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POSITIVE SOLUTIONS OF A BOUNDARY VALUE PROBLEM

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For the moment, let $\mathcal{K}$ be a cone in $\mathbf{R}^n$. Then it is easy to prove that if

$$\begin{aligned} -u''(t) &\in \mathcal{K}, \qquad t \in [a,b], \\ u(a) &\in \mathcal{K}, \qquad u(b) \in \mathcal{K}, \end{aligned}$$

then $u(t) \in \mathcal{K}$ for $t \in [a,b]$. This result was used in the work of Schmitt and Smith [**3**] on extremal solutions. Our main goal is to prove a generalization of this result.

First, we give some preliminary definitions and results. Let $\mathcal{X}$ be a Banach space. A closed subset $\mathcal{K} \subseteq \mathcal{X}$ is said to be a *cone* provided

(i) if $u, v \in \mathcal{K}$, then $\alpha u + \beta v \in \mathcal{K}$ for all $\alpha, \beta \geq 0$,

(ii) if $u, -u \in \mathcal{K}$, then $u = \theta$ (the zero element of $\mathcal{X}$).

A cone $\mathcal{K}$ is *solid* if its interior $\mathcal{K}^\circ \neq \varnothing$. As in [**2**], if $u, v \in \mathcal{X}$, we write $u \leq v$ in case $v - u \in \mathcal{K}$, and we write $u \ll v$ in case $v - u \in \mathcal{K}^\circ$.

LEMMA 1. *Let $\mathcal{K}$ be a cone in a Banach space $\mathcal{X}$. If $y(t)$ is the solution of the boundary value problem*

$$\begin{aligned} y^{(n)}(t) &= \theta, \qquad t \in [a,b], \\ y^{(i)}(a) &= \theta, \qquad 0 \leq i \leq k-1, \\ y^{(i)}(b) &= \theta, \qquad 0 \leq i \leq n-k-1, \ i \neq j, \\ (-1)^j y^{(j)}(b) &= \beta_j \in \mathcal{K}, \end{aligned}$$

where j is a fixed integer with $0 \leq j \leq n-k-1$, then

$$y(t) \in \mathcal{K}, \qquad t \in [a,b].$$

PROOF. Using the boundary conditions at a we get that

$$y(t) = \sum_{i=k}^{n-1} c_i \frac{(t-a)^i}{i!},$$

where $c_i \in \mathcal{X}$, $k \le i \le n-1$. It follows that $y(t)$ satisfies the differential equation

$$y^{(n-1)}(t) = c_{n-1}, \qquad t \in [a,b].$$

We now show that

$$(-1)^{n-k-1} c_{n-1} \in \mathcal{K}. \tag{1}$$

From the boundary conditions at b we get that

$$c_{n-1} = (-1)^{n-k+1} \frac{W_1}{W_2} \beta_j, \tag{2}$$

where W_2 is the Wronskian of $(t-a)^k/k!, \dots, (t-a)^{n-2}/(n-2)!$ evaluated at b and W_1 is the determinant obtained from W_2 by deleting the $(j+1)$-st row and last column. It is well known that $W_2 > 0$. To see that $W_1 > 0$, interchange the rows and columns, then shuffle the rows (i-th row becomes the $(n-k-i)$-th row, $1 \le i \le k$), and, finally, shuffle the columns to get a determinant which is well known to be positive. Hence, (1) follows from (2) and the fact that $\beta_j \in \mathcal{K}$.

Note that $y(t)$ is a solution of the boundary value problem

$$\begin{aligned} (-1)^{n-k-1} y^{(n-1)}(t) &= (-1)^{n-k-1} c_{n-1}, \\ y^{(i)}(a) &= \theta, \qquad 0 \le i \le k-1, \\ y^{(i)}(b) &= \theta, \qquad 0 \le i \le n-k-1, \ i \ne j. \end{aligned}$$

Hence,

$$y(t) = \int_a^b G_j(t,s)(-1)^{n-k-1} c_{n-1}\, ds, \tag{3}$$

where $G_j(t,s)$ is the Green's function for the scalar boundary value problem

$$\begin{aligned} (-1)^{n-k-1} u^{(n-1)}(t) &= h(t), \\ u^{(i)}(a) &= 0, \qquad 0 \le i \le k-1, \\ u^{(i)}(b) &= 0, \qquad 0 \le i \le n-k-1, \ i \ne j. \end{aligned}$$

But, for example, by results in [**1**], the Green's function for this boundary value problem where we "skip" a condition at b satisfies

$$G_j(t,s) > 0, \text{ on } (a,b)^2. \tag{4}$$

It follows from (3), (4), and (1) that $y(t) \in \mathcal{K}$ for $t \in [a,b]$. □

LEMMA 2. *Let $\mathcal{K}$ be a cone in the Banach space $\mathcal{X}$. If $y(t)$ is the solution of the boundary value problem*

$$\begin{aligned} y^{(n)}(t) &= \theta, \qquad t \in [a,b],\\ y^{(i)}(a) &= \theta, \qquad 0 \le i \le k-1,\ i \ne j,\\ y^{(j)}(a) &= \alpha_j \in \mathcal{K},\\ y^{(i)}(b) &= \theta, \qquad 0 \le i \le n-k-1, \end{aligned}$$

where $0 \le j \le k-1$ is fixed, then

$$y(t) \in \mathcal{K}, \qquad t \in [a,b].$$

The proof of Lemma 2 is similar to the proof of Lemma 1 and will be omitted. We now can state and prove our main result.

THEOREM 1. *Assume $\mathcal{K}$ is a cone in the Banach space $\mathcal{X}$ and $y \in C^n([a,b],\mathcal{X})$ satisfies*

$$\begin{aligned} (-1)^{n-k} y^{(n)}(t) &\in \mathcal{K}, \qquad t \in [a,b],\\ y^{(i)}(a) &\in \mathcal{K}, \qquad 0 \le i \le k-1,\\ (-1)^i y^{(i)}(b) &\in \mathcal{K}, \qquad 0 \le i \le n-k-1. \end{aligned}$$

Then $y(t) \in \mathcal{K}$ for $t \in [a,b]$. If, in addition, $\mathcal{K}$ is a solid cone and one of

(i) $(-1)^{n-k} y^{(n)}(t_0) \in \mathcal{K}^\circ$ *for some* $t_0 \in [a,b]$,

(ii) $y^{(j)}(a) \in \mathcal{K}^\circ$ *for some* $0 \le j \le k-1$, *or*

(iii) $(-1)^j y^{(j)}(b) \in \mathcal{K}^\circ$ *for some* $0 \le j \le n-k-1$

holds, then $y(t) \in \mathcal{K}^\circ$ for $t \in (a,b)$.

PROOF. Set

$$\alpha_i = y^{(i)}(a), \quad 0 \le i \le k-1,$$
$$\beta_i = (-1)^i y^{(i)}(b), \quad 0 \le i \le n-k-1,$$

and

$$h(t) = (-1)^{n-k} y^{(n)}(t), \qquad t \in [a,b].$$

Then $\alpha_i \in \mathcal{K}$, $0 \le i \le k-1$, $\beta_j \in \mathcal{K}$, $0 \le j \le n-k-1$, and $h(t) \in \mathcal{K}$ for $t \in [a,b]$.

Let $y_j(t)$, $0 \le j \le n-k-1$, be the solution of the boundary value problem in Lemma 1, and let $z_j(t)$, $0 \le j \le k-1$, be the solution of the boundary value problem in Lemma 2. Then

$$y(t) = \sum_{j=0}^{n-k-1} y_j(t) + \sum_{j=0}^{k-1} z_j(t) + w(t), \tag{5}$$

where

$$w(t) = \int_a^b G(t,s)h(s)\,ds$$

and $G(t,s)$ is the appropriate Green's function, which is well known to be positive on $(a,b)^2$. It follows that

$$w(t) \in \mathcal{K}, \qquad t \in [a,b].$$

Using Lemmas 1 and 2, we get from (5) that $y(t) \in \mathcal{K}$ for $t \in [a,b]$.

Now fix $t \in (a,b)$. To complete the proof, we need to show that if $\mathcal{K}$ is a solid cone and any of (i)–(iii) hold, then $y(t) \in \mathcal{K}^\circ$. Note that if $\theta \ll u \le v$, then $v \in \mathcal{K}^\circ$. Hence, it suffices to show that one of the terms in the expression (5) is in $\mathcal{K}^\circ$.

First, suppose that (i) holds; that is, there exists $t_0 \in [a,b]$ such that $h(t_0) \in \mathcal{K}^\circ$. By continuity, there is a $t_1 \in (a,b)$ such that $h(t_1) \in \mathcal{K}^\circ$. Since $t \in (a,b)$, we have $G(t,t_1) > 0$ and so $G(t,t_1)h(t_1) \in \mathcal{K}^\circ$. Let B be a ball about $G(t,t_1)h(t_1)$ such that $\overline{B} \subseteq \mathcal{K}^\circ$. By continuity, there is an interval $[c,d] \subseteq (a,b)$ including t_1, such that $G(t,s)h(s) \in B$ for $s \in [c,d]$. Consider the Riemann sum for $1/(d-c)\int_c^d G(t,s)h(s)\,ds$, given by

$$\frac{1}{d-c}\sum_{i=1}^{m} G(t,s_i)h(s_i)\Delta s_i = \sum_{i=1}^{m} G(t,s_i)h(s_i)\frac{\Delta s_i}{d-c},$$

where $c = s_0 < \cdots < s_m = d$ is a partition of $[c, d]$ and $\Delta s_i = s_i - s_{i-1}$ for $1 \le i \le m$. This sum is in the convex hull of B, hence in B since B is convex. It follows that

$$\frac{1}{d-c}\int_c^d G(t,s)h(s)\,ds \in \overline{B} \subseteq \mathcal{K}^\circ$$

so that $\int_c^d G(t,s)h(s)\,ds \in \mathcal{K}^\circ$. Since

$$\theta \ll \int_c^d G(t,s)h(s)\,ds \le \int_a^b G(t,s)h(s)\,ds = w(t),$$

it follows that $w(t) \in \mathcal{K}^\circ$. Then $\theta \ll w(t) \le y(t)$, which implies that $y(t) \in \mathcal{K}^\circ$. Since $t \in (a, b)$ was arbitrary, we have $y(t) \in \mathcal{K}^\circ$ for $t \in (a, b)$.

Now suppose that (iii) holds. The proof of Lemma 1 shows that

$$y_j(t) = \int_a^b G_j(t,s)\beta\,ds$$

for some $\beta \in \mathcal{K}^\circ$. Using the sign condition on $G_j(t,s)$ and arguments similar to those above, it can be shown that $y_j(t) \in \mathcal{K}^\circ$. Since $y_j(t) \le y(t)$ it follows that $y(t) \in \mathcal{K}^\circ$.

The proof in the case that (ii) holds is similar to the case for (iii) and will be omitted. □

A direct application of the theorem gives the following comparison result.

COROLLARY 1. *Assume $\mathcal{K}$ is a cone in the Banach space $\mathcal{X}$. If the functions $y, z \in C^n([a,b], \mathcal{X})$ satisfy*

$$\begin{aligned}
(-1)^{n-k}y^{(n)}(t) &\le (-1)^{n-k}z^{(n)}(t), && t \in [a,b],\\
y^{(i)}(a) &\le z^{(i)}(a), && 0 \le i \le k-1,\\
(-1)^i y^{(i)}(b) &\le (-1)^i z^{(i)}(b), && 0 \le i \le n-k-1,
\end{aligned}$$

then $y(t) \le z(t)$ for $t \in [a,b]$. If, in addition, one of the following holds:

(i) $(-1)^{n-k}y^{(n)}(t_0) \ll (-1)^{n-k}z^{(n)}(t_0)$ *for some* $t_0 \in [a,b]$,

(ii) $y^{(j)}(a) \ll z^{(j)}(a)$ *for some* $0 \le j \le k-1$, *or*

(iii) $(-1)^j y^{(j)}(b) \ll (-1)^j z^{(j)}(b)$ *for some* $0 \le j \le n-k-1$,

then $y(t) \ll z(t)$ *for* $t \in (a,b)$.

As an example of these results, consider the sequence space $\mathcal{X} = l^2$ and the cone $\mathcal{K} = \{(x_j) \in \mathcal{X} : x_j \ge x_{j+1} \ge 0\}$. Let $n = 4$, $k = 2$, and suppose $y(t)$ satisfies

$$y^{(4)}(t) = \left(\frac{1}{j^4}\sin\frac{t}{j}\right), \qquad t \in \left[0, \frac{\pi}{2}\right],$$
$$y(0) = \left(\frac{1}{j}\right), \qquad y'(0) = \left(\frac{1}{\sqrt{j}\log(j+1)}\right),$$
$$y\left(\frac{\pi}{2}\right) = \left(\frac{1}{j^2}\right), \qquad y'\left(\frac{\pi}{2}\right) = (-e^{-j}).$$

It is easy to verify that

$$(-1)^{n-k}y^{(n)}(t) \in \mathcal{K}, \qquad t \in \left[0, \frac{\pi}{2}\right],$$
$$y^{(i)}(0) \in \mathcal{K}, \qquad 0 \le i \le k-1,$$
$$(-1)^i y^{(i)}\left(\frac{\pi}{2}\right) \in \mathcal{K}, \qquad 0 \le i \le n-k-1.$$

Hence, by Theorem 1 it follows that $y(t) \in \mathcal{K}$ for $t \in [0, \pi/2]$. □

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