rule

$$f \to \frac{1}{n}\sum_{i=1}^{n} f\Big(\frac{2i}{2n+1}\Big)$$

has minimal maximal error $1/(2n+1)$.

2. A probability measure Q on BV. Let H be the space of all homeomorphisms h from $[0,1]$ onto itself with $h(0) = 0$ and $h(1) = 1$ equipped with the topology of uniform convergence.

Received by the editors on February 26, 1987.

In [**1**] a natural probability measure P on the Borel field of H was investigated. A typical P-random homeomorphism h can be generated by the following construction. First choose the value $h(1/2)$ according to the uniform distribution on $[0,1]$. Then choose $h(1/4)$ according to the uniform distribution on $[0,h(1/2)]$ and independently $h(3/4)$ according to the uniform distribution on $[h(1/2),1]$. Continue this process. With probability one, the function constructed in that way on the dyadic rationals extends to an element of H. The measure P is characterized by the formula (see [**1**, Theorem 3.7])

$$\int_H G(h)\,dP(h) = \int_H \int_H \int_0^1 G([f,g]_y)\,dy\,dP(f)\,dP(g), \tag{2.1}$$

for each integrable $G : H \to \mathbf{R}$, where

$$[f,g]_y(t) = \begin{cases} yf(2t), & t \le 1/2 \\ y + (1-y)g(2t-1), & t \ge 1/2. \end{cases} \tag{2.2}$$

Using P it is easy to define a Borel probability measure Q on the space

$$BV = \{f : [0,1] \to \mathbf{R}\,|\,f \text{ continuous},\ f(0) = 0, \operatorname{Var}(f) \le 1\}$$

with the uniform topology. Here $\operatorname{Var}(f)$ denotes the variation of f, i.e.,

$$\operatorname{Var}(f) = \sup \sum_{i=1}^{n} |f(x_{i+1}) - f(x_i)|,$$

where the supremum is taken over all families $0 \le x_1 < x_2 < \cdots < x_n \le 1$. We observe that

$$BV \text{ contains } \{c_1h_1 - c_2h_2 | c_i \ge 0, c_1 + c_2 \le 1, h_i \in H\} \tag{2.3}$$

as a dense subset. Now we define Q on BV by

$$\int_{BV} G(f)\,dQ(f) = 2 \int_D \int_H \int_H G(c_1h_1 - c_2h_2)\,dP(h_1)\,dP(h_2)dc_1dc_2, \tag{2.4}$$

where $D = \{(c_1,c_2)|c_i \ge 0, c_1 + c_2 \le 1\}$ and $G : BV \to \mathbf{R}_+$ is measurable.

REMARKS 2.1. (a) [**1**, Remark 2.15] shows that the measure P has full support. Thus it follows from the definition of Q and (2.3) that Q also has full support.

(b) The measure P satisfies [**1**, Remark 4.17(a)]

$$\int_H f(x)\,dP(f) = x \quad \text{for all } x \in [0,1].$$

3. The average error for the trapezoidal rule. Let $n \in \mathbf{N}$ and

$$T_n(f) = \frac{1}{2n} \cdot f(1) + \frac{1}{n} \cdot \sum_{i=1}^{n-1} f\Big(\frac{i}{n}\Big), \quad f \in BV,$$

be the n-th trapezoidal rule. We define the average error of T_n on BV by

$$\Delta_Q(T_n) = \bigg(\int_{BV} (I(f) - T_n(f))^2 dQ(f)\bigg)^{1/2},$$

where $I(f) = \int_0^1 f(x)dx$. The number $\Delta_P(T_n)$ is defined in the same way (where Q is replaced by P). We will calculate the value of $\Delta_Q(T_n)$ for $n = 2^m, m \in \mathbf{N}_0$.

THEOREM 3.1. *For* $m \in \mathbf{N}_0$,

$$\Delta_Q(T_{2^m}) = 120^{-1/2} \cdot 6^{-m/2}.$$

PROOF. By (2.4),

$$\begin{aligned}
\Delta_Q(T_n)^2 & \\
&= 2 \cdot \int_D \int_H \int_H (c_1 I(h_1) - c_2 I(h_2) - c_1 T_n(h_1) \\
&\qquad\qquad + c_2 T_n(h_2))^2 dP(h_1) dP(h_2) dc_1 dc_2 \\
&= 2 \int_D \bigg[\int_H c_1^2 (I(h_1) - T_n(h_1))^2 dP(h_1) \\
&\qquad + \int_H c_2^2 (I(h_2) - T_n(h_2))^2 dP(h_2)\bigg] dc_1 dc_2.
\end{aligned}$$

The last equality holds because it follows from Remark 2.1(b) that

$$\int_H T_n(f)\,dP(f) = \frac{1}{2} = \int_H I(f)\,dP(f).$$

Because $\int_D c_1^2 + c_2^2 dc_1 dc_2 = 1/6$, we get

$$\Delta_Q(T_n)^2 = \frac{1}{3}\cdot\Delta_P(T_n)^2. \tag{3.1}$$

For $n = 1$ we obtain

$$\begin{aligned}\Delta_Q(T_n)^2 &= \frac{1}{3}\cdot\int_H (I(h)-1/2)^2 dP(h)\\ &= \frac{1}{12}-\frac{1}{6}+\frac{1}{3}\cdot\int_H I(h)^2 P(h).\end{aligned} \tag{3.2}$$

By means of (2.1), deduce

$$\begin{aligned}&\int_H I(h)^2 dP(h)\\ &\qquad = \int_H\int_H\int_0^1 I([f,g]_y)^2 dy dP(f) dP(g)\\ &\qquad = \int_H\int_H\int_0^1 \frac{1}{4}\cdot(yI(f)+y+(1-y)I(g))^2 dy dP(f) dP(g)\\ &\qquad = \int_H\int_H\int_0^1 \frac{1}{4}\cdot(y^2 I(f)^2+y^2+(1-y)^2 I(g)^2+2y^2 I(f)\\ &\qquad\quad + 2y(1-y)I(f)I(g)+2y(1-y)I(g)\,dy\,dP(f)\,dP(g)\\ &\qquad = \int_H \frac{1}{6}I(h)^2 dP(h)+\frac{11}{48}.\end{aligned}$$

This implies

$$\int_H I(h)^2 dP(h) = \frac{11}{40}. \tag{3.3}$$

From (3.2) we deduce that

$$\Delta_Q(T_1)^2 = -\frac{1}{12}+\frac{11}{120} = \frac{1}{120}. \tag{3.4}$$

For $n \in \mathbf{N}$, by (3.1) and (2.1),

$$\begin{aligned}\Delta_Q(T_{2n})^2 &= \frac{1}{3}\cdot\Delta_P(T_{2n})^2\\ &= \frac{1}{3}\cdot\int_H\int_H\int_0^1 \Big(\frac{1}{2}yI(f)+\frac{1}{2}y+\frac{1}{2}(1-y)I(g)\\ &\qquad -\frac{1}{4n}(y+(1-y)g(1))-\frac{1}{2n}\cdot\sum_{i=1}^{n} y\cdot f\Big(\frac{i}{n}\Big)\\ &\qquad -\frac{1}{2n}\cdot\Big(\sum_{i=n+1}^{2n-1} y+(1-y)\cdot g(\frac{i}{n}-1)\Big)\Big)^2 dydP(f)dP(g)\\ &= \frac{1}{3}\cdot\int_H\int_H\int_0^1 \Big(\frac{1}{2}y\cdot(I(f)-T_n(f))+\frac{2n-1-1-2n+2}{4n}y\\ &\qquad\qquad +\Big(\frac{1}{2}(1-y)\cdot(I(g)-T_n(g))\Big)^2 dydP(f)dP(g)\\ &= \frac{1}{18}\cdot\Delta_P(T_n)^2=\frac{1}{6}\cdot\Delta_Q(T_n)^2.\end{aligned}$$

By induction we obtain, for $m \geq 0$,

$$\Delta_Q(T_{2^m})^2 = \frac{1}{120}\cdot\big(\frac{1}{6}\big)^m. \tag{3.5}$$

4. The best possible rate of convergence for the average error. For $0 \leq a_1 < \cdots < a_n \leq 1$, let $a = (a_1,\dots,a_n)$ and $N_a : BV \to \mathbf{R}^n$ be defined by

$$N_a(f) = (f(a_1),\dots,f(a_n)).$$

Let $A_n = \{\phi\circ N_a | a_i \in [0,1], \phi : \mathbf{R}^n \to \mathbf{R}\}$, $n \in \mathbf{N}$. For formal reasons we define $A_0 = \{M : BV \to \mathbf{R} | M \text{ constant}\}$. Define the average error of $M \in A_n$ by

$$\Delta_Q(M) = \left(\int_{BV}^{*} (I(f)-M(f))^2 dQ(f)\right)^{1/2},$$

where $\int^*$ denotes the upper integral.

Let $e_n^Q = \inf\{\Delta_Q(M) | M \in A_n\}$ be the n-th average case error bound. The numbers $\Delta_P(M)$ and e_n^P are defined analogously.

We are interested in the rate of convergence of $e_n^Q (n \to \infty)$.

THEOREM 4.1. There exist constants c, C in $(0, \infty)$ with

$$c \cdot n^{-\log 6/2 \log 2} \le e_n^Q \le C \cdot n^{-\log 6/2 \log 2}$$

for $n \in \mathbf{N}$.

For the proof of the theorem we will use some lemmas.

LEMMA 4.2.

$$\left(e_n^Q\right)^2 \ge \frac{1}{3}\left(e_n^P\right)^2$$

PROOF. First we want to show that, in the definition of e_n^P and e_n^Q, it is enough to consider measurable functions $M \in A_n$. We will prove this for Q, the proof for P is similar. Let $a = (a_1, \ldots, a_n)$ be fixed and let μ be the image of Q with respect to N_a. By the disintegration theorem (see for instance [**5**]) there exists a family $\mu_s, s \in N_a(BV)$, of probability measures on BV such that the following conditions hold:

(i) For Borel measurable $B \subset BV$, the function $s \to \mu_s(B)$ is μ-measurable.

(ii) $\int_A \mu_s(B) d\mu(s) = Q(B \cap N_a^{-1}(A))$ for all Borel measurable A, B $(A \subset \mathbf{R}^n, B \subset BV)$.

(iii) For μ - a.e. s the measure $\mu_s(N_a^{-1}(s))$ equals 1.

For $\phi : \mathbf{R}^n \to \mathbf{R}$, it follows from (ii) that

$$\begin{aligned}
&\int_{BV}^* (I(f) - \phi(f(a_1), \ldots, f(a_n))^2 dQ(f) \\
&\qquad \ge \int_{\mathbf{R}^n}^* \int_{BV}^* (I(f) - \phi(f(a_1), \ldots, f(a_n))^2 d\mu_s(f) d\mu(s) \\
&\qquad \ge \int_{\mathbf{R}^n}^* \int_{BV}^* (I(f) - \int I d\mu_s))^2 d\mu_s(f) d\mu(s).
\end{aligned}$$

The last inequality holds because of (iii) and since the function $c \to \int_{BV}^{*}(I(f)-c)^2 d\mu_s$ attains its smallest value for $c = \int I\, d\mu_s$.

By property (i) the function $s \to \int I d\mu_s = \phi_0(s)$ is measurable and hence

$$\begin{aligned}\int_{BV}^{*} &(I(f)-\phi(f(a_1),\dots,f(a_n))^2 dQ(f)\\ &\geq \int_{\mathbf{R}^n}\int_{BV}\Big(I(f)-\int I d\mu_s\Big)^2 d\mu_s(f)d\mu(s)\\ &\overset{\text{(ii)}}{=} \int_{BV}(I(f)-\phi_0(f(a_1),\dots,f(a_n)))^2 dQ(f).\end{aligned}$$

Thus our claim is proved.

For $0 \leq a_1 < \cdots < a_n \leq 1$ and measurable $\phi : \mathbf{R}^n \to \mathbf{R}$, we obtain, by (2.4),

$$\begin{aligned}\int_{BV} &(I(f)-\phi(f(a_1),\dots,f(a_n)))^2 dQ(f)\\ &= 2\cdot\int_D\int_H\int_H (I(c_1h_1-c_2h_2)-\phi((c_1h_1-c_2h_2)(a_1),\dots,\\ &\qquad (c_1h_1-c_2h_2)(a_n)))^2 dP(h_1)dP(h_2)dc_1dc_2.\end{aligned}$$

By an argument similar to the one above we see that the last expression is greater than or equal to

$$\begin{aligned}2\int_D\int_H\int_H &(c_1I(h_1)-c_2I(h_2)-c_1\tilde{\phi}(h_1(a_1),\dots,h_1(a_n))\\ &+c_2\tilde{\phi}(h_2(a_1),\dots,h_2(a_n)))^2 dP(h_1)dP(h_2)dc_1dc_2,\end{aligned}$$

where $\tilde{\phi}$ is induced by the conditional expectation of I with respect to the σ-field generated by N_a. An easy calculation shows this to be equal to

$$2\int_D c_1^2+c_2^2 dc_1dc_2\int_H (I(h)-\tilde{\phi}(h(a_1),\dots,h(a_n)))^2 dP(h) \geq \frac{1}{3}\cdot(e_n^P)^2.$$

Thus our lemma is proved. □

LEMMA 4.3.

$$\left(e_n^P\right)^2 \geq \inf_{n=k+m} \frac{1}{12} \cdot \left(\left(e_m^P\right)^2 + \left(e_k^P\right)^2\right).$$

PROOF. For measurable ϕ and $0 \leq a_1 < \cdots < a_k \leq 1/2 < a_{k+1} \cdots < a_n \leq 1$ we have, by (2.1),

$$\begin{aligned}
&\int_H (I(h) - \phi(h(a_1), \ldots, h(a_n)))^2 dP(h) \\
&\quad = \int_0^1 \int_H \int_H \Big(\frac{1}{2} y I(f) + \frac{1}{2} y + \frac{1}{2}(1-y) I(g) - \phi(y f(2a_1), \ldots, \\
&\qquad\qquad y f(2a_k), y + (1-y) g(2a_{k+1} - 1), \ldots, \\
&\qquad\qquad y + (1-y) g(2a_n - 1))\Big)^2 dP(g) dP(f) dy \\
&\quad \geq \int_0^1 \int_H \int_H \Big(\frac{1}{2} y I(f) + \frac{1}{2} y + \frac{1}{2}(1-y) I(g) - \frac{1}{2} y \phi_1(f(2a_1), \ldots, \\
&\qquad\qquad f(2a_k)) - \frac{1}{2} y - \frac{1}{2}(1-y) \phi_2(g(2a_{k+1} - 1), \ldots, \\
&\qquad\qquad g(2a_n - 1))\Big)^2 dP(g) dP(f) dy,
\end{aligned}$$

where ϕ_1 and ϕ_2 are induced by the conditional expectations of I with respect to the σ-fields generated by $N_{2a_1,\ldots,2a_k}$ and $N_{2a_{k+1}-1,\ldots,2a_n-1}$, respectively. The last expression is equal to

$$\begin{aligned}
&\frac{1}{12} \cdot \int_H (I(f) - \phi_1(f(2a_1), \ldots, f(2a_k)))^2 dP(f) + \\
&\qquad + \frac{1}{12} \cdot \int_H (I(g) - \phi_2(2a_{k+1}, \ldots, g(2a_n)))^2 dP(g) \\
&\quad \geq \frac{1}{12}\left(e_k^P\right)^2 + \frac{1}{12}\left(e_{n-k}^P\right)^2.
\end{aligned}$$

Thus the lemma is proved. □

PROOF OF THEOREM 4.1. We define $a_0 = (e_0^P)^2$ and

$$a_n = \inf_{0 \leq k \leq n} \frac{1}{12}(a_k + a_{n-k}). \tag{4.1}$$

Since a_n is obviously nonincreasing and $a_2 = (1/66) \cdot a_0$ we know that, for $n \geq 2$,

$$a_n < \frac{1}{12}(a_0 + a_n),$$

and hence

$$a_n = \inf_{0<k<n} \frac{1}{12}(a_k + a_{n-k}). \tag{4.2}$$

We will show that, for $n \in \mathbf{N}$,

$$a_n \geq a_1 \cdot n^{-\alpha},$$

where $\alpha = \log 6/\log 2$. Assume that $a_k \geq a_1 \cdot k^{-\alpha}$ for $k = 1, \ldots, n$. By (4.2) there exists a $k, 0 < k < n$, with

$$\begin{aligned} a_{n+1} &= \frac{1}{12}(a_k + a_{n+1-k}) \\ &\geq \frac{1}{12} a_1 (k^{-\alpha} + (n+1-k)^{-\alpha}) \\ &\geq \frac{1}{6} a_1 \Big(\frac{n+1}{2}\Big)^{-\alpha} \\ &= a_1 (n+1)^{-\alpha}. \end{aligned}$$

By (4.1) and Lemma 4.3 we get $(e_n^P)^2 \geq a_1 n^{-\alpha}$. Thus Lemma 4.2 implies

$$\big(e_n^Q\big)^2 \geq \frac{1}{3} a_1 n^{-\alpha}.$$

Using Theorem 3.1, we prove the upper bound:

$$\big(e_n^Q\big)^2 \leq \big(e_{2^m}^Q\big)^2 \leq (\Delta_Q(T_{2^m}))^2 = \frac{1}{120} \cdot 6^{-m} \leq \frac{1}{20} n^{-\frac{\log 6}{\log 2}}$$

for $m \in \mathbf{N}$ with $2^m \leq n < 2^{m+1}$. □

REFERENCES

1. S. Graf, R.D. Mauldin and S.C. Williams, *Random homeomorphisms,* Adv. in Math. **60** (1986), 239–359.

2. F.M. Larkin, *Gaussian measure in Hilbert space and applications in numerical analysis,* Rocky Mountain J. Math. **2** (1972), 379–421.

3. D. Lee and G.W. Wasilkowski, *Approximation of linear functionals on a Banach space with a Gaussian measure,* J. Complexity **2** (1986), 12–43.

4. E. Novak, *Deterministic and stochastic error bounds in numerical analysis,* Lecture Notes in Math. **1349** (1988).

5. K.R. Parthasarathy, *Probability measures on metric spaces,* Academic Press, New York.

6. A.V. Suldin, *Wiener measure and its applications to approximation methods* I, II. Izv. Vyssh. Uchebn. Zaved. Mat. **13**, **18** (1959, 1960), 145–158, 165–179.

7. G.W. Wasilkowski, *Average case optimality,* J. Complexity **1** (1985), 107–117.

8. S. Zubrzycki, *Some approximate integration formulas of statistical interest,* Coll. Math. **11** (1963/64), 123–135.

Fakultät Für Mathematik und Informatik Innstr. 33, D-8390 Passau, West Germany

Mathematisches Institut, Bismarckstr. 1 1/2, D-8520 Erlangen, West Germany