ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 3, Summer 1990

COMPLEX TRANSFORMATIONS OF SOLUTIONS OF GENERALIZED INITIAL VALUE HEAT PROBLEMS

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ABSTRACT. Let $\{z_i\}_{i=1}^n$ denote n complex variables. Let $z = (z_1, \ldots, z_n), D = (D_1, \ldots, D_n)$ with $D_j\phi(z) = \partial\varphi(z)/\partial z_j$, and let $P(D)$ be a multinomial partial differential operator in the D_j. Using complex translations and quasi inner products, solutions are constructed for initial value generalized heat problems of the form

$$H_t(z,t) = P(D)H(z,t),\ H(z,0) = \varphi(z)$$

in which $\varphi(z)$ is entire with suitable growth. Growth bounds are obtained for the $H(z,t)$ and these, in turn, are used to construct solutions of higher order and other types of well-posed and ill-posed evolution problems. Applications are given for special equations.

1. Introduction. Let $z_j = x_j + iy_j$ denote a complex variable, $1 \le j \le n$. Let $z = (z_1, \ldots, z_n), D = (D_1, \ldots, D_n)$ in which $D_j\varphi(z) = \partial\varphi(z)/\partial z_j$ (when $n = 1, z$ and D are assigned their usual meanings). Let $P(D)$ be a multinomial partial differential operator in the D_j with constant coefficients. We shall primarily be concerned with the following:

(a) The construction of a solution of the generalized heat problem

$$H_t(z,t) = P(D)H(z,t), \quad H(z,0) = \varphi(z) \tag{1.1}$$

in which $\varphi(z)$ is entire and of suitable growth, and

(b) the construction of solutions of higher order and other well-posed and ill-posed evolution type problems in terms of the $H(z,t)$.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35C15; Secondary 35E15, 35C10.
Keywords and phrases. Quasi inner product, transmutations, Bergman kernel, entire functions, generalized heat problem, function of an operator, growth bounds, hypergeometric problems, Sobolev type equation.
Received by the editors on February 26, 1987.

The mathematical tools to be used for these purposes are complex translations, the quasi inner product [**8**] and entire and analytic functions of differential operators. The growth bounds on the $\varphi(z)$ (and hence the $H(z,t)$) will play a vital role in these considerations. Applications will be made to a polynomial representation of a solution of a generalized heat problem, the solution of a generalized Euler-Poisson-Darboux problem, and to a Sobolev type problem. Lastly, two examples will be given in which the complex transformations reduce to real transformations (when the z_j are taken to be real) and the entireness requirements on $\varphi(z)$ can then be removed. These transformations apply to their counterparts in Banach spaces.

In order to view this research in the broad scheme of partial differential equations, it is useful to give a brief sketch of the historical development of complex and real transmutations. Stated simply, a transmutation is an integral transformation that connects the solution of one problem in partial differential equations to the solution of another such problem. The origins of complex transmutation methods can be found in the studies of E.T. Whittaker [**33**] (also, see [**34**]). His methods were extensively researched and broadened in scope by S. Bergman (see [**2**]). The Whittaker-Bergman operator now takes on a central role in function theoretic methods (see [**23**] for a discussion of this, its applications, and other related methods). More recent developments in the function theoretic approach and their applications can be found in [**1**]. The Cauchy integral formula and numerous modifications of it are at the heart of the function theoretic approach. The notion of a real transmutation was introduced by J. Delsarte [**19, 20**] and extended and applied by J.L. Lions [**25, 26**]. In 1967, the author and J.W. Dettman initiated the development of the method of related partial differential equations for constructing a wide class of transmutations ([**6**], [**10**–**12**], [**21**]). The "heat" equation takes on a central role in this work. Other methods for constructing transmutations include generalized translations (see Carroll [**17**–**18**]) and the method of separation of variables [**16**]. Real transmutations have found application in the construction of fundamental solutions of partial differential equations ([**5**], [**11**–**12**]), the construction of solutions of abstract differential equations ([**17**–**18**], [**22**]), the solution of control problems [**32**], and the development of function theories for a variety of standard and singular partial differential equations [**13**–**14**].

As will be seen, the quasi inner product (*qip*) is simply another, albeit convenient to use, version of the Cauchy integral formula. The use of *qip*'s along with the solutions of the generalized heat problem (1.1) for building up to solutions of other types of evolution problems combines some of the key ingredients of the function theoretic approach and the method of related partial differential equations. It will be seen that this approach permits us to handle a wide variety of solution representation questions for complex and real Cauchy problems by reducing them to questions about the solutions of associated ordinary differential equations. While this paper will focus upon the solution representation problem, the approach should also be viewed as providing a framework for carrying out further studies in such areas as (i) the well and ill-posedness of Cauchy problems, (ii) polynomial representations of solutions of Cauchy problems, (iii) the weakening of the analyticity (entireness) requirements on the data and (iv) the usual related questions when the variables z_j are taken to be real.

In §2, we review the definition and properties of the *qip* and then use it to express certain exponential and other functions in forms that will be more suitable for later purposes. Results on the entireness of properties of *qips* will be given in §3 and these will be invoked in subsequent sections to infer the existence of integrals in the complex transformations. The generalized heat problem will be treated in §4. Solution representations and growth bounds will be obtained and a proof will be given to establish the validity of the solution construction technique. A series representation problem closely related to one considered in [**30**] will be discussed, and an example that involves an operator $P(D)$ with variable coefficients will also be given to indicate that this approach can be extended to apply to numerous other types of "heat" problems. The result of §4 will be applied in §5 to construct formulas for evaluating entire and analytic functions of the operator $P(D)$ acting on entire data. Throughout §4 and §5, symbolic operators will be used to simplify the exposition. §6 is concerned with hypergeometric type Cauchy problems (including a generalized Euler-Poisson-Darboux problem), and §7 covers the application of a related ordinary differential equation technique for obtaining solution representations for two additional Cauchy problems, one of which has the Sobolev type. In the case of the Sobolev problem, three different ways of expressing the solution operator will lead to three different solution representations. Finally, §8 provided examples of related equations in which the transformations involved reduce to real ones. In these cases, the entireness requirement on the data is dropped and results for abstract versions of these problems are given.

2. Quasi inner products and associated identities. Let $f(z_1)$ and $g(z_2)$ denote analytic functions of z_1 and z_2 in open disks D_1 and D_2, each centered at the origin, with

$$f(z_1) = \sum_{n=0}^{\infty} a_n z_1^n \quad \text{and} \quad g(z_2) = \sum_{n=0}^{\infty} b_n z_2^n.$$

For $z_1 \in D_1, z_2 \in D_2$, we define the quasi inner product (*qip*) of $f(z_1)$ and $g(z_2)$, namely $f(z_1) \circ g(z_2)$, by means of the relation

$$(2.1)\quad f(z_1) \circ g(z_2) = (2\pi)^{-1} \int_0^{2\pi} f(z_1 e^{i\theta}) g(z_2 e^{-i\theta}) d\theta = \sum_{n=0}^{\infty} a_n b_n z_1^n z_2^n.$$

As was noted in [**8**], this defines a type of convolution of $f(z_1)$ and $g(z_2)$. If $f(z_1)$ and/or $g(z_2)$ are analytic functions of two or more variables, we use underscores to designate the variables or parameters being used in the formation of the *qip*. Thus, we write

$$f(\underline{z}_1, z_2) \circ g(z_3, \underline{z}_4, z_5)$$

to denote

$$(2\pi)^{-1} \int_0^{2\pi} f(z_1 e^{i\theta}, z_2) g(z_3, z_4 e^{-i\theta}, z_5) d\theta.$$

In this case, the a_n's and the b_n's are, respectively, functions of z_2 and of z_3 and z_5.

With the complex change of variables $s = e^{i\theta}$ in (2.1), that relation becomes

$$(2.2)\qquad f(z_1) \circ g(z_2) = (2\pi i)^{-1} \int_{|s|=1} f(z_1 s) g\Big(\frac{z_2}{s}\Big) \frac{ds}{s}$$

which is just a form of the Cauchy integral formula. An analogous formula was used by J. Hadamard to discuss the singularities of the analytic function $\sum_{n=0}^{\infty} a_n b_n z^n$ in terms of the singularities of $f(z)$ and $g(z)$ [**31**]. From [**8**], we note that the binary operation $\circ$ is commutative and distributive over addition but is not, in general, associative.

Similarly, if p and q are positive integers with $(p, q) = 1$, we define the generalized *qip*

$$f(z_1)_p \circ {}_q g(z_2)$$

by the relation

$$\begin{aligned} f(z_1)_p \circ {}_q g(z_2) &= (2\pi)^{-1} \int_0^{2\pi} f(z_1 e^{pi\theta}) g(z_2 e^{-qi\theta}) d\theta \\ &= \sum_{n=0}^{\infty} a_{qn} b_{pn} z_1^{qn} z_2^{pn}. \end{aligned} \tag{2.3}$$

The operation ${}_p \circ {}_q$ is distributive over the addition but is not commutative or associative. However, we do have

$$f(z_1)_p \circ {}_q g(z_2) = g(z_2)_q \circ {}_p f(z_1). \tag{2.4}$$

We will provide theorems on the entireness properties of these *qips* in §3.

Now, let a and b denote complex parameters with $|b| < 1$. We note that

$$e^{ab} = \sum_{n=0}^{\infty} \frac{a^n b^n}{n!} = e^{\underline{a}} \circ \frac{1}{1 - \underline{b}}.$$

But since $|b| < 1$, we have $1/(1-b) = \int_0^{\infty} e^{-\sigma} e^{b\sigma} d\sigma$. After interchanging orders of integration, we can write

$$e^{ab} = \int_0^{\infty} e^{-\sigma} (e^{\underline{a}} \circ e^{\underline{b}\sigma}) d\sigma. \tag{2.5}$$

Also, we see that

$$e^{ab^p} = e^{\underline{a}}{}_p \circ {}_1 \frac{1}{1 - \underline{b}} = \int_0^{\infty} e^{-\sigma} (e^{\underline{a}}{}_p \circ {}_1 e^{\underline{b}\sigma}) d\sigma. \tag{2.6}$$

If $f(z)$ is entire in z of small growth (say $\rho < 1$), we have

(2.7)

$$\text{(a)} \qquad f(ab) = f(\underline{a}) \circ \frac{1}{1-\underline{b}} = \int_0^\infty e^{-\sigma}(f(\underline{a}) \circ e^{\underline{b}\sigma})\,d\sigma,$$

$$\text{(b)} \qquad f(ab^p) = \int_0^\infty e^{-\sigma}(f(\underline{a})_p \circ {}_1 e^{\underline{b}\sigma})\,d\sigma.$$

The conditions on $f(z)$, as they pertain to Cauchy problems will be considered in §5. Let us note further that

(2.8)

$$\text{(i)} \qquad f(ab) = \int_0^\infty e^{-\sigma}(f(\underline{a}\lambda) \circ e^{\underline{b}\sigma/\lambda})\,d\sigma,$$

$$\text{(ii)} \qquad f(ab^p) = \int_0^\infty e^{-\sigma}(f(\underline{a}\lambda)_p \circ {}_1 e^{\underline{b}\sigma/\lambda^{1/p}})\,d\sigma,$$

in which λ is a positive parameter with $|b|\,|\lambda < 1$ or $|b||\,\lambda^{1/p} < 1$. It is also easy to check that

(2.9)

$$\text{(a)} \qquad f(\underline{a}) \circ g(\underline{b}c) = f(\underline{a}) \circ g(b\underline{c}),$$

$$\text{(b)} \qquad f(\underline{a}b) \circ g(\underline{c}) = f(\underline{a}) \circ g(b\underline{c}),$$

with appropriate restrictions on b in (2.9). These will prove to be convenient later in writing solution operators in alternative ways. Finally, we note some special cases of (2.6) and (2.8) that will be useful in §8:

(2.10)

$$\text{(a)} \qquad \begin{aligned} I_0(ab) &= e^{(a\underline{b})/2} \circ e^{(a\underline{b})/2} \\ &= (2\pi)^{-1} \int_0^{2\pi} e^{ab\cos\theta}\,d\theta, \end{aligned}$$

$$\text{(b)} \qquad \begin{aligned} e^{ab^2} &= \int_0^\infty e^{-\sigma}(e^{a\underline{b}} \circ e^{\underline{b}\sigma})\,d\sigma \\ &= \int_0^\infty e^{-\sigma} I_0(2\sqrt{a\sigma}\,b)\,d\sigma, \quad a > 0. \end{aligned}$$

In these, I_0 denotes the usual modified Bessel function of index 0.

3. Entireness and QIP's. In [**8**], we noted the following results: (a) If $f(z_1)$ and $g(z_2)$ are analytic functions in the open disk's D_1 and D_2 respectively, then $f(z_1) \circ g(z_2)$ is analytic in $z_1 z_2$ (for $z_1 \in D_1, z_2 \in D_2$) and, similarly, $f(z_1)_p \circ {}_q g(z_2)$ is analytic in $z_1^q z_2^p$ and (b) if $f(z_1)$ is analytic in z_1 in D_1 and $g(z_2)$ is entire in z_2, then, for $z_1 \in D_1, f(z_1) \circ g(z_2)$ is entire in $z_1 z_2$. Our study of complex transformations of solutions of heat equations will require that we make use of entire functions of several complex variables. Such functions will enter as data functions and they will appear in the formation of various *qip*'s. We will primarily be concerned with the growth properties of these functions. There are, however, two distinct notions of growth associated with these: growth in the individual variables and overall growth (See [**28**, Chapter 3]). Since we will require both notions, we provide results about them in this section. Let us first recall the following:

Definition 3.1. Let $f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$ in which ξ denotes a single complex variable. Then $f(\xi)$ is entire and of growth (ρ, τ) if

$$\limsup_{n\to\infty} \frac{n}{e\rho} |a_n|^{\rho/n} = \tau \qquad ([\mathbf{3}], [\mathbf{24}]). \tag{3.1}$$

This implies the existence of a positive constant M so that

$$|f(\xi)| \leq M e^{\tau|\xi|^{\rho}} \quad \forall \text{ complex } \xi. \tag{3.2}$$

Throughout this section, *we assume that the ρ's and the τ's are strictly positive.*

We now give results on the overall growth:

Theorem 3.1. *Let $f(z_1)$ be entire in z_1 of growth (ρ_1, τ_1), and let $g(z_2)$ be entire in z_2 of growth (ρ_2, τ_2). If $\rho_1 = \alpha\rho_2$ with $0 < \alpha \leq 1$, then $f(z_1) \circ g(z_2)$ is entire in $z_1 z_2$ of growth (ρ, τ), where*

$$\rho = \rho_1 \rho_2 / (\rho_1 + \rho_2)$$

and

$$\tau = \left(\frac{\rho_1+\rho_2}{\rho_2}\right)\tau_1^{\rho_2/(\rho_1+\rho_2)}\left(\frac{\rho_2\tau_2}{\rho_1}\right)^{\rho_1/(\rho_1+\rho_2)}.$$

PROOF. Given $\varepsilon > 0$, it follows by (3.1) that there exists a positive integer N such that if $n \geq N$, then

$$\begin{aligned} |a_n| &\leq (\rho_1^{n/\rho_1}[e(\tau_1+\varepsilon)]^{n/\rho_1})/n^{n/\rho_1} \\ |b_n| &\leq (\rho_2^{n/\rho_2}[e(\tau_2+\varepsilon)]^{n/\rho_2})/n^{n/\rho_2}. \end{aligned} \tag{3.3}$$

If we multiply these together, use the value of α and simplify, we get

$$|a_n b_n| \leq \frac{(\alpha\rho_2)^{n/\alpha\rho_2}\rho_2^{n/\rho_2}e^{n(1/n\rho_2+1/\rho_2)}(\tau_1+\varepsilon)^{n/\alpha\rho_2}(\tau_2+\varepsilon)^{n/\rho_2}}{n^{n(1/\alpha\rho_2+1/\rho_2)}} \tag{3.4}$$

for $n \geq N$. If we let $\omega = \alpha/(1+\alpha)$, then some algebraic manipulations permit us to show that the second member of (3.4) can be expressed in the form

$$\begin{aligned} &\frac{[(\rho_2\omega)^{n/\rho_2\omega}e^{n/\rho_2\omega}(1+\alpha)^{n/\rho_2\omega}(\tau_1+\varepsilon)^{n/\rho_2\alpha}\left(\frac{\tau_2+\varepsilon}{\alpha}\right)^{n/\rho_2}}{n^{n/\rho_2\omega}} \\ &= \frac{\left\{(\rho_2\omega)^{n/\rho_2\omega}\left[e(1+\alpha)(\tau_1+\varepsilon)^{1/(1+\alpha)}\left(\frac{\tau_2+\varepsilon}{\alpha}\right)^{\omega}\right]^{n/\rho_2\omega}\right\}}{n^{n/\rho_2\omega}}. \end{aligned} \tag{3.5}$$

The stated result follows from the last member of this by replacing ω by $\alpha/(\alpha+1)$ and α by ρ_1/ρ_2. $\square$

EXAMPLE. We note that e^{z_1} and e^{z_2} are both entire of growth $(1,1)$. By Theorem 3.1, $e^{z_1} \circ e^{z_2}$ has growth $(1/2, 2)$. But $e^{z_1} \circ e^{z_2} = I_0(2z_1z_2)$ and it is easy to check, using Definition 3.1 directly, that $I_0(2z_1z_2)$ has growth $(1/2, 2)$ in z_1z_2.

By analogous but simpler computation, one can prove

THEOREM 3.2. *Let $f(z_1)$ be analytic in z_1 for $|z_1| < R, 0 < R < \infty$, and let $g(z_2)$ be entire of growth (ρ, τ) in z_2. Then, for $|z_1| < R, f(z_1) \circ g(z_2)$ is entire in z_1z_2 of growth $(\rho, \tau/R^\rho)$.*

In the proof of Theorem 3.2, the restriction imposed on z_1 is necessitated by the requirement that the integral in (2.1) be well defined for all $z_1 z_2$. In practice, one can select $z_1 \in D_1, z_1 \neq 0$, and then choose z_2 so that $z_1 z_2$ takes on the value one wishes to use.

By using calculations similar to those employed in the proof of Theorem 3.1, we can prove

THEOREM 3.3. *Let $f(z_1)$ be entire in z_1 of growth (ρ_1, τ_1), and let $g(z_2)$ be entire in z_2 of growth (ρ_2, τ_2), where $\rho_1 = \alpha\rho_2$ with $0 < \alpha \leq 1$. Then $f(z_1)_p \circ {}_q g(z_2)$ is entire in $z_1^q z_2^p$ of growth (ρ, τ), where*

$$\rho = \rho_1 \rho_2 / [p\rho_1 + q\rho_2]$$

and

$$\tau = \frac{(p\rho_1 + q\rho_2)}{\rho_2} \left(\frac{\tau_1}{q}\right)^{q\rho_2/(p\rho_1 + q\rho_2)} \left(\frac{\rho_2 \tau_2}{p\rho_1}\right)^{p\rho_1/(p\rho_1 + q\rho_2)}.$$

THEOREM 3.4. *Let $f(z_1)$ be analytic in z_1 for $|z_1| < R$, and let $g(z_2)$ be entire in z_2 of growth (ρ, τ). Then, for $|z_1| < R, f(z_1)_p \circ {}_q g(z_2)$ is entire in $z_1^q z_2^p$ of growth $(\rho/p, \tau/(R)^{\rho q})$.*

For the case of the growth in the individual variables, we make use of (3.2)

THEOREM 3.5. *Let $f(z_1)$ be entire in z_1 of growth (ρ_1, τ_1), and let $g(z_2)$ be entire in z_2 of growth (ρ_2, τ_2). Then there exists a positive constant M such that*

$$|f(z_1)_p \circ {}_q g(z_2)| \leq M e^{\tau_1 |z_1|^{\rho_1} + \tau_2 |z_2|^{\rho_2}}.$$

PROOF. If

$$|f(z_1)| \leq M_1 e^{\tau_1 |z_1|^{\rho_1}} \quad \text{and} \quad |g(z_2)| \leq M_2 e^{\tau_2 |z_2|^{\rho_2}},$$

then it readily follows that

$$\begin{aligned}|f(z_1)_p \circ {}_q g(z_2)| &\le (2\pi)^{-1}\int_0^{2\pi} |f(z_1 e^{pi\theta})|\,|g(z_2 e^{-qi\theta})|d\theta \\ &\le (2\pi)^{-1}\int_0^{2\pi} (M_1 e^{\tau_1 |z_1 e^{pi\theta}|^{\rho_1}})(M_2 e^{\tau_2 |z_2 e^{-qi\theta}|^{\rho_2}})d\theta \\ &= M e^{\tau_1|z_1|^{\rho_1}+\tau_2|z_2|^{\rho_2}} \quad \text{with } M = M_1M_2. \ \square\end{aligned}$$

In the sections to follow, we will employ entire data functions $\varphi(z_1,\dots,z_n)$ that satisfy a condition of the form

$$|\varphi(z_1,\dots,z_n)| \le M e^{\sum_{j=1}^n \tau_j |z_j|^{\rho_j}}, \tag{3.6}$$

with $0<\rho_j<1, j=1,2,\dots,n$, and $\tau_j>0, j=1,\dots,n$.

We denote the class of all such entire functions by $\propto(z)$. Since we can construct a variety of elements of $\propto(z)$ by applying *qips* to entire functions of single complex variables, it follows from Theorem 3.5 that $\propto(z)\neq\varnothing$.

In view of the fact that $|z_j+\xi_j|^{\rho_j} \le |z_j|^{\rho_j}+|\xi_j|^{\rho_j}$ for z_j and ξ_j complex and $0<\rho_j<1$, it follows that if $\varphi(z_1,\dots,z_n)\in\propto(z)$ then

$$|\varphi(z_1+\xi_1,\dots,z_n+\xi_n)| \le M e^{(\sum_{j=1}^n \tau_j|z_j|^{\rho_j}+\sum_{j=1}^n \tau_j|\xi_j|^{\rho_j})} \tag{3.7}$$

for some positive constant M and some set of $\rho_j, j=1,\dots,n$, with $0<\rho_j<1$. The inequality (3.7) will be used in §4 for imposing bounds on complex translations of $\varphi(z_1,\dots,z_n)$. In some cases of special interest, one can select some of the ρ_j to have the value 1 provided that the corresponding τ_j are suitably restricted (see the remark in §4).

4. Generalized heat problems. Taking z, D, and $P(D)$ as in the introduction, we now wish to construct a solution of the complex generalized heat problem (1.1) in which $\varphi(z)\in\propto(z)$. To simplify the exposition, we will use symbolic operators to denote solutions of (1.1). Thus, we can write

$$H(z,t) = e^{tP(D)}\varphi(z) \tag{4.1}$$

to denote such a solution. If $P(D)$ can be expressed in the form $\sum_{l=1}^{N} P_l(D)$ in which the $P_l(D)$ are multinomials in the D_j, then we write, as usual,

$$
\begin{aligned}
H(z,t) &= \prod_{l=1}^{N} e^{tP_l(D)}\varphi(z) \\
&= \Big\{ \prod_{l=1}^{N-1} e^{tP_l(D)} \Big\} \big\{ e^{tP_N(D)}\varphi(z) \big\}.
\end{aligned} \tag{4.2}
$$

The $P_l(D)$ in this may involve positive integer powers of some one of the D_j, or they may involve products of powers of different D_j's. For our purposes, it suffices to show how one can construct a solution of (1.1) corresponding to the following cases: (I.) $P(D) = \mu D_1^p$ and (II.) $P(D) = \mu D_1^p D_2^q$. In these, μ is a complex number and p and q are positive integers. In view of the commutativity of the exponentials $e^{tP_l(D)}$ in (4.2), repeated applications of our methods will permit the evaluation of the last member of (4.2).

Our task in this section is to re-express the right hand member of (4.1) and/or (4.2) in terms of standard analytical tools that will permit us to establish rigorously that we do obtain a solution of (1.1). The expression obtained will also permit us to give growth bounds on $|H(z,t)|$. For these purposes, we will make use of the following relations:

(4.3)

(a)
$$
e^{aD_j} \cdot \varphi(z_1, \dots, z_j, \dots z_n) = \varphi(z_1, \dots, z_j + a, \dots, z_n)
$$
a complex (generalized translation),

(b)
$$
\begin{aligned}
&e^{aD_j^p} \cdot \varphi(z_1, \dots, z_j, \dots, z_n) \\
&= (2\pi)^{-1} \int_0^\infty e^{-\sigma} \Big\{ \int_0^{2\pi} e^{ae^{pl\theta}} \varphi(z_1, \dots, z_j + \sigma e^{-i\theta}, \dots, z_n) d\theta \Big\} d\sigma,
\end{aligned}
$$
a complex, p a positive integer.

Formally, the second of these follows by replacing b in (2.6) by D_j and then interpreting $e^{\sigma e^{-i\theta} D_j} \cdot \varphi(z)$ by means of (4.3a). The validity of (4.3b) and analogous formulas for $\varphi(z) \in \propto$ will be established in Theorem 4.1.

A. *Solution construction procedure.*

Case I. The relation (4.3b) with $a = \mu t$ and $j = 1$ for this $P(D)$ leads to

(4.4)
$$H(z,t) = e^{t\mu D_1^p}\varphi(z_1,\dots,z_n) = (2\pi)^{-1}\int_0^\infty e^{-\sigma}\Big\{\int_0^{2\pi} e^{\mu t e^{pi\theta}}\varphi(z_1+\sigma e^{-i\theta},\dots,z_n)d\theta\Big\}d\sigma.$$

This can also be written in the more compact form

$$H(z_1,t) = \int_0^\infty e^{-\sigma}(e^{\mu t}{}_p \circ {}_1\varphi(z_1+\underline{\sigma}, z_2,\dots,z_n))d\sigma. \tag{4.5}$$

With the entireness condition on $\varphi(z)$, the results of §3 show that the repeated integral in the last member of (4.4) exists and that the orders integration in that integral can be exchanged (see part B below). We now prove:

THEOREM 4.1 *The function $H(z,t)$ defined by the last member of* (4.4) *is a solution of* (1.1) *corresponding to $P(D) = \mu D_1^p$ and $\varphi(z) \in \propto(z)$.*

PROOF. At $t = 0$, the inner integral in the last member of (4.4) becomes
$$\int_0^{2\pi}\varphi(z_1+\sigma e^{-i\theta},\dots,z_n)d\theta,$$
and, since $\varphi(z) \in \propto$, this clearly has the value $2\pi\varphi(z)$. It then readily follows that the initial condition of (1.1) is satisfied by the last member of (4.4).

Next, we differentiate with respect to t to get

(4.6)
$$\partial H(z,t)/\partial t = \mu\int_0^\infty e^{-\sigma}\Big\{(2\pi)^{-1}\int_0^{2\pi} e^{pi\theta}e^{\mu t e^{pi\theta}}\varphi(z_1+\sigma e^{-i\theta},\dots,z_n)d\theta\Big\}d\sigma.$$

Also,

(4.7)
$$\partial H(z,t)/\partial z_1 = (2\pi)^{-1}\int_0^{2\pi} e^{\mu t e^{pi\theta}}\Big\{\int_0^\infty e^{-\sigma}\frac{\partial\varphi}{\partial z_1}(z_1+\sigma e^{-i\theta},\dots,z_n)d\sigma\Big\}d\theta.$$

Upon replacing $\partial\varphi/\partial z_1$ in this by $e^{i\theta}\partial\varphi/\partial\sigma$, it follows by an integration by parts, that

(4.8)
$$\begin{aligned}\partial H(z,t)/\partial z_1 &= (2\pi)^{-1}\int_0^{2\pi} e^{\mu t e^{pi\theta}} e^{i\theta}\Big\{\int_0^\infty e^{-\sigma}\frac{\partial\varphi}{\partial\sigma}d\sigma\Big\}d\theta\\ &= (2\pi)^{-1}\int_0^{2\pi} e^{\mu t e^{pi\theta}} e^{i\theta}\{e^{-\sigma}\varphi(z_1+\sigma e^{-i\theta},\dots,z_n)|_0^\infty\}d\theta\\ &\quad + (2\pi)^{-1}\int_0^{2\pi} e^{\mu t e^{pi\theta}} e^{i\theta}\Big\{\int_0^\infty e^{-\sigma}\varphi(z_1+\sigma e^{-i\theta},\dots,z_n)d\sigma\Big\}d\theta.\end{aligned}$$

The growth condition on $\varphi(z)$ shows that the first term in the left member of (4.8) reduces to

$$-(2\pi)^{-1}\varphi(z)\int_0^{2\pi} e^{\mu t e^{pi\theta}} e^{i\theta}d\theta,$$

and this clearly vanishes since the expansion of the integrand leads to positive integer powers of $e^{i\theta}$ and $\int_0^{2\pi} e^{qi\theta}d\theta = 0$ for q a positive (or negative) integer. Hence, (4.8) becomes

$$\partial H(z,t)/\partial z_1 = (2\pi)^{-1}\int_0^{2\pi} e^{\mu t e^{pi\theta}}\cdot e^{i\theta}\Big\{\int_0^\infty e^{-\sigma}\varphi(z_1+\sigma e^{-i\theta},\dots,z_n)d\sigma\Big\}d\theta.$$

This right member of this is just like the last member of (4.4) except that it contains the additional factor $e^{i\theta}$. Repetitions of this argument show that each additional differentiation of H with respect to z_1 introduces one additional factor $e^{i\theta}$ into the integrand. After p differentiations of $H(z,t)$ with respect to z_1, we obtain, except for the factor μ, the right hand member of (4.6). Thus, the relation (4.3b) has permitted us to construct a solution of (1.1).

Case II. If $P(D) = \mu D_1^p D_2^q$, we require a procedure for peeling off the effect of the differential operator D_2^q to get a reduction to case I. Again, we call upon (2.6) with p replaced by q. If we then replace a by $\mu t D_1^p$ and b by D_2, we get

(4.9)

$$\begin{aligned}&e^{t\mu D_1^p D_2^q}\varphi(z)\\&= \int_0^\infty e^{-\sigma}\Big\{(2\pi)^{-1}\int_0^\infty e^{t\mu e^{pi\theta}D_1^p}\varphi(z_1, z_2+\sigma_1 e^{-i\theta_1},\dots,z_n)d\theta_1\Big\}d\sigma.\end{aligned}$$

The procedure of Case I can now be used in the right member of this to complete the solution construction. This leads to a 4-fold integral. The proof of Case I can be carried over to Case II. For a somewhat different approach to solving generalized heat problems with mixed derivatives, see [**7**].

B. *Growth bounds on solutions.* To obtain growth bounds on the $H(z,t)$, suppose we first take $P(D) = \mu D_1^p$ as in Case I above with $\varphi(z) = \varphi(z_1)$ and $|\varphi(z_1)| \le M e^{\tau|z_1|^\rho}, \rho < 1$. Then

$$H(z,t) = \int_0^\infty e^{-\sigma}\Big\{(2\pi)^{-1}\int_0^{2\pi} e^{\mu t e^{pi\theta}}\cdot\varphi(z_1+\sigma e^{i\theta})d\theta\Big\}d\sigma.$$

Hence

(4.10)

$$\begin{aligned}|H(z_1,t)| &\le \int_0^\infty e^{-\sigma}\Big\{(2\pi)^{-1}\int_0^{2\pi}|e^{\mu t e^{pi\theta}}|\cdot|\varphi(z_1+\sigma e^{-i\theta})|d\theta\Big\}d\sigma\\&\le \int_0^\infty e^{-\sigma}\Big\{(2\pi)^{-1}\int_0^{2\pi} e^{|\mu t|}Me^{\tau[|z_1|^\rho+|\sigma|^\rho]}d\theta\Big\}d\sigma \text{ (by (3.7))}\\&= M^*(\rho,\tau)e^{|\mu t|+\tau|z_1|^\rho},\end{aligned}$$

where

$$M^*(\rho,\tau) = M\int_0^\infty e^{-\sigma+\tau|\sigma|^\rho}d\sigma,\ 0<\rho<1.$$

Thus, we see that the bound on $H(z,t)$ depends upon the constants ρ and τ but not upon the power p of the operator D_1. Repeated applications of the type of estimate used to obtain (4.10) permits us to show, in general, that if $H(z,t)$ satisfies (1.1) with $\varphi(z)\in\propto(z)$ then

$$|H(z,t)| \le \tilde{M}(\rho,\tau)e^{K|t|+\sum_{j=1}^n \tau_j|z_j|^{\rho_j}}, \tag{4.10*}$$

where K is the sum of the absolute values of the coefficients of the multinomial and $\tilde{M}(\rho, \tau)$ is a generic constant depending only on the ρ_i's and the τ_i's.

REMARK. We could select $\rho = 1$ in (4.10) provided that the associated τ satisfies $0 < \tau < 1$. In this case, the M^* in (4.10) has the value $(1-\tau)^{-1}$. Thus, we could have taken $\propto(z)$ to be a somewhat larger class to include functions $\varphi(z)$ that have one or more of the $\rho_j = 1$ provided that the sum of the associated τ_j have a sum that is less than 1.

C. *Generalized heat series representations.* The bounds (4.10) and (4.10*) can be applied to the problem of representing solutions of cases of (1.1) in terms of special polynomials or multinomials. In order to provide an example of this while keeping the notation simple, we restrict ourselves to the one dimensional problem

$$H_t(z,t) = D^p H(z,t), H(z,0) = \varphi(z) \tag{4.11}$$

in which z denotes a single complex variable and $\varphi(z)$ is entire of growth (ρ, τ), $0 < \rho < 1$. Let $P_n(z,t)$ denote a polynomial solution of (4.11) corresponding to the condition $H(z,0) = z^n$. Now, $P_n(z,t)$ is easily seen to be defined by the generating relation

$$e^{az+a^p t} = \sum_{n=0}^{\infty} \frac{P_n(z,t) a^n}{n!}$$

and, using (4.5), is given by

$$P_n(z,t) = \int_0^\infty e^{-\sigma} (e^{\underline{t}}{}_p \circ {}_1 (x + \underline{\sigma})^n) d\sigma.$$

The evaluation of this integral finally gives

$$P_n(z,t) = \sum_{l=0}^{[n/p]} \frac{n! z^{n-pl} t^l}{l!(n-pl)!}. \tag{4.12}$$

By considering the maximum of $\xi^n e^{-T\xi^\rho}$ for $\xi \geq 0$ and $T > 0$, it is easy to show that

$$|z|^n \leq \left(\frac{n}{\rho T}\right)^{n/\rho} e^{-n/\rho} e^{T|z|^\rho}. \tag{4.13}$$

Then, from (4.10) with $\mu = 1$, we get

$$|P_n(z,t)| \le \left(\frac{n}{\rho T}\right)^{n/\rho} e^{-n/\rho} e^{|t|+T|z|^\rho} k(\rho,\tau), \tag{4.14}$$

where

$$k(\rho,\tau) = \int_0^\infty e^{-\sigma+T|\sigma|^\rho} d\sigma.$$

Now, let $\varphi(z) = \sum_{n=0}^\infty a_n z^n$ be an entire of growth $(\rho,\tau), \rho < 1$. Select $T > \tau$ in (4.14) and select $\varepsilon > 0$ such that $\tau + \varepsilon < T$. Using (3.1), there exists a positive integer N such that if $n \ge N$, then

$$|a_n| \le \left(\frac{e\rho}{n}\right)^{n/\rho} (\tau+\varepsilon)^{n/\rho}.$$

Finally, let

$$H(z,t) = \sum_{n=0}^\infty a_n P_n(z,t). \tag{4.15}$$

THEOREM 4.2. *The series* (4.15) *converges for all complex* z *and real* t *and uniformly so in compact subsets of* (z,t) *space.*

PROOF. From (4.15), we have

$$\begin{aligned} |H(z,t)| &\le \sum_{n=0}^N |a_n|\ |P_n(z,t)| \\ &\quad + \sum_{n=N+1}^\infty |a_n| \cdot |P_n(z,t)|. \end{aligned} \tag{4.16}$$

Using the bound (4.16) on $|P_n(z,t)|$ and the estimate on $|a_n|$, we see that the bound on the second sum in (4.16) is given by

$$K(\rho,T) e^{|t|+T|z|^\rho} \sum_{n=N+1}^\infty \left(\frac{\tau+\varepsilon}{T}\right)^{n/\rho}.$$

Since the series of constants in this converges, the theorem follows. □

When $p = 2$, the equation (4.11) is the classical heat equation. The representation of its solutions in terms of heat polynomials ((4.12) with $p = 2$) with z real was treated in [**22**].

D. *A radially symmetric operator.* In the introduction, we selected $P(D)$ to be a multinomial operator in the D_j with constant coefficients. We need not, however, so restrict $P(D)$. As an example, consider the equation

$$H_t(z,t) = \Delta_\mu^p H(z,t), \quad H(z,0) = \varphi(z) \tag{4.17}$$

in which z is a single complex variable, p is a positive integer, $\Delta_\mu = D^2 + \frac{\mu}{z} D_z (\mu > 0)$, and $\varphi(z)$ is entire in z^2 of growth (ρ, τ) with $\rho < 1/2$. Under these conditions, the radial heat problem

$$\omega_\eta(z,\eta) = \Delta_\mu \omega(z,\eta), \quad \omega(z,0) = \varphi(z) \tag{4.18}$$

has a solution that can be represented in terms of the radial heat polynomials $R_n(z,\eta)$ throughout (z,η) space, η real (see [**4**]). Moreover, this solution function satisfies a growth condition of the form

$$|\omega(z,\eta)| \leq M(\rho,\tau) e^{|\eta| + \tau |z^2|^\rho}.$$

Then a solution of (4.17) can be expressed as

$$\begin{aligned} H(z,t) &= e^{t\Delta_\mu^p} \varphi(z) \\ &= (2\pi)^{-1} \int_0^\infty e^{-\sigma} \Big\{ \int_0^{2\pi} e^{te^{pi\theta}} \big(e^{\sigma e^{-i\theta} \Delta_\mu} \varphi(z) \big) d\theta \Big\} d\sigma. \end{aligned} \tag{4.19}$$

Using the function $\omega(z,\eta)$, we finally obtain

$$H(z,t) = \int_0^\infty e^{-\sigma} \big(e^{\underline{t}}{}_p \circ {}_1 \omega(z, \underline{\sigma}) \big) d\sigma. \tag{4.20}$$

5. Functions of differential operators. For the moment, let $f(\xi)$ be an entire function of the single complex variable ξ. With t real,

$P(D)$ as in the introduction, and $\varphi(z) \in \propto(z)$ we define $f(tP(D))\varphi(z)$ by means of the relations

$$
(5.1)\quad f(tP(D))\varphi(z) = \begin{cases} \text{(i)}\ \displaystyle\int_0^\infty e^{-\sigma}(f(\underline{t}) \circ H(z,\underline{\sigma}))d\sigma \\ \quad \text{or} \\ \text{(ii)}\ \displaystyle\int_0^\infty e^{-\sigma}\big(f(\underline{\lambda}) \circ H\big(z, \frac{\underline{t}\sigma}{\lambda}\big)\big)d\sigma, \quad \lambda > 0, \end{cases}
$$

in which $H(z,t)$ is a solution of (1.1) corresponding to the operator $P(D)$ and the data function $\varphi(z)$. The definition (5.1)(i) is motivated by (2.8a) by replacing a in that formula by t, b by $P(D)$, and then formally applying the heat solution operator $e^{\underline{\sigma}P(D)}$ to $\varphi(z)$ to obtain $H(z,\underline{\sigma})$. The formula (5.16) is an alternative version of this (see formula (2.8)(i)).

We now make an appraisal of the integral (5.1)(ii) to determine when it makes sense and to determine if we can relax the conditions on f. Suppose we let $h(\lambda) = \max_{0\le\theta\le 2\pi}|f(\lambda e^{i\theta})|$. Then it follows that

$$
(5.2)\qquad \left|\left(f(\underline{\lambda}) \circ H\left(z, \frac{\underline{t}\sigma}{\lambda}\right)\right)\right| \le h(\lambda)M^*(\rho,\tau)e^{\frac{\kappa|t|\sigma}{\lambda}}e^{\sum_{j=1}^n \tau_j|z_j|^{\rho_j}}.
$$

Hence,

$$
(5.3)\qquad \begin{aligned} &\left|\int_0^\infty e^{-\sigma}\left(f(\underline{\lambda}) \circ H\left(z, \frac{\underline{t}\sigma}{\lambda}\right)\right)d\sigma\right| \\ &\qquad\qquad \le h(\lambda)M^*(\rho,\tau)e^{\sum_{j=1}^n \tau_j|z_j|^{\rho_j}} \int_0^\infty e^{-\sigma(1-\frac{k|t|}{\lambda})}d\sigma. \end{aligned}
$$

The integral on the right side of this converges provided that $1 - K|t|/\lambda > 0$. If $f(\xi)$ is entire and $|t|$ is large, we can select λ so large that $K|t|/\lambda < 1$. Under these circumstances, we see that (5.1)(ii) is defined for all $|t|$.

Next, suppose that $f(\xi)$ is analytic in a disk of radius R centered at the origin. Then the λ in (5.3) must be restricted so that $0 < \lambda < R$. The integral in the right member of (5.3) then converges only if $|t| < R/K$, i.e., in a time strip. If we know that $H(z, t\sigma/\lambda)$ has an overall growth $(\rho,\tau), \rho < 1$, then Theorem 3.2 shows that $(f(\underline{\lambda}) \circ H(z, \underline{t}\sigma/\lambda))$ has an overall growth $(\rho, \tau/R^\rho)$ so that (5.1)(ii) makes sense for all $|t|$.

If we make use of (2.8)(ii), we can also show that, for k a positive integer,

$$
\begin{aligned}
(5.4)\qquad & f(t(P(D))^k)\varphi(z) \\
&= \int_0^\infty e^{-\sigma}\Big\{(2\pi)^{-1}\int_0^{2\pi} f(\lambda^k e^{ki\theta})H\big(z, \frac{\sigma t^{1/k}}{\lambda}e^{-i\theta}\big)\, d\theta\Big\} d\sigma.
\end{aligned}
$$

In the work to follow, $f(tP(D))$ (or $f(t(P(D)^k))$ is taken to be a formal solution operator for some Cauchy problem in which the initial conditions involve one piece of non zero data $\varphi(z)$ and 0's. To construct f, one can use an associated problem in ordinary differential equations. This is constructed from the Cauchy problem by replacing $P(D)$ by a parameter ν (we usually take $\nu < 1$), $\varphi(z)$ by 1 and 0 data by 0's. The solution of this associated problem yields $f(t\nu)$ (or$f(t\nu^k)$). One then obtains the required formal solution operator for the Cauchy problem by replacing ν by $P(D)$.

The Example 7B on a Sobolev type problem will illustrate that we have, in fact, much more flexibility in constructing solution representations of Cauchy problems than would be indicated here. Depending on the form of $f(t\nu)$, it may be possible to obtain various solution representations through a judicious *qip* decomposition of $f(t\nu)$ or through some other means of expressing $f(t\nu)$.

6. Hypergeometric type cauchy problems. Let p and q be non-negative integers with $p < q+1$ and consider the Cauchy problem

$$
(6.1)\qquad \Big[tD_t\prod_{j=1}^{q}(tD_t+\beta_j-1) - tP(D)\prod_{l=1}^{p}(tD_t+\alpha_l)\Big]\nu(z,t) = 0
$$

$$
\nu(z,0) = \varphi(z), \quad \varphi(z) \in \propto(z).
$$

in which the $\beta_j > 0$. A solution of this is given formally by

$$
(6.2)\qquad \nu(z,t) = {}_pF_q(\alpha_1\dots,\alpha_p;\beta_1,\dots,\beta_q;tP(D))\cdot\varphi(z).
$$

Using (5.3)(ii), this can be expressed as

$$
(6.3)\qquad \nu(z,t) = \int_0^\infty e^{-\sigma}\Big(F_q(\alpha_1\dots\alpha_p;\beta_1,\dots,\beta_q;\underline{\lambda})\circ H\big(z,\frac{\sigma \underline{t}}{\lambda}\big)\Big)d\sigma
$$

with $H(z,t)$ as in (1.1). If the problems (1.1) and (6.1) are well-posed when $z_j = x_j,\ x_j$ real $(j = 1, \dots, n)$ and $t > 0$, then one could use the methods of [**6**] to construct a set of transmutations (involving inverse Laplace transforms and convolutions) to relate the solution of (6.1) to the solution of (1.1). For entire data, the complex transformation provides a more direct means for solving (6.1).

Now, suppose we have $p = q + 1$. The hypergeometric function appearing in the integrand of (6.3) converges absolutely for $\lambda < 1$, diverges for $\lambda > 1$, and has a variety of possible behaviors at $\lambda = 1$ (depending upon the values of the parameters α_l and β_j). In this case, the ${}_pF_q$ function is analytic with $R = 1$ and the problem (6.1) can be guaranteed to have a solution only in the time strip $|t| < 1/K$. The behavior of solutions as one approaches the boundaries of this time strip are not considered here. The interested reader is referred to, for example, [**27**] for relevant information.

We complete this section by considering two examples of Cauchy problems having solutions that tie in the hypergeometric functions. These will call upon the construction and solution of associated problems in ordinary differential equations.

Problem 6A.

$$\begin{aligned} & t\frac{\partial^p W(z,t)}{\partial t^p} + a\frac{\partial^{p-1} W(z,t)}{\partial t^{p-1}} - t\{P(D)\}^p W(z,t) = 0, \\ & \qquad\qquad\qquad\qquad a > -1,\ t \neq 0 \\ & W(z,0) = \varphi(z), \quad \partial^j W(z,t)/\partial t^j|_{t=0} = 0, \quad j = 1, 2, \dots, p-1, \end{aligned} \tag{6.4}$$

where $\varphi(z) \in \propto(z)$. This is a generalization of the familiar Euler-Poisson-Darboux problem.

A problem in ordinary differential equations associated with problem (6.4) is given by

$$\begin{aligned} & tS^{(p)}(t) + aS^{(p-1)}(t) - tv^p S(t) = 0 \\ & S(0) = 1, \quad S^{(j)}(0) = 0, \quad j = 1, \dots, p-1. \end{aligned} \tag{6.5}$$

It is not difficult to show that this has the series solution

$$S(t) = \sum_{m=0}^{\infty} \frac{(1)_m \left(\frac{1}{p}\right)_m}{\left(\frac{1+a}{p}\right)_m m!} \frac{(vt)^{mp}}{(mp)!} \tag{6.6}$$

in which $(A)_m = A(A+1)\dots(A+m-1), m \geq 1$, and $(A)_0 = 1$. Taking $0 < \alpha < 1$, the series (6.6), using the *qip* (2.3), can be rewritten as

$$S(t) = {}_2F_1\left(a, \frac{1}{p}; \frac{1+a}{p}; \underline{\alpha}\right)_p \circ_1 e^{\underline{t}v/\alpha^{1/p}}. \tag{6.7}$$

Then a solution of (6.4) can be expressed symbolically as $S(tP(D))\varphi(z)$. From our discussion in §5, this leads to

$$W(z,t) = {}_2F_1\left(1, \frac{1}{p}; \frac{1+a}{p}; \underline{\alpha}\right)_p \circ_1 H(z, \underline{t}/\alpha^{1/p}). \tag{6.8}$$

In view of the fact that $S(t)$ was constructed to satisfy (6.5), one can use arguments similar to the ones used in the proof of Theorem 4.1 and the proofs in [**6**] to show that the function $W(z,t)$ defined by (6.8) is, indeed, a solution of (6.4).

If we take $n = 1, z_1 = z$ and $P(D) = D_1 = D$ in (6.4), it follows from our discussion in §5 that if $\varphi(z)$ has growth $(\rho, \tau), 0 < \rho < 1$, then

$$|W(z,t)| \leq M^*(\rho,\tau)\ {}_2F_1\left(1, \frac{1}{p}; \frac{1+a}{p}; \alpha\right) e^{\tau|z|^p + |t|/\alpha^{1/p}}. \tag{6.9}$$

Note that when $p = 2$, the problem (6.4) reduces to the classical E.P.D. problem. An alternative function theoretic technique for constructing its solution may be found in [**1**].

Problem 6B.

$$\begin{aligned} &\frac{\partial^p W(z,t)}{\partial t^p} - t^q\{P(D)\}^k W(z,t) = 0, \quad t > 0 \\ &W(z,0) = \varphi(z), \quad \partial^j W(z,t)/\partial t^j|_{t=0} = 0, \quad j = 1, \dots, p-1, \end{aligned} \tag{6.10}$$

in which p, q, k are positive integers with $k < p$. The associated problem in ordinary differential equations is given by

$$y^{(p)}(t) - v^k t^q y(t) = 0, \quad y(0) = 1, \quad y^{(j)}(0) = 0, \quad j = 1, \dots, p-1. \tag{6.11}$$

After some lengthy calculations, one obtains

$$y(t) = {}_0F_{p-1}\left(\_; 1 - \frac{1}{p+q}, \dots, 1 - \frac{p-1}{p+q}; Tv^k\right)_k \circ_1 \frac{1}{1-\underline{\nu}} \tag{6.12}$$

where $T = t^{p+q}/(p+q)^p$. Using (2.7b), we get

(6.13)
$$y(t) = \int_0^\infty e^{-\sigma}\left\{{}_0F_{p-1}\left(\_;1-\frac{1}{p+q},\dots,1-\frac{p-1}{p+q};\underline{T}\right)_k \circ {}_1e^{v\underline{\sigma}}\right)\right\}d\sigma$$

provided $0 < v < 1$. A formal solution operator for Problem 6B follows by replacing v in (6.13) by $P(D)$. Finally, using the results of §5, we obtain

(6.14)
$$W(z,t) = \int_0^\infty e_0^{-\sigma}\left(F_{p-1}\left(\_;1-\frac{1}{p+q},\dots,1-\frac{p-1}{p+q};\underline{T}\right)\ {}_k\circ_1 H(z,\underline{\sigma})\right)d\sigma$$

as a solution of (6.10).

7. Additional examples. In this section, we provide two further examples of Cauchy problems. The main concern in the first of these relates to the growth bounds. A real parameter η is introduced into the solution function and growth bounds on the solution are obtained in terms of η. For a fixed t, a critical equation is obtained for η that will minimize the growth bounds. The second example involves a Sobolev type equation.

PROBLEM 7A.

$$\begin{aligned}(7.1)\quad &\frac{\partial^p W(z,t)}{\partial t^p} - (P(D))^q W(z,t) = 0, \quad (p,q)=1,\\ &W(z,0)=\varphi(z), \quad \partial^j W(z,t)/\partial t^j|_{t=0} = 0, \quad j=1,2,\dots,p-1.\end{aligned}$$

Using the now familiar method of associated O.D.E.'s, we find that the required function $f(\nu t)$ for obtaining a solution operation for the problem (7.1) is given by

$$(7.2)\qquad f(t\nu) = (2\pi)^{-1}\int_0^\infty e^{-\sigma}\left\{\int_0^{2\pi} e^{tqi\theta}e^{\nu\sigma e^{-pi\theta}}d\theta\right\}d\sigma.$$

Upon replacing ν in this by $P(D)$ and operating on $\varphi(z)$, we obtain

$$(7.3)\qquad W(z,t) = \int_0^\infty e^{-\sigma}(e^{\underline{t}/\eta^p}\ {}_q\circ_p H(z,\eta^p\underline{\sigma}))d\sigma$$

after introducing the parameter $\eta, 0 < \eta < 1$.

Now, suppose we take the dimension $n = 1$ and let $P(D) = D$. If $|\varphi(z)| \le Me^{\tau|z|^\rho}, 0 < \rho < 1$, then, as usual,

$$|H(z,t)| \le M^*(\rho,\tau)e^{|\sigma|+\tau|z|^p}.$$

Using this in (7.3), we get

$$\begin{aligned}|W(z,t)| &\le \int_0^\infty e^{-\sigma}\Big\{\frac{1}{2\pi}\int_0^{2\pi} e^{|t|/\eta^q}M^*e^{|\sigma|\eta^p+\tau|x|^\rho}d\theta\Big\}d\sigma \\ &= M^*e^{|t|/\eta^q+\tau|z|^\rho}\int_0^\infty e^{-\sigma(1-\eta^p)}d\sigma.\end{aligned} \tag{7.4}$$

Since $0 < \eta < 1$, the integral in the last member of this converges to $(1-\eta^p)^{-1}$ and we obtain a bound on $|W(z,t)|$. A useful choice for η is the one that minimizes the last member of (7.4). For fixed $|t|$, the critical equation for this choice of η is given by $p\eta^{p+q} + |t|\eta^p - |t| = 0$. It is readily checked that there is at least one critical point η_0 in the interval $(0,1)$.

Problem 7B. Here, we wish to construct solutions of the following version of a Sobolev problem:

$$\begin{aligned}&W_t(z,t) - \beta t\Delta_n W_t(z,t) = \Delta_n W(z,t), \quad t > 0,\ \beta > 0 \\ &W(z,0) = \varphi(z), \quad \varphi(z) \in \propto(z),\ \Delta_n = \sum_{j=1}^n D_j^2.\end{aligned} \tag{7.5}$$

(See [**9**] for a slightly different Sobolev problem). It is readily shown, using an associated problem, that a solution of (7.5) can be expressed symbolically as

$$W(z,t) = f(t\Delta_n)\varphi(z) \tag{7.6}$$

in which

$$f(t\nu) = (1-\beta\nu t)^{-1/\beta}, \quad 0 < \nu < 1. \tag{7.7}$$

The form of the solution representation for $W(z,t)$ that results from (7.6) is dependent upon the nature of the transformation that connects the solution operator $f(t\Delta_n)$ to the solution operator $e^{t\Delta_n}$ of the heat problem.

$$H_t(z,t) = \Delta_n H(z,t); \quad H(z,0) = \varphi(z). \tag{7.8}$$

In the following, we provide three different ways for obtaining an integral representation for $W(z,t)$. The first of these uses formula (5.1)(ii) directly while the other two result from using other types of formulas for connecting $f(t\nu)$ to the exponential function. The last procedure we use leads to a solution formula that is meaningful in more abstract cases.

Procedure 1. Applying (5.1)(ii) directly in (7.6), we get

$$W(z,t) = \int_0^\infty e^{-\sigma}\left((1-\underline{\eta})^{-\beta} \circ H\left(z, \frac{\beta \underline{t}\sigma}{\eta}\right)\right) d\sigma, \quad 0 < \eta < 1. \tag{7.9}$$

Procedure 2. We first rewrite (7.7) in the form

$$f(tv) = \int_0^\infty e^{-\sigma}\big((1-\underline{\eta})^{-\beta} \circ e^{(\beta \underline{t}\sigma v)/\eta}\big) d\sigma. \tag{7.10}$$

From the definition (2.1), the *qip* in the integrand of this has the evaluation ${}_1F_1(1/\beta; 1; \beta t\sigma v)$ (a hypergeometric function). Hence, (7.10) becomes

$$f(tv) = \int_0^\infty e^{-\sigma}\ {}_1F_1\left(\frac{1}{\beta}; 1; \beta\tau\sigma v\right) d\sigma \tag{7.11}$$

and the solution of (7.5) can be written

$$W(z,t) = \int_0^\infty e^{-\sigma}\Big[{}_1F_1\left(\frac{1}{\beta}; 1; \beta t\sigma \Delta_\eta\right)\varphi(z)\Big] d\sigma. \tag{7.12}$$

If $\beta > 1$, the integrand term in braces can be evaluated using Theorem 3.3 of [**6**]. In fact, we get

$$\begin{aligned} &{}_1F_1\left(\frac{1}{\beta}; 1; \beta\sigma t\Delta_n\right)\varphi(z) \\ &\qquad = \frac{1}{\Gamma\big(\frac{1}{\beta}\big)\Gamma\big(1-\frac{1}{\beta}\big)} \int_0^1 \xi^{1/\beta - 1}(1-\xi)^{-1/\beta} H(z, \beta\sigma t\xi) d\xi. \end{aligned} \tag{7.13}$$

We can now insert this in the integrand in the right member of (7.12) to obtain $W(z,t)$.

Procedure 3. Observe that if $\beta\lambda t < 1$ in (7.7), then we have

$$\begin{aligned} f(t\lambda) &= \frac{1}{\Gamma(1/\beta)} \int_0^\infty \sigma^{1/\beta - 1} e^{-\sigma(1-\beta t\lambda)} d\sigma \\ &= \frac{1}{\Gamma(1/\beta)} \int_0^\infty \sigma^{1/\beta - 1} e^{-\sigma} e^{\sigma\beta\lambda t} d\sigma \end{aligned} \tag{7.14}$$

If we replace λ in this by Δ_n and apply both sides to $\varphi(z)$, we get

$$W(z,t) = \frac{1}{\Gamma(1/\beta)} \int_0^\infty \sigma^{1/\beta - 1} e^{-\sigma} H(z, \beta t\sigma) d\sigma. \tag{7.15}$$

The representation is valid for $\beta > 0$ but requires that $\varphi(z) \in \propto(z)$. The transmutation formula (7.13) holds only if $\beta > 1$ while (7.15) holds for all $\beta > 0$. In view of the fact that the integrands in (7.13) and (7.15) do not require complex arguments, the z_j's may be replaced by real variables $x_j, j = 1, \ldots, n$, and the data function $\varphi(z) = \varphi(x)$ selected in (7.5) no longer need to be restricted to the class $\propto$. The formula (7.15), in fact, motivates the following generalization in a Banach Space:

THEOREM 7.1. *Let $\underline{X}$ be a Banach space and let A be the infinitesimal generator of a semigroup of operators $\{T_A(t)\}$ in $\underline{X}$. Let $\varphi \in D(A)$. Then the function*

$$W(t) = \frac{1}{\Gamma(1/\beta)} \int_0^\infty \sigma^{1/\beta - 1} e^{-\sigma} \{T_A(\beta t\sigma)\varphi\} d\sigma, \quad \beta > 0, \tag{7.16}$$

is a solution of the Sobolev problem

$$W_t(t) - \beta t A W_t(t) = AW(t), \quad t > 0, \beta > 0, \quad W(0+) = \varphi. \tag{7.17}$$

The proof of this theorem is relatively straightforward and is left to the reader.

The important point illustrated in this case is the flexibility one has in expressing various functions of operators in terms of the exponential operator. This will be particularly useful in future projected studies on the well-posedness and ill-posedness of Cauchy problems and the weakening of requirements on the data. §8 to follow will provide two further examples where an appropriate writing of the solution operator leads to a real transmutation formula.

8. Examples of real transmutations. For z real, the problems in this section have been discussed in other papers using real arguments. Nevertheless, they serve to illustrate how the appropriate formation of a *qip* can quickly lead to known results.

PROBLEM 8A. We first consider the Euler-Poisson-Darboux problem (taking $n = 1$)

$$\begin{aligned} &W_{tt}(z,t) + \frac{1}{t}W_t(z,t) = D^2 W(z,t), \\ &W(z,0) = \varphi(z), \quad W_t(z,0) = 0; \quad \varphi(z) \in \propto(z). \end{aligned} \tag{8.1}$$

Using associated equation techniques, a solution of (8.1) can be expressed formally as

$$W(z,t) = I_0(tD_z)\varphi(z). \tag{8.2}$$

From the results of §5 we have

$$W(z,t) = \int_0^\infty e^{-\sigma}(I_0(\underline{t}) \circ H(z,\underline{\sigma}))d\sigma. \tag{8.3}$$

However, if we make use of the *qip* formula (2.10a) with $a = t$ and $b = D_z$, we see that

$$\begin{aligned} W(z,t) &= (2\pi)^{-1}\int_0^{2\pi} e^{t(\cos\theta)D_z}\varphi(z)d\theta \\ &= (2\pi)^{-1}\int_0^{\infty} \varphi(z + t\cos\theta)d\theta. \end{aligned} \tag{8.4}$$

This is the much simpler form of the solution. Moreover, this formula permits the following generalization:

Let $\underline{X}$ be a Banach space and let B denote the generator of a continuous group $\{G_B(t)\}$ of operators in $\underline{X}$. Suppose that $W(t)$ satisfies the problem

$$\begin{aligned} &W''(t) + \frac{1}{t}W'(t) = B^2 W(t), \quad t > 0, \\ &W(0) = \varphi, \quad W'(0) = 0 \quad \text{where } \varphi \in D(B^2). \end{aligned} \tag{8.5}$$

Then

$$W(t) = (2\pi)^{-1} \int_0^{2\pi} [G_B(t\cos\theta)\varphi] d\theta. \tag{8.6}$$

PROBLEM 8B. We conclude with an example that involves the classical heat problem in 1 space variable, namely

$$\begin{gathered} H_t(z,t) = D^2 H(z,t) \\ H(z, 0+) = \varphi(z), \ \varphi \in \propto. \end{gathered} \tag{8.7}$$

The solution of this can be formally expressed as in (4.1):

$$H(z,t) = e^{tD^2}\varphi(z). \tag{8.8}$$

Rather than use the results of §5, we go directly to the qip formula (2.10b). Taking $a = t$ and $b = D$, we get

$$H(z,t) = \int_0^\infty e^{-\sigma} I_0(2\sqrt{t\sigma}D)\varphi(z) d\sigma. \tag{8.9}$$

From (8.2), we see that $I_0(2\sqrt{t\sigma}D)\varphi(z)$ is simply the function $W(z, 2\sqrt{t\sigma})$, and (8.9) becomes

$$H(z,t) = \int_0^\infty e^{-\sigma} W(z, 2\sqrt{t\sigma}) d\sigma, \tag{8.10}$$

where $W(z,t)$ satisfies (8.1). We leave it to the reader to write the abstract generalizations of problems (8.7) and (8.1) whose solutions are connected by a formula of the type (8.10).

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