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SOLVABILITY OF TWO-POINT BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS OF THE FORM $\mathbf{y}'' = \mathbf{g}(\mathbf{t}, \mathbf{y}, \mathbf{y}', \mathbf{y}'')$

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Introduction. Two authors of this paper have proved in [**7**] a number of existence results for systems of differential inclusions

$$y'' \in F(t, y, y'), \quad a.e.\ t \in I, \tag{1}$$

subject to various boundary conditions. In the above, I stands for either a finite interval or a half-line, F is an "admissible" convex-valued multifunction, $y \in C^1(I; \mathbf{R}^n)$ and $y'' \in L^2(I; \mathbf{R}^n)$ ($y'' \in L^2_{\text{loc}}(I; \mathbf{R}^n)$ in case $I = [0, \infty)$.

With the help of those results, we establish the solvability of the same boundary value problems for systems of differential equations

$$y'' = g(t, y, y', y''), \quad a.e.\ t \in I, \tag{2}$$

where g is a Caratheodory function which, loosely speaking, satisfies Bernstein-Nagumo-type conditions, with respect to (y, y') and is non-expansive in y''. Unlike the contraction principle, fixed point theorems for nonexpansive maps do not guarantee the unique solvability of $y'' = g(t, y, y', y'')$ with respect to y'', so we cannot reduce (2) to the classical equation

$$y'' = F(t, y, y'), \quad a.e.\ t \in I,$$

with $F(t, y, y')$ being the fixed point. The set of fixed points of a nonexpansive mapping is, however, convex, and, by denoting it by $F(t, y, y')$, we reduce (2) to (1). In Section 1, we prove that this "implicit multifunction" $F(t, y, y')$ has the required properties so that the results of [**7**] may be applied. In Section 2 we derive from those results analogous conclusions about boundary value problems for (2).

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We remark that our "implicit multifunction" technique was used in [**8**] but, unlike this paper, a very strong assumption on a sublinear growth at infinity was imposed on $g(t, y, y', y'')$.

As a simple example, consider the BVP

$$y'' = y^3 + y'^2 + 1 + \alpha \sin y'' - h(t), \quad 0 \le t \le 1,$$
$$y(0) = y(1) = 0$$

where $|\alpha| \le 1$. This problem was considered by Petryshyn in [**10**] for $0 < \alpha < 1$. The results we obtain apply to the above equation with more general boundary conditions.

Another method for treating problems of this type was proposed by Bielawski and Górniewicz [**2**] (see also [**3**]).

1. Let Ω be a domain in Euclidean space. Let us recall that a function $y : \Omega \times \mathbf{R}^m \to \mathbf{R}^N$ is said to be *a Caratheodory function* of variables $x \in \Omega$ and $u \in \mathbf{R}^m$ if it is measurable in x for all u and continuous in u for all x (by "measurable" we will always mean Lebesgue measurable). We propose the following definition of a *Caratheodory multifunction*, somewhat more general than the usual one:

DEFINITION. A multifunction $F : \Omega \times \mathbf{R}^m \to \mathbf{R}^n$ with nonempty compact convex values is called a *Caratheodory multifunction* if it satisfies the following two conditions:

(C1). For each measurable $u : \Omega \to \mathbf{R}^m$, the map $x \mapsto F(x, u(x))$ has measurable single-valued selections;

(C2). For *a.e.* $x \in \Omega$, the map $u \mapsto F(x, u)$, $u \in \mathbf{R}^m$ is upper semicontinuous.

We note that the usual condition of measurability of the multifunction $x \mapsto F(x, u)$ (c.f. [**5**]), for each given $u \in \mathbf{R}^n$, together with (C2), implies (C1) but it is really (C1) which is used in applications. Some authors do not assume the convexity of the values, but then the stronger condition of the continuity of $u \mapsto F(x, u)$ must be imposed in (C2) (cf. [**1**]).

We recall from [**7**] that a multifunction $F : \Omega \times \mathbf{R}^m \to \mathbf{R}^n$ is called *admissible* if the multivalued mapping $F^* : C(\bar{\Omega}; \mathbf{R}^m) \to L^2(\Omega; \mathbf{R}^n)$, given by

$$F^*(u) := \{v \in L^2(\Omega; \mathbf{R}^n) : v(x) \in F(x, u(x)) \text{ for } a.e.\ x \in \Omega\},$$

is well defined with nonempty convex values such that the composed multivalued map $K \circ F^*$ is upper semicontinuous and maps bounded sets to relatively compact sets, for any completely continuous linear operator $K : L^2(\Omega; \mathbf{R}^n) \to C(\bar{\Omega}; \mathbf{R}^m)$.

The following lemma is a direct consequence of Proposition 1.7 in [**11**].

LEMMA 1. *Suppose that a Caratheodory multifunction $F : \Omega \times \mathbf{R}^m \to \mathbf{R}^n$ satisfies the condition*

(C3). *For any bounded $B \subset \mathbf{R}^m$ there exists $\varphi_B \in L^2(\Omega; [0, \infty))$ such that $\|F(x, u)\| \leq \varphi_B(x)$ for a.e. $x \in \Omega$ and all $u \in B$, where $\|F(x, u)\| = \sup\{\|v\| : v \in F(x, u)\}$.*

Then F is admissible. □

The following classical result will be used below.

THEOREM 1. (cf. [**6**], Corollary 1.6 in Chapter I, §2). *Let H be a Hilbert space, $B_r = \{x \in H : \|x\| \leq r\}$, and let $f : B_r \to H$ be a nonexpansive mapping. Assume that, for all x with $\|x\| = r$, one of the following four conditions holds*:

(a). $\|f(x)\| \leq r$;

(b). $\|f(x)\| \leq \|x - f(x)\|$;

(c). $\|f(x)\|^2 \leq r^2 + \|x - f(x)\|^2$;

(d). $x \cdot f(x) \leq r^2$, *where $x \cdot f(x)$ stands for the scalar product.*

Then f has a fixed point in B_r.

We will actually need Theorem 1 for $H = \mathbf{R}^n$ only.

Let now $g : \Omega \times \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ be a Caratheodory function of variables $x \in \Omega$ and $(u, v) \in \mathbf{R}^m \times \mathbf{R}^n$. We shall state the following hypothesis on g:

(H0). For *a.e.* $x \in \Omega$ and all $u \in \mathbf{R}^m$, there exists $r = r(x,u)$ such that the function $f : B_r \to \mathbf{R}^n$, $B_r \subset \mathbf{R}^n$, $f(v) = g(x,u,v)$ satisfies the assumptions of Theorem 1. Moreover, $r = r(x,u)$ can be chosen as a Caratheodory function of x and u.

REMARK. The following two conditions guarantee the hypothesis (H0):

(i) There exists a Caratheodory function $\alpha : \Omega \times \mathbf{R}^m \to [0,\infty)$ and a constant $c < 1$ such that

$$\|g(x,u,v)\| \le \alpha(x,u) + c\|v\|$$

for *a.e.* $x \in \Omega$ and all $(u,v) \in \mathbf{R}^m \times \mathbf{R}^n$;

(ii) $\|g(x,u,v_1) - g(x,u,v_2)\| \le \|v_1 - v_2\|$ for *a.e.* $x \in \Omega$, all $u \in \mathbf{R}^m$ and all $v_1, v_2 \in B_r \subset \mathbf{R}^n$, where $r = \alpha(x,u)/(1-c)$.

LEMMA 2. *Suppose that a Caratheodory function* $g : \Omega \times \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ *of variables* $x \in \Omega$ *and* $(u,v) \in \mathbf{R}^m \times \mathbf{R}^n$ *satisfies the condition* (H0). *We define the multifunction* $F : \Omega \times \mathbf{R}^m \to \mathbf{R}^n$ *by the formula*

$$F(x,u) = \{v \in B_{r(x,u)} : v = g(x,u,v)\}. \tag{3}$$

Then F *is a Caratheodory multifunction.*

PROOF. Given $x \in \Omega$ and $u \in \mathbf{R}^m$, $F(x,u) \neq \phi$ by Theorem 1, $F(x,u)$ is convex and, obviously, closed as the set of fixed points of a nonexpansive map. It is also bounded by definition, therefore compact. For (C1), let $u : \Omega \to \mathbf{R}^m$ be measurable. By elementary measure-theoretic arguments, $r(x,u(x))$ is measurable in x. Using approximation algorithms for fixed points of nonexpansive maps (cf. [**9**] or [**4**] together with the proof of Theorem 1.5, Chapter 1, §2 in [**6**]), one constructs a fixed point $v(x) \in F(x,u(x))$ which, as the function of the parameter x, is a pointwise limit of a sequence of measurable functions $\{v_n(x)\}$. Hence, $v(x)$ is measurable. For (C2), let x be given. In order to prove that $F(x,u)$ is u.s.c. in u, it is sufficient to show that it maps bounded sets in $\mathbf{R}^m$ to relatively compact sets in $\mathbf{R}^n$ and that its graph is closed in $\mathbf{R}^m \times \mathbf{R}^n$. The first conclusion follows from the continuity of $r(x,u)$ in u and the fact that any bounded set in $\mathbf{R}^n$ is relatively compact. The second conclusion is an immediate consequence of the continuity of functions $r(x,u)$ in u and $g(x,u,v)$ in (u,v), and of the definition of F. □

2. Let $I=[a_0,a_1]$ be a closed bounded interval in $\mathbf{R}$ and let $g:[a_0,a_1]\times\mathbf{R}^{3n}\to\mathbf{R}^n$ be a Caratheodory function of variables $t\in[a_0,a_1]$ and $(y,y',y'')\in\mathbf{R}^{3n}$, where $y,y',y''\in\mathbf{R}^n$. We will assume that g satisfies (H0) with $u=(y,y')$ and $v=y''$. We shall next state the following hypotheses on g:

(H1). There exists a constant $R>0$ such that if $\|y_0\|>R$ and $y_0\cdot y_0'=0$, then, for some $\delta>0$, we have

$$0<\operatorname{ess}\inf_{t\in I}\inf\{y\cdot g(t,y,y',y''):\|(y,y')-(y_0,y_0')\|\le\delta \text{ and } \|y''\|\le r(t,y,y')\};$$

(H2). There exists a function $\varphi:[0,\infty)\to(0,\infty)$ with $s/\varphi(s)\in L^\infty_{\text{loc}}[0,\infty)$, $\int_0^\infty(s/\varphi(s))ds=\infty$ such that

$$\|g(t,y,y',y'')\|\le\varphi(\|y'\|)$$

for *a.e.* $t\in I$ and all $(y,y',y'')\in\mathbf{R}^{3n}$ with $\|y\|\le R$ and $\|y''\|\le r(t,y,y')$;

(H3). There exist constants K, $\alpha>0$ such that

$$\|g(t,y,y',y'')\|\le 2\alpha(y\cdot g(t,y,y',y'')+\|y'\|^2)+K$$

for *a.e.* $t\in I$ and all $(y,y',y'')\in\mathbf{R}^{3n}$ with $\|y\|\le R$ and $\|y''\|\le r(t,y,y')$.

Let $G_i:\mathbf{R}^{4n}\to\mathbf{R}^n$, $i=0,1$, be continuous functions and let $A_0,A_1\in GL(n,\mathbf{R})$ be two (nonsingular) matrices. We introduce the following conditions:

(N1). One of the following inequalities is satisfied for all $u_0,u_0',u_1,u_1'\in\mathbf{R}^n$:

$$(-1)^i[u_i\cdot u_i'\pm A_iG_i(u_0,u_0',u_1,u_1')\cdot u_i']\ge 0.$$

(N2). One of the following inequalities is satisfied for all $u_0,u_0',u_1,u_1'\in\mathbf{R}^n$:

$$(-1)^i[u_i\cdot u_i'\pm A_iG_i(u_0,u_0',u_1,u_1')\cdot u_i]\ge 0.$$

(N3). One of the following relations is satisfied for all $\lambda \in [0,1]$ and all $u_0, u_0', u_1, u_1' \in \mathbf{R}^n$ such that $\|u_i\| > R$:

$$u_i \neq \lambda[u_i \pm A_i G_i(u_0, u_0', u_1, u_1')],$$

where $i = 0, 1$.

Given a function $y \in C^1([a_0, a_1]; \mathbf{R}^n)$, we denote by $y \in \mathcal{B}$ the boundary conditions

$$G_i(y(a_0), y'(a_0), y(a_1), y'(a_1)) = 0, \quad i = 0, 1,$$

where $G_i : \mathbf{R}^{4n} \to \mathbf{R}^n$, $i = 0, 1$, are continuous functions satisfying one of the conditions (N1), (N2) or (N3). The boundary conditions $\mathcal{B}$ were studied in [**7**], where it was proved that any of the following sets of boundary conditions belongs to $\mathcal{B}$:

(I) $y(a_0) = r_0$, $y(a_1) = r_1$,

(II) $y'(a_0) = 0$, $y'(a_1) = 0$,

(III) $-A_0 y(a_0) + B_0 y'(a_0) = r_0$, $-A_1 y(a_1) + B_1 y'(a_1) = r_1$,

(IV)(a) $y(a_0) = r_0$, $-A_1 y(a_1) + B_1 y'(a_1) = r_1$,

(IV)(b) $-A_0 y(a_0) + B_0 y'(a_0) = r_0$, $y(a_1) = r_1$,

(V)(a) $y'(a_0) = 0$, $-A_1 y(a_1) + B_1 y'(a_1) = r_1$,

(V)(b) $-A_0 y(a_0) + B_0 y'(a_0) \geq r_0$, $y'(a_1) = 0$,

where A_0, B_0, $-A_1$, B_1 are nonnegative definite symmetric $n \times n$-matrices, $r_0, r_1 \in \mathbf{R}^n$, and we suppose that if $y(a_i) = r_i$, then $\|r_i\| \leq R$, and if $-A_i y(a_i) + B_i y'(a_i) = r_i$, then $\|B_i^{-1}\| \|A_i^{-1} B_i\| \cdot \|r_i\| \leq R$.

Denote by $\mathcal{O}(n, \mathbf{R})$ the group of real orthogonal $n \times n$ matrices. We shall say that two maps $A : \mathbf{R}^{4n} \to \mathcal{O}(n, \mathbf{R})$ and $B : \mathbf{R}^{4n} \to GL(n, \mathbf{R})$ have the *property* (P) if $u \cdot v \leq 0$ implies $u \cdot A(x)[B(x)]^{-1} v \leq 0$ for all $x \in \mathbf{R}^{4n}$, $u, v \in \mathbf{R}^n$. Let A and B be two continuous maps having the property (P). Given $y \in C^1([a_0, a_1], \mathbf{R}^n)$, we denote by $y \in \mathcal{P}$ the boundary condition

$$\begin{cases} y(a_0) = A(y(a_0), y(a_1), y'(a_0),\ y'(a_1)) y(a_1) \\ y'(a_0) = B(y(a_0), y(a_1), y'(a_0),\ y'(a_1)) y'(a_1) \end{cases}.$$

Now we can state our existence result.

THEOREM 2. *Suppose that a Caratheodory function* $g : I \times \mathbf{R}^{3n} \to \mathbf{R}^n$ *satisfies the conditions* (H0), (H1), (H2) *and* (H3). *Then the boundary value problem*

$$(4) \qquad \begin{cases} y'' = g(t, y, y', y'') \ \textit{a.e.}\ t \in [a_0, a_1] \\ y \in \mathcal{B} \ \textit{ or } y \in \mathcal{P} \end{cases}$$

has a solution $y \in H^2([a_0, a_1]; \mathbf{R}^n)$.

PROOF. Let $F(t, y, y')$ be the multifunction defined for y as in Lemma 2. Then any solution of the problem

$$(5) \qquad \begin{cases} y'' \in F(t, y, y') \ \textit{a.e.}\ t \in [a_0, a_1] \\ y \in \mathcal{B} \ \text{ or } y \in \mathcal{P} \end{cases}$$

also is a solution of (4). It now follows from Lemma 1, (H2) and Lemma 2 that F is admissible. Easy verification shows that F satisfies the assumptions made in Theorem (5.3), Corollary (5.4) and Theorem (5.6) in [**7**], hence the conclusion. □

We next consider $I = [0, \infty)$ and a Caratheodory function $g : [0, \infty) \times \mathbf{R}^{3n} \to \mathbf{R}^n$ satisfying (H0). Assume:

(I1). There exists a constant $R > 0$ such that, if $\|y_0\| > R$ and $y_0 \cdot y_0' = 0$, then, for any $a > 0$, there is a $\delta > 0$ with

$$0 < \operatorname{ess} \inf_{t \in I} \inf\{y \cdot g(t, y, y', y'') : \|(y, y') - (y_0, y_0')\| < \delta \text{ and } \|y''\| < r(t, y, y')\};$$

(I2). There exists a function $\varphi : [0, \infty) \to (0, \infty)$ with $s/\varphi(s) \in L^\infty_{\text{loc}}[0, \infty)$, $\int_0^\infty (s/\varphi(s)) ds = \infty$ such that

$$\|g(t, y, y', y'')\| \le \varphi(\|y'\|)$$

for *a.e.* $t \in [0, \infty)$ and all $(y, y', y'') \in \mathbf{R}^{3n}$ with $\|y''\| \le r(t, y, y')$;

(I3). There exist constants α and k such that

$$\|g(t, y, y', y'')\| \le 2\alpha(y \cdot g(t, y, y', y'') + \|y'\|^2) + k$$

for *a.e.* $t \in [0, \infty)$ and all $(y, y', y'') \in \mathbf{R}^{3n}$ with $\|y''\| \le r(t, y, y')$.

Given $y \in C^1([0,\infty), \mathbf{R}^n)$, we denote by $y \in \mathcal{A}$ the boundary conditions:

(i) $y(0) = r$.

(ii) $Ay(0) - By'(0) = r$, where A and B are symmetric nonnegative definite $n \times n$-matrices such that if $r = 0$, then at least one of these matrices is nonsingular; otherwise, both of them are nonsingular.

(iii) $G(y(0), y'(0)) = 0$, where $G : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function which satisfies one of the conditions (N1) or (N3) for $i = 0$, where $a_0 = 0$.

(iv)$y'(0) = 0$.

(v) $G(y(0), y'(0)) = 0$, where $G : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function which satisfies the condition (N2) for $i = 0$, where $a_0 = 0$.

THEOREM 3. *Let* $g : [0,\infty) \times \mathbf{R}^{3n} \to \mathbf{R}^n$ *be a Caratheodory function satisfying* (H0), (I1), (I2), (I3). *Then the boundary value problem*

$$\begin{cases} y'' = g(t, y, y', y''), & a.e.\ t \in [0,\infty) \\ y \in \mathcal{A} \end{cases}$$

has a solution $y \in H^2_{\text{loc}}([0,\infty), \mathbf{R}^n)$. *Moreover, if* $\varphi \in L^\infty_{\text{loc}}[0,\infty)$, *then* $y \in W^{2,\infty}([0,\infty), \mathbf{R}^n)$.

PROOF. The proof is based on Theorem (7.1) in [**7**] and is analogous to the proof of Theorem 2 above.

REMARK. Conditions (I2) and (I3) may be relaxed somewhat, as in [**7**].

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