ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 1, Winter 1991

APPLICATIONS OF SZEGÖ POLYNOMIALS TO DIGITAL SIGNAL PROCESSING

WILLIAM B. JONES AND OLAV NJÅSTAD

Dedicated to W.J. Thron on the occasion of his 70th birthday

ABSTRACT. Applications of Szegö polynomials, moment theory and two-point Padé approximants to problems in digital signal processing are described. The frequency analysis problem consists of determining unknown frequencies in a signal which is the sum of a finite number of cosine waves superimposed to white noise. The problem of filter design is to construct a causal filter T with finite energy, which has a prescribed amplitude response function $\Phi(\theta)$. Examples are given to illustrate each of the two applications.

1. Introduction. Connections between Szegö polynomials (orthogonal on the unit circle), the trigonometric moment problem and two-point Padé approximants are well known and have been given, for example, in [**2, 12, 13, 18, 19** and **21**]. The purpose of this expository article is to describe important applications of these topics to two problems involved with digital filters and the processing of digital signals.

In the frequency analysis problem, we consider a signal $u = \{u(k)\}$, superimposed on white noise, where $u(k)$ has the form

$$u(k) = \lambda_0 + \sum_{j=1}^{I} \lambda_j \cos(\omega_j k + \varphi_j), \quad k = 0, \pm 1, \pm 2, \dots, \tag{1.1}$$
$$1 \leq I < \infty, \quad \lambda_0 \geq 0, \quad \lambda_j > 0, \quad \omega_j, \varphi_j \in \mathbf{R} \quad \text{for } 1 \leq j \leq I.$$

We wish to determine the unknown frequencies $\omega_1, \omega_2, \dots, \omega_I$. The linear prediction method of Wiener [**28**] and Levinson [**24**] used for this problem is described in Section 3. Also included there for illustration

The research of the first author was supported in part by the U.S. National Science Foundation under Grant No. DMS-8700498.
Received by the editors on January 10, 1989, and in revised form on April 5, 1989.

are results from three numerical experiments. In order to make this presentation reasonably self-contained we have summarized in Section 2 a number of definitions, notations and known results that are subsequently used. Among the topics included are digital filters, Levinson's algorithm, and positive PC-fractions and their relation to the two-point Padé table, Szegö polynomials and the trigonometric moment problem.

The problem of designing digital filters is dealt with in Section 6. In particular, we consider a given real-valued function $\Phi(\theta)$ on $[-\pi,\pi]$ such that

$$\Phi(\theta) \geq 0 \quad \text{and} \quad \Phi(-\theta) = \Phi(\theta) \quad \text{for } -\pi \leq \theta \leq \pi \tag{1.2a}$$

and

$$\int_{-\pi}^{\pi} [\Phi(\theta)]^2 \, d\theta < \infty. \tag{1.2b}$$

A method is described for constructing a function $K_0(z)$, defined and analytic for $|z| > 1$, such that $K_0(z)$ is the transfer function of a causal filter T with finite energy, satisfying

$$\lim_{\rho\to 1^+} |K_0(\rho e^{i\theta})| = \Phi(\theta) \quad \text{a.e. on } [-\pi,\pi]. \tag{1.3}$$

It is shown (Theorem 6.1) that a function $K_0(z)$ of the above type exists provided

$$\int_{-\pi}^{\pi} \ln \psi'(\theta) \, d\theta > -\infty \qquad \text{(Szegö's condition)} \tag{1.4}$$

where

$$\psi(\theta) := \int_{-\pi}^{\theta} [\Phi(t)]^2 \, dt + \sigma(\theta), \tag{1.5}$$

where $\sigma(\theta)$ is an arbitrary singular distribution function.

The function $K_0(z)$ is seen to be (cf. (6.14)) the limit of the sequence $\{A_n(z)\}$, where

$$A_n(z) := \frac{1}{\sqrt{2\pi}\,\overline{(\varphi_n^*(1/\bar{z}))}},$$

where $\varphi_n^*(z) := z^n\overline{\varphi_n(1/\bar{z})}$, $\varphi_n(z)$ being the normalized n-th Szegö polynomial with respect to the distribution function $\psi(\theta)$. For illustration, some examples are described at the end of Section 6, where $G_n(\theta) := |A_n(e^{i\theta})|$ is used to approximate $\Phi(\theta)$.

In order to make this part of the paper self-contained, we have included in Section 4 some basic results on the theory of Szegö polynomials. Closely related to this is the discussion of deterministic, weakly stationary stochastic processes given in Section 5. We have included proofs of certain results which were felt to be necessary for self-containment. References are given for these and for other proofs that are omitted.

2. Background. This section is used to summarize material employed in subsequent parts of the paper. First we describe connections between positive PC-fractions, the trigonometric moment problem, and Szegö polynomials (Theorems 2.1, 2.2, 2.3). The Levinson algorithm is described and references are given for other fast algorithms to solve real positive definite Toeplitz systems. The section concludes with a brief summary of basic concepts about digital filters.

Positive PC-fractions. A double sequence of complex numbers $\{\mu_k\}_{k=-\infty}^{\infty}$ is called *hermitian positive definite* if

$$\mu_{-k} = \bar{\mu}_k, \quad k = 0, 1, 2, \dots \tag{2.1a}$$

and

$$\Delta_n := \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n-1} & \cdots & \mu_0 \end{vmatrix} > 0, \quad n = 0, 1, 2, \dots . \tag{2.1b}$$

A continued fraction

$$\delta_0 - \frac{2\delta_0}{1} + \frac{1}{\bar{\delta}_1 z} + \frac{(1-|\delta_1|^2)z}{\delta_1} + \frac{1}{\bar{\delta}_2 z} + \frac{(1-|\delta_2|^2)z}{\delta_2} + \cdots \tag{2.2a}$$

is called a *positive PC-fraction* (*positive Perron-Carathéodory continued fraction*) if

$$\delta_0 > 0 \quad \text{and} \quad |\delta_n| < 1, \quad n = 1, 2, 3, \dots . \tag{2.2b}$$

The n-th numerator P_n and denominator Q_n of (2.2) are defined by the *difference equations*

$$P_0 := \delta_0, \qquad P_1 := -\delta_0, \qquad Q_0 := Q_1 := 1, \tag{2.3a}$$

(2.3b)
$$\begin{pmatrix} P_{2n}(z) \\ Q_{2n}(z) \end{pmatrix} := \bar{\delta}_n z \begin{pmatrix} P_{2n-1}(z) \\ Q_{2n-1}(z) \end{pmatrix} + \begin{pmatrix} P_{2n-2}(z) \\ Q_{2n-2}(z) \end{pmatrix}, \quad n = 1, 2, 3, \dots,$$

(2.3c)
$$\begin{pmatrix} P_{2n+1}(z) \\ Q_{2n+1}(z) \end{pmatrix} := \delta_n \begin{pmatrix} P_{2n}(z) \\ Q_{2n}(z) \end{pmatrix} + (1 - |\delta_n|^2) z \begin{pmatrix} P_{2n-1}(z) \\ Q_{2n-1}(z) \end{pmatrix},$$
$$n = 1, 2, 3, \dots .$$

From these it follows that, for $n \geq 1$, $P_{2n}(z), Q_{2n}(z), P_{2n+1}(z)$ and $Q_{2n+1}(z)$ are polynomials in z of degrees at most n, with $Q_{2n}(0) = 1$ and $Q_{2n+1}(z) = z^n + \cdots + \delta_n$. Connections between hermitian positive definite sequences $\{\mu_k\}$ and positive PC-fractions are summarized by the following theorem, a proof of which can be found in [**19**, Theorems 2.1, 2.2., 3.1 and 3.2] where PC-fractions were introduced. Here the symbol $O(z^r)$ is used to denote a formal power series (fps) in increasing powers of z, starting with a power not less than r. If R is a rational function, then the symbols $\Lambda_0(R)$ and $\Lambda_\infty(R)$ denote the Taylor and Laurent series expansions of R about 0 and ∞, respectively.

Theorem 2.1. (A) *Let* (2.2) *be a given positive PC-fraction. Then there exists a unique pair* (L_0, L_∞) *of fps*

$$L_0 := \mu_0 + 2\sum_{k=1}^{\infty} \mu_k z^k, \qquad L_\infty := -\mu_0 - 2\sum_{k=1}^{\infty} \mu_{-k} z^{-k} \tag{2.4}$$

such that, for $n = 0, 1, 2, \dots,$

$$L_0 - \Lambda_0\left(\frac{P_{2n}}{Q_{2n}}\right) = O(z^{n+1}) \tag{2.5a}$$

$$L_\infty - \Lambda_\infty\left(\frac{P_{2n+1}}{Q_{2n+1}}\right) = O\left(\left(\frac{1}{z}\right)^{n+1}\right) \tag{2.5b}$$

and

$$Q_{2n}L_0 - P_{2n} = O(z^{n+1}), \qquad Q_{2n}L_\infty - P_{2n} = O(1), \tag{2.6a}$$

$$Q_{2n+1}L_0 - P_{2n+1} = O(z^n), \qquad Q_{2n+1}L_\infty - P_{2n+1} = O\left(\frac{1}{z}\right). \tag{2.6b}$$

Also, for $n = 1, 2, 3, \ldots,$

$$\mu_0 > 0, \qquad \mu_{-n} = \bar{\mu}_n, \qquad \Delta_n > 0, \tag{2.7a}$$

$$1 - |\delta_n|^2 = \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}, \tag{2.7b}$$

$$\delta_0 = \mu_0 > 0, \qquad \delta_n = \frac{(-1)^n}{\Delta_{n-1}} \begin{vmatrix} \mu_{-1} & \mu_0 & \cdots & \mu_{n-2} \\ \mu_{-2} & \mu_{-1} & \cdots & \mu_{n-2} \\ \vdots & \vdots & & \vdots \\ \mu_{-n} & \mu_{-n+1} & \cdots & \mu_{-1} \end{vmatrix}, \tag{2.7c}$$

and

$$Q_{2n}(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \vdots & \vdots & & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1 \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix},$$

$$Q_{2n+1}(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_{-1} \\ 1 & z & \cdots & z^n \end{vmatrix}. \tag{2.7d}$$

Moreover, for $|z| < 1$ $(|z| > 1)$, $P_{2n}/Q_{2n}(z)$ $(P_{2n+1}(z)/Q_{2n+1}(z))$ *converges to a holomorphic function* $f(z)$ $(g(z))$ *such that*

$$\operatorname{Re} f(z) \geq 0 \quad \textit{for } |z| < 1 \quad \textit{and} \quad \operatorname{Re} g(z) \leq 0 \quad \textit{for } |z| > 1 \tag{2.8a}$$

and

$$f(z) = \overline{-g(1/\bar{z})} \quad \textit{for } |z| < 1. \tag{2.8b}$$

The convergence is uniform on compact subsets of $|z| < 1$ $(|z| > 1)$ *and* L_0 (L_∞) *is the Taylor series expansion of* $f(z)$ $(g(z))$ *about* 0 (∞).

(B) *Conversely, let* (L_0, L_∞) *be a given pair of fps* (2.4) *such that* (2.7a) *holds. Let* $\{\delta_n\}_0^\infty$ *be defined by* (2.7c). *Then* (2.2b) *and* (2.7b) *hold so that* (2.2a) *is a positive PC-fraction. Moreover, the positive PC-fraction* (2.2a) *corresponds to* (L_0, L_∞) *in the sense that* (2.5), (2.6) *and* (2.7d) *hold.*

The correspondence properties (2.6) insure that P_{2n}/Q_{2n} and P_{2n+1}/Q_{2n+1} are the weak (n, n) two-point Padé approximants for (L_0, L_∞) of orders $(n+1, n)$ and $(n, n+1)$, respectively (see, for example, [**19**]).

Trigonometric moment problem. A bounded, nondecreasing function ψ will be called a *distribution function.* The *trigonometric moment problem* (TMP) can be stated as follows: For a given double sequence $\{\mu_k\}_{-\infty}^\infty$ of complex numbers, find necessary and sufficient conditions for the existence of a distribution function $\psi(\theta)$ with infinitely many points of increase on $-\pi \le \theta \le \pi$, such that

$$\mu_n = \int_{-\pi}^{\pi} e^{-in\theta}\, d\psi(\theta), \quad n = 0, \pm 1, \pm 2, \dots . \tag{2.9}$$

Such a function ψ will be called a *solution* of the TMP. It is well known that, if a solution of the TMP exists, then it is unique except at points of discontinuity [**2**, pp. 180–181].

The class $\mathcal{C}$ of *normalized carathéodory functions* is defined by

$$\begin{aligned}\mathcal{C} := [f : f(0) > 0 \quad &\text{and} \quad f(z) \quad \text{is holomorphic and} \\ &\operatorname{Re} f(z) > 0 \quad \text{for } |z| < 1].\end{aligned} \tag{2.10}$$

We consider the decomposition $\mathcal{C} = \mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c$, where $\mathcal{C}_a$ consists of all constant functions equal to a positive constant; $\mathcal{C}_b := \bigcup_{n=1}^\infty \mathcal{C}_n$ where $\mathcal{C}_n$ denotes the class of all rational functions of the form

$$\sum_{m=1}^{n} \lambda_m \frac{e^{i\theta_m} + z}{e^{i\theta_m} - z}, \qquad \lambda_m > 0, \qquad -\pi \le \theta_1 < \theta_2 < \dots < \theta_n \le \pi; \tag{2.11}$$

and $\mathcal{C}_c$ consists of all elements of $\mathcal{C}$ not in $\mathcal{C}_a \cup \mathcal{C}_b$.

Connections between the TMP, the class $\mathcal{C}_c$, hermitian positive definite sequences, and positive PC-fractions are summarized by the following theorem; proofs can be found in, for example, [**18, 19, 21**].

Theorem 2.2. *Let $\{\mu_k\}_{-\infty}^{\infty}$ be a given double sequence of complex numbers such that*

$$\mu_0 > 0 \quad \text{and} \quad \mu_{-k} = \bar{\mu}_k \quad \text{for } k = 1, 2, 3, \dots . \tag{2.12}$$

Let L_0 be the fps defined by

$$L_0 := \mu_0 + 2\sum_{k=1}^{\infty} \mu_k z^k. \tag{2.13}$$

Then the following four statements are equivalent:

(i) *$\{\mu_k\}_{-\infty}^{\infty}$ is hermitian positive definite.*

(ii) *There exists a solution ψ to the trigonometric moment problem for $\{\mu_k\}$, and L_0 is the Taylor series expansion at $z = 0$ of the holomorphic moment generating function*

$$f(z) := \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\psi(\theta), \quad |z| < 1. \tag{2.14}$$

(iii) *There exists a positive PC-fraction* (2.2) *corresponding to the pair (L_0, L_∞) of fps* (2.4) *in the sense of Theorem* 1.

(iv) *The fps L_0 converges for $|z| < 1$ to a normalized Carathéodory function $f(z)$ in the class $\mathcal{C}_c$.*

Szegö polynomials. We denote by $\Phi_\infty[-\pi, \pi]$ the family of all distribution functions $\psi(\theta)$ with infinitely many points of increase on $-\pi \le \theta \le \pi$. It can then be seen that each $\psi \in \Phi_\infty[-\pi, \pi]$ defines an inner product on $\Lambda \times \Lambda$ by

$$(f, g) := \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\psi(\theta) \quad \text{for } f, g \in \Lambda. \tag{2.15}$$

Here Λ denotes the linear space of all *Laurent polynomials* (*L*-

polynomials)

$$\sum_{n=p}^{q} c_n z^n, \quad c_n \in \mathbf{C}, \quad -\infty < p \leq n \leq q < \infty.$$

The following theorem describes connections between positive PC-fractions and Szegö polynomials orthogonal on the unit circle with respect to the inner product (2.15).

Theorem 2.3. *Let* (2.2) *be a given positive PC-fraction and denote its* n*-th denominator by* $Q_n(z)$ *and the distribution function of Theorem* 2.2(*ii*) *by* $\psi(\theta)$*. Let* $\{\rho_n\}$ *and* $\{\rho_n^*(z)\}$ *be defined by*

$$\rho_n(z) = Q_{2n+1}(z) \quad \text{and} \quad \rho_n^*(z) = Q_{2n}(z), \quad n = 0, 1, 2, \dots. \tag{2.16}$$

Then, for $n = 0, 1, 2, \dots,$

(2.17a)
$$\begin{aligned}(\rho_n, z^m) :=& \int_{-\pi}^{\pi} \rho_n(e^{i\theta}) e^{-im\theta}\, d\psi(\theta) \\ &= \begin{cases} 0, & \text{if } m = 0, 1, \dots, n-1, \\ \Delta_n/\Delta_{n-1}, & \text{if } n = m, \end{cases}\end{aligned}$$

$$(\rho_n^*, z^m) = \begin{cases} \Delta_n/\Delta_{n-1}, & \text{if } m = 0 \\ 0, & \text{if } m = 1, 2, \dots, n, \end{cases} \tag{2.17b}$$

$$(\rho_n, \rho_n) = (\rho_n, z^n) = \Delta_n/\Delta_{n-1}, \tag{2.17c}$$

$$\rho_n^*(z) = z^n \overline{\rho_n(1/\bar{z})}, \tag{2.18}$$

and, for $n = 1, 2, 3, \dots,$

$$\rho_n(z) = z\rho_{n-1}(z) + \delta_n \rho_{n-1}^*(z) \tag{2.19a}$$

$$\rho_n^*(z) = \bar{\delta}_n z \rho_{n-1}(z) + \rho_{n-1}^*(z) \tag{2.19b}$$

and

$$\delta_n = -\frac{(z\rho_{n-1}, 1)}{(\rho_{n-1}^*, 1)} = -\frac{\sum_{j=0}^{n-1} q_j^{(n-1)} \mu_{-j-1}}{\sum_{j=0}^{n-1} \overline{q_j^{(n-1)}} \mu_{j+1-n}}, \tag{2.20a}$$

where

$$\rho_n(z) =: \sum_{j=0}^{n} q_j^{(n)} z^j, \qquad q_n^{(n)} := 1. \tag{2.20b}$$

Remarks on the proof of Theorem 2.3. The orthogonality and normality conditions (2.17) can be derived easily from (2.7), (2.15) and (2.16). The reciprocity conditions (2.18) and recurrence relations (2.19) follow directly from the difference equations (2.3). Finally (2.20) is a simple consequence of (2.17) and (2.19).

The polynomials ρ_n defined by (2.16) are called the *monic Szegö polynomials with respect to the distribution function* ψ. It is clear from (2.17) why one says that the Szegö polynomials are orthogonal on the unit circle $\Gamma := [z \in \mathbf{C} : |z| = 1]$. Szegö considered the polynomials

$$\varphi_n(z) := \alpha_n \rho_n(z), \qquad \alpha_n := \sqrt{\Delta_{n-1}/\Delta_n} \tag{2.21}$$

normalized so that $(\varphi_n, \varphi_n)_\psi = 1$, $n \geq 0$. If we are given the moments $\{\mu_k\}$ for a distribution function $\psi \in \Phi_\infty[-\pi, \pi]$ (see (2.9)), then the Szegö polynomials ρ_n (or φ_n) can be computed by various means. The importance of this computation for this paper is shown in Sections 3 and 6. We therefore discuss some procedures by which the computation can be carried out.

Levinson's algorithm. One method is to compute the coefficients δ_n in (2.19) by *Levinson's algorithm* [**24**] described as follows (see also (3.29)). Suppose that, for some integer n, one has computed δ_{n-1} and the coefficients $q_j^{(n-2)}$, $j = 0, 1, \dots, n-2$, defined by (2.20b). One then has from (2.19)

$$q_0^{(n-1)} = \delta_{n-1} \quad \text{and} \quad q_j^{(n-1)} = q_{j-1}^{(n-2)} + \delta_{n-1} \overline{q_{n-2-j}^{(n-2)}}, \qquad j = 1, 2, \dots, n-2, \tag{2.22}$$

from which δ_n can be computed by (2.20a). The process can be repeated to compute $\delta_{n+1}, \delta_{n+2}, \dots$. It can be seen that the number of operations needed to compute $\delta_1, \delta_2, \dots, \delta_n$ is $O(n^2)$. Some evidence for the numerical stability of Levinson's algorithm is given by [**7**] and [**8**]. The latter reference also describes two other methods to compute the δ_n coefficients: Schur's algorithm and a quotient-difference algorithm (see also [**20**] and [**27**]). The quotient-difference algorithms in this context can be derived from properties of the PC-fractions.

The Szegö polynomials ρ_n can also be constructed by solving systems of linear equations for the coefficients $q_j^{(n)}$ in (2.20b). We restrict ourselves here to the case in which the moments μ_k are all real. Such systems of equations can be obtained by using the well-known property that, for each $n = 0, 1, 2, \dots$, the set

$$[(R_n, R_n) : R_n(z) \text{ is a monic polynomial of degree } n]$$

attains its minimum value for the unique polynomial $R_n = \rho_n$ [**13**, Section 2.2]. We write

$$R_n(z) = \sum_{j=0}^{n} r_j^{(n)} z^j, \qquad r_j^{(n)} \in \mathbf{R}\,, r_n^{(n)} = 1. \tag{2.23}$$

It follows that the system of equations

$$\frac{\partial (R_n, R_n)}{\partial r_m^{(n)}} = 0, \quad m = 0, 1, \dots, n-1,$$

has the unique solution $r_j^{(n)} = q_j^{(n)}$, $j = 0, 1, \dots, n-1$. Hence, the $q_j^{(n)}$ satisfy the positive definite Toeplitz system of equations

$$\sum_{j=0}^{n-1} r_j^{(n)} \mu_{m-j} = -\mu_{m-n}, \quad m = 0, 1, \dots, n-1. \tag{2.24}$$

In deriving the *normal equations* (2.24) we make use of

$$(R_n, R_n) = \int_{-\pi}^{\pi} |R_n(e^{i\theta})|^2 \, d\psi(\theta) = \sum_{j,k=0}^{n} r_j^{(n)} \overline{r_k^{(n)}} \mu_{k-j}, \tag{2.25}$$

which is a consequence of (2.9), (2.15) and (2.23). The system (2.24) can be solved by Gaussian elimination, which requires $O(n^3)$ arithmetic operations. Much faster algorithms for solving real, positive definite

Toeplitz systems (2.24) have been found recently, using divide-and-conquer techniques, by [**3, 4, 5** and **6**]. These methods require only $O(n \log_2^2 n)$ operations.

Digital filters. Since our interest here is in discrete filters, we consider the linear space ℓ of all real double sequences

$$\ell := [u = \{u(k)\}_{k=-\infty}^{\infty} : u(k) \in \mathbf{R}, \quad k = 0, \pm 1, \pm 2, \dots].$$

An element $u \in \ell$ is called a (discrete) *signal*. We are concerned with linear transformations $T : \ell_D \to \ell_R$ that map subsets ℓ_D of ℓ into subsets ℓ_R. One such transformation is the *shift operator* S defined by

$$(Su)(k) := u(k-1) \quad \text{for all } u \in \ell, \quad k = 0, \pm 1, \pm 2, \dots .$$

A transformation T is called *shift-invariant* if

$$ST = TS. \tag{2.26}$$

We note that (2.26) implies $S^m T = TS^m$ for $m = 0, \pm 1, \pm 2, \dots$. A linear shift-invariant (LSI) transformation $T : \ell_D \to \ell_R$ is called a *digital filter*.

The signal δ defined by

$$\delta(k) := \begin{cases} 0, & k \neq 0 \\ 1, & k = 0 \end{cases}$$

is called the *unit pulse* and its image $h = T\delta$ is called the *unit pulse response* for a digital filter T. The following theorem indicates the manner in which the unit pulse response can be used to represent a digital filter. We employ the standard terminology for normed linear spaces

$$\ell_1 := \left[u \in \ell : \|u\|_1 := \sum_{k=-\infty}^{\infty} |u(k)| < \infty \right]$$

and

$$\ell_\infty := \left[u \in \ell : \|u\|_\infty := \sup_{k \in \mathbf{Z}} |u(k)| < \infty \right].$$

We also make use of the notation for the *convolution* $u*h$ of two signals $u, h \in \ell$ defined by

$$u * h := \left\{ \sum_{m=-\infty}^{\infty} u(m)h(k-m) \right\}_{k=-\infty}^{\infty}, \tag{2.27}$$

provided the sums in (2.27) exist.

Theorem 2.4. *Let $h \in \ell_1$ be given. Then the sums in* (2.27) *are all convergent if $u \in \ell_\infty$; moreover,*

$$Tu := u * h = h * u, \quad u \in \ell_\infty \tag{2.28}$$

defines a digital filter $T : \ell_\infty \to \ell_\infty$, T is a continuous transformation, and h is the unit pulse response $h = T\delta$.

A filter T is called *BIBO* (*bounded input bounded output*) *stable* if the sequence Tu is bounded whenever the input sequence u is bounded. Clearly, the filter (2.28) of Theorem 2.4 is BIBO stable.

A signal $u \in \ell$ is said to be *causal* if

$$u(k) = 0 \quad \text{for } k < 0.$$

A digital filter T is said to be *causal* if it maps causal signals into causal signals. It can be shown that T is a causal filter iff

$$u(k) = v(k) \quad \text{for } k < m \Longrightarrow (Tu)(k) = (Tv)(k) \quad \text{for } k < m.$$

It is easily seen that if $T : \ell_\infty \to \ell_\infty$ is a filter defined by $Tu = h * u$ where $h \in \ell_1$ and h is causal, then T is causal.

The Z-transform is a useful concept in the theory of digital filters. For each $u \in \ell$, the *Z-transform* $U(z)$ of u is defined by the formal series

$$U(z) := \sum_{m=-\infty}^{\infty} u(m)z^{-m}.$$

In our notation we use a cap letter for the Z-transform of a signal denoted by the corresponding lower case letter, and we write

$$U(z) \circ\!\!\xrightarrow{\;z\;}\!\!\circ\, u = \{u(m)\}_{-\infty}^{\infty}$$

to indicate the correspondence. It is readily seen that

$$H(z)U(z) \circ\!\!\overset{z}{\longrightarrow}\!\!\circ\, h * u,$$

provided the sums in $h * u$ converge. Thus, we have

Theorem 2.5. *If $h \in \ell_1$ and*

$$y := Tu := h * u \quad \textit{for } u \in \ell_\infty \tag{2.29}$$

then

$$Y(z) = H(z)U(z). \tag{2.30}$$

Since $h \in \ell_1$, the series $H(z) := \sum_{-\infty}^{\infty} h(m)z^{-m}$ is convergent at least for $|z| = 1$ to a function H. If $h \in \ell_1$ and h is causal, then the series $H(z)$ converges at least for $|z| \geq 1$ to a function H holomorphic in $|z| > 1$. The function H in Theorem 2.5 is called the *transfer function* of the filter T; $H(e^{i\theta})$, $|H(e^{i\theta})|$ and $\arg H(e^{i\theta})$ are called, respectively, the *frequency response, magnitude response* and *phase response functions* of T. The importance of these functions is made clear by the following theorem, which is an immediate consequence of the preceding results.

Theorem 2.6. *Let $u = \{u(k)\}$ be a given signal of the form, for $k = 0, \pm 1, \pm 2, \dots,$*

$$u(k) = \sum_{j=-I}^{I} \alpha_j e^{i\omega_j k} = \lambda_0 + \sum_{j=1}^{I} \lambda_j \cos(\omega_j k + \varphi_j), \tag{2.31}$$

where $1 \leq I < \infty$, $\alpha_0 = \lambda_0 \geq 0$, $\omega_0 = 0$, and, for $1 \leq j \leq I$, $\lambda_j > 0$, $\omega_{-j} = -\omega_j \in \mathbf{R}$, $\varphi_{-j} = -\varphi_j \in \mathbf{R}$, and $\alpha_{-j} = \bar{\alpha}_j = (1/2)\lambda_j e^{-i\varphi_j}$. Let $T : \ell_\infty \to \ell_\infty$ be defined by $Tu := h * u$, $h \in \ell_1$. Then

(2.32)

$$\begin{aligned}(Tu)(k) &= \sum_{j=-I}^{I} \alpha_j H(e^{i\omega_j}) e^{i\omega_j k} \\ &= \lambda_0 H(1) + \sum_{j=1}^{I} \lambda_j |H(e^{i\omega_j})| \cos(\omega_j k + \varphi_j + \arg H(e^{i\omega_j})).\end{aligned}$$

It follows from Theorem 2.6 that each term of the input signal u with frequency ω_j appears in the output signal Tu with the multiplicative factor $H(e^{i\omega_j})$. Thus, the frequency response function $H(e^{i\theta})$ controls the filtering of the individual terms in the input. In particular, we see that

(2.33)
$$H(e^{i\omega_j}) = 0 \quad \text{for } 0 \le j \le I \Longrightarrow (Tu)(k) = 0 \quad \text{for } k = 0, \pm 1, \pm 2, \dots .$$

Therefore, the entire signal is filtered out if the frequency response function vanishes at the points $e^{i\omega_j}$ on the unit circle. This property is of special interest in the problem of frequency analysis (Section 3). Theorem 2.6 also helps us understand the role of $|H(e^{i\theta})|$ in the problem of constructing filters treated in Section 6. We are ready now to consider the frequency analysis problem in the following section.

3. Frequency analysis. The problem of frequency analysis considered here is the following. For a given signal $u = \{u(k)\}_{k=-\infty}^{\infty}$ of the form

$$u(k) = \sum_{j=-I}^{I} \alpha_j e^{i\omega_j k}, \quad k = 0, \pm 1, \pm 2, \dots , \tag{3.1}$$

where $\alpha_0 \ge 0$, $\omega_0 = 0$, $\omega_{-j} = -\omega_j \in \mathbf{R}$, $\alpha_{-j} = \bar{\alpha}_j$ for $j = 1, 2, \dots , I$, we wish to find (or approximate) the frequencies $\omega_1, \omega_2, \dots , \omega_I$. By Theorem 2.5 we see that if we could find a digital filter T of the form $Tv = h * v$ with $h \in \ell_1$ and $v \in \ell_\infty$, such that $Tu = \{0\}$, the zero signal, then we would expect that the zeros of the transfer function $H(z) \bullet\!\!\xrightarrow{z}\!\!\circ$ are $e^{i\omega_j}$, $j = 1, 2, 3, \dots$. We describe here a method that yields a sequence of filters $\{T_n\}$ with transfer functions $\{H_n(z)\}$ of the form

$$H_n(z) = \sum_{j=0}^{n} h_j^{(n)} z^{-j}, \qquad h_j^{(n)} \in \mathbf{R}, h_0^{(n)} = 1, \tag{3.2}$$

such that $T_n u = \{\varepsilon_k^{(n)}\}$, and we determine the $h_j^{(n)}$ so as to minimize the sum of squares $\sum_{k=-\infty}^{\infty} [\varepsilon_k^{(n)}]^2$. It will be seen that the zeros $z_m^{(n)}$, $m = 1, 2, \dots , n$, of H_n all lie in the unit disk $|z| < 1$. We choose the ones nearest to the unit circle to approximate the values $e^{i\omega_j}$, from which $e^{i\omega_j}$ can be determined, $|j| \le I$.

Since our computations can involve only a finite number of terms in the signal (sequence) u, we shall work with truncated signals $u_N = \{u_N(k)\}_{-\infty}^{\infty}$ defined as follows:

$$u_N(k) = \begin{cases} u(k), & \text{for } k = 0, 1, \dots, N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

In practice, the sample size N will be much larger than the degree n in (3.2). The following autocorrelation theorem is basic for this purpose.

Theorem 3.1. *Let $x = \{x(k)\}_{-\infty}^{\infty}$ be a given real signal such that*

$$x(k) = 0 \quad \textit{for } k < 0 \quad \textit{and for } k \geq N, \tag{3.4a}$$

and

$$x(k^*) \neq 0 \quad \textit{for some } k^* \quad \textit{such that } 0 \leq k^* \leq N-1. \tag{3.4b}$$

Let

$$\mu_k := \sum_{m=-\infty}^{\infty} x(m)x(m+k), \quad k = 0, \pm 1, \pm 2, \dots . \tag{3.5}$$

Then $\{\mu_k\}_{-\infty}^{\infty}$ is a hermitian positive definite sequence; that is, it satisfies, for $k = 1, 2, \dots,$

$$\mu_0 > 0, \mu_{-k} = \mu_k \quad \textit{and} \quad \Delta_n := \det(\mu_{j-k})_{j,k=0}^{n} > 0 \tag{3.6}$$

(*see* (2.1)).

Proof. For each $k = 0, \pm 1, \pm 2, \dots,$ we see that, by (3.5),

$$\mu_{-k} := \sum_{m=-\infty}^{\infty} x(m)x(m-k) = \sum_{j=-\infty}^{\infty} x(j+k)x(j) = \mu_j.$$

By (3.4),

$$\mu_0 = \sum_{m=0}^{N-1} [x(m)]^2 \geq [x(k^*)]^2 > 0.$$

Since

$$(x(0), x(1), \dots, x(N-1))^T \neq (0, 0, \dots, 0)^T \in \mathbf{R}^N,$$

there exists t_0 such that $0 \le t_0 \le N-1$ and

$$x(t_0) \neq 0 \quad \text{and} \quad x(t) = 0 \quad \text{for } t < t_0. \tag{3.7}$$

Let $n \ge 0$ be given. We then let $(u_0, u_1, \dots, u_n)^T$ denote an arbitrary nonzero vector in $\mathbf{R}^{n+1}$. Hence, there exists j_0 such that $0 \le j_0 \le n$ and

$$u_{j_0} \neq 0 \quad \text{and} \quad u_j = 0 \quad \text{for } j_0 < j \le n \quad \text{if } j_0 < n. \tag{3.8}$$

Setting $m_0 := t_0 - j_0$, we obtain from (3.8), (3.6) and the definitions of m_0 and j_0

$$\begin{aligned} \sum_{j=0}^{n} u_j x(m_0 + j) &= \sum_{j=0}^{j_0} u_j x(m_0 + j) \\ &= \sum_{j=0}^{j_0} u_j x(t_0 - j_0 + j) \\ &= u_{j_0} x(t_0) \neq 0. \end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\begin{aligned} \sum_{j,k=0}^{n} u_j u_k \mu_{k-j} &= \sum_{j,k=0}^{n} u_j u_k \left[\sum_{m=-\infty}^{\infty} x(m+j) x(m+k) \right] \\ &= \sum_{m=-\infty}^{\infty} \sum_{j,k=0}^{n} u_j x(m+j) u_k x(m+k) \\ &= \sum_{m=-\infty}^{\infty} \left[\sum_{j=0}^{n} u_j x(m+j) \right]^2 \\ &\ge \left[\sum_{j=0}^{n} u_j x(m_0 + j) \right]^2 = [u_{j_0} x(t_0)]^2 > 0. \end{aligned} \tag{3.10}$$

We can deduce $\Delta_n > 0$ for $n \ge 0$ from (3.10) and the theory of positive definite Toeplitz forms [**17**, Section 9.3, Theorem 6]. $\square$

We now describe the Wiener-Levinson linear prediction method and apply it to a signal $u_N = \{u_N(k)\}_{-\infty}^{\infty}$ of the form given by (3.1) and (3.3) for some given positive integer N. For each $n = 1, 2, 3, \ldots$ we seek a predictor $\hat{u}_N(k)$ of $u_N(k)$ of the form

$$\hat{u}_N(k) := \begin{cases} -\sum_{j=1}^{n} h_j^{(n)} u_N(k-j), & k \geq 1, \quad h_j^{(n)} \in \mathbf{R}, \\ 0, & k \leq 0. \end{cases} \tag{3.11}$$

Its residual is then

$$\varepsilon_k^{(n)} := u_N(k) - \hat{u}_N(k) = \sum_{j=0}^{n} h_j^{(n)} u_N(k-j), \quad h_0^{(n)} := 1, \tag{3.12}$$

and, hence,

$$\begin{aligned} \varepsilon^{(n)} := \{\varepsilon_k^{(n)}\}_{k=-\infty}^{\infty} &= \left\{ \sum_{j=0}^{n} h_j^{(n)} u_N(k-j) \right\}_{k=-\infty}^{\infty} \\ &= \{h_j^{(n)}\} * \{u_N(j)\} = h^{(n)} * u_N. \end{aligned} \tag{3.13}$$

It follows that $\varepsilon^{(n)}$ is the output from a filter T_n with unit pulse response $h^{(n)} = \{h_j^{(n)}\}_{j=-\infty}^{\infty}$, where $h_j^{(n)} = 0$ for $j < 0$ and $j > n$; hence,

$$T_n u_n = \varepsilon^{(n)} = h^{(n)} * u_N \tag{3.14}$$

and thus the transfer function H_n of T_n is given by (3.2). Clearly, $h^{(n)} \in \ell_1$ and $u_n \in \ell_\infty$. Following the ideas stated at the beginning of this section, we wish to choose the coefficients $h_j^{(n)}$ so as to make the residuals $\varepsilon_k^{(n)}$ small in magnitude. In fact, we shall determine the $h_j^{(n)}$ in such a manner as to minimize the sum of squares of the residuals $\|\varepsilon^{(n)}\|_2^2 := \sum_{k=-\infty}^{\infty} [\varepsilon_k^{(n)}]^2$. To achieve that end we write

$$\begin{aligned} \|\varepsilon^{(n)}\|_2^2 &= \sum_{k=-\infty}^{\infty} \left[\sum_{j=0}^{n} h_j^{(n)} u_N(k-j) \right]^2 \qquad \text{by (3.12)} \\ &= \sum_{j=0}^{n} \sum_{m=0}^{n} h_j^{(n)} h_m^{(n)} \sum_{k=-\infty}^{\infty} u_N(k-j) u_N(k-m) \\ &= \sum_{j=0}^{n} \sum_{m=0}^{n} h_j^{(n)} h_m^{(n)} \mu_{j-m}, \end{aligned} \tag{3.15}$$

where we define

$$\mu_k := \sum_{m=-\infty}^{\infty} u_N(m)u_N(m+k), \quad k = 0, \pm 1, \pm 2, \dots . \tag{3.16}$$

It follows from Theorem 3.1 that $\{\mu_k\}_{-\infty}^{\infty}$ is hermitian positive definite. Therefore, by Theorem 2.2, there exists a solution ψ to the trigonometric moment problem for $\{\mu_k\}$, and hence

$$\mu_k = \int_{-\pi}^{\pi} e^{-ik\theta}\, d\psi(\theta), \quad k = 0, \pm 1, \pm 2, \dots . \tag{3.17}$$

Combining this with (3.15) yields

$$\begin{aligned} \|\varepsilon^{(n)}\|_2^2 &= \sum_{j=0}^{n}\sum_{m=0}^{n} h_j^{(n)} h_m^{(n)} \int_{-\pi}^{\pi} e^{i(m-j)\theta}\, d\psi(\theta) \\ &= \int_{-\pi}^{\pi} \left| \sum_{j=0}^{n} h_j^{(n)} e^{-ij\theta} \right|^2 d\psi(\theta) \\ &= (H_n, H_n) = \|H_n\|^2 \end{aligned} \tag{3.18}$$

in the notation of (2.15). If we define $\sigma_n(z) := z^n H_n(z)$, then it is readily verified from (3.18) that

$$\|\varepsilon^{(n)}\|_2^2 = (H_n, H_n) = (z^{-n}\sigma_n, z^{-n}\sigma_n) = (\sigma_n, \sigma_n). \tag{3.19}$$

Since σ_n is a monic polynomial in z of degree n, it follows from a well-known theorem on Szegö polynomials [**13**, Section 2.2] that

$$E_n := \min_{h_j^{(n)} \in \mathbf{R}} \|\varepsilon^{(n)}\|_2^2 = (\rho_n, \rho_n), \tag{3.20}$$

where $\{\rho_n\}$ is the sequence of monic Szegö polynomials with respect to the distribution ψ. The preceding results are summarized in

Theorem 3.2. *Let* $u_N = \{u_N(k)\}$ *be a given signal of the form* (3.3) *and* (3.1). *Let* $h^{(n)} = \{h_j^{(n)}\}$ *be such that*

$$h_0^{(n)} = 1 \quad \textit{and} \quad h_j^{(n)} \in \mathbf{R}, \quad j = \pm 1, \pm 2, \dots \tag{3.21a}$$

and

$$h_j^{(n)} = 0 \quad \textit{for } j < 0 \quad \textit{and for } j > n. \tag{3.21b}$$

Let $T_n : \ell_\infty \to \ell_\infty$ *denote the digital filter*

$$T_n u_N = \varepsilon^{(n)} := h^{(n)} * u_N = \left\{ \sum_{j=0}^{n} h_j^{(n)} u_N(k-j) \right\}_{k=-\infty}^{\infty}. \tag{3.22}$$

For $k = 0, \pm 1, \pm 2, \dots,$ *let* $\mu_k := \sum_{m=-\infty}^{\infty} u_N(m) u_N(m+k)$, *so that* $\{\mu_k\}$ *is positive-definite hermitian (Toeplitz). Let* ψ *be the solution to the trigonometric moment problem for* $\{\mu_k\}$ *and let* $\{\rho_n\}$ *denote the sequence of monic Szegö polynomials with respect to the distribution* ψ. *Then*

(A)

$$\min_{h_j^{(n)} \in \mathbf{R}} \|\varepsilon^{(n)}\|_2^2 = \min_{h_j^{(n)} \in \mathbf{R}} \sum_{k=-\infty}^{\infty} \left[\varepsilon_k^{(n)}\right]^2 = (\rho_n, \rho_n) \tag{3.23}$$

is attained by

$$H_n(z) = \sum_{j=0}^{n} h_j^{(n)} z^{-j} = z^{-n} \rho_n(z). \tag{3.24}$$

(B) *The normal equations* $\partial \|\varepsilon^{(n)}\|_2^2 / \partial h_m^{(n)} = 0$ *are equivalent to the positive-definite Toeplitz system*

$$\sum_{j=1}^{n} h_j^{(n)} \mu_{m-j} = -\mu_m, \quad m = 1, 2, \dots, n. \tag{3.25}$$

Numerical illustrations. We describe here some numerical results that illustrate both the computational procedures and the types of results obtainable with the Wiener-Levinson method of frequency analysis. Our observed signals u_N consist of superpositions of sine waves and white noise

$$u_N(k) := \begin{cases} \sum\limits_{j=1}^{4} a_j \sin(\omega_j k) + R_\sigma(k), & k = 0, 1, \dots, N-1 \\ 0, & k < 0 \text{ or } k \geq N, \end{cases} \tag{3.26}$$

with $a_j \geq 0$. It is readily seen that

$$\sum_{j=1}^{4} a_j \sin(\omega_j k) = \sum_{j=-4}^{4} \alpha_j e^{i\omega_j k}$$

where $\alpha_0 = 0$, $\omega_{-j} = -\omega_j$, $\alpha_{-j} = \bar{\alpha}_j$, $\varphi_j := \arg \alpha_j = -\pi/2$, and $a_j = 2|\alpha_j|$, for $j = 1, 2, 3, 4$. Hence, (3.21) has the form given by (3.1) and (3.3) with $I = 4$. The component $R_\sigma(k)$ consists of white noise and was formed by

$$R_\sigma(k) := \left(\frac{R_k - \mu}{s}\right)\sigma, \quad k = 0, 1, \ldots, N-1, \tag{3.27a}$$

where the R_k are random whole numbers taken from [**1**, pp. 991–995], and

$$\mu := \frac{1}{N}\sum_{k=0}^{N-1} R_k, \qquad s := \sqrt{\frac{1}{N}\sum_{k=0}^{N-1}(R_k - \mu)^2}. \tag{3.27b}$$

It follows that $R_\sigma(k)$ has sample mean equal to zero and variance σ^2.

We then compute the autocorrelation coefficients (3.16) by

$$\mu_k := \sum_{m=0}^{N-k-1} u_N(m) u_N(m+k), \quad k = 0, 1, 2, \ldots, K, \tag{3.28}$$

and set $\mu_{-k} := \mu_k$, $k = 1, 2, \ldots, K$. We can now apply the

Levinson algorithm. Given $\mu_0, \mu_1, \ldots, \mu_K$, we compute $\delta_0, E_0, \delta_1, E_1, \ldots, \delta_K, E_K$ successively. Set initially

$$\delta_0 = 1, \quad E_0 = \mu_0, \quad \delta_1 = -\mu_1/\mu_0, \quad q_0^{(1)} = \delta_1, \quad q_1^{(1)} = 1. \tag{3.29a}$$

Then, for $k = 2, 3, \ldots, K$, compute

$$\begin{aligned} E_{k-1} &= \sum_{j=0}^{k-1} q_j^{(k-1)} \mu_{k-1-j}, \\ \delta_k &= -\frac{\sum_{j=0}^{k-1} q_j^{(k-1)} \mu_{j+1}}{E_{k-1}}, \\ q_j^{(k)} &= \delta_k q_{k-1-j}^{(k-1)} + q_{j-1}^{(k-1)}, \quad j = 1, 2, \ldots, k-1, \\ q_k^{(k)} &= 1, \qquad q_0^{(k)} = \delta_k. \end{aligned} \tag{3.29b}$$

Finally,

$$E_K = \sum_{j=0}^{n} q_j^{(k)} \mu_{K-j}. \tag{3.29c}$$

We consider three examples of signals of the form (3.26), where the sample size $N = 200$ and the variance of white noise $\sigma^2 = 0.02$. The amplitudes a_j and frequencies ω_j are chosen as follows:

Example 1. $N = 200$, $\sigma^2 = 0.02$.

j	1	2	3	4
a_j	1	0	0	0
ω_j	$\frac{\pi}{4} \doteq .785398$			

Example 2. $N = 200$, $\sigma^2 = 0.02$.

j	1	2	3	4
a_j	1	1	0	0
ω_j	$\frac{\pi}{4} \doteq .785398$	$\frac{\pi}{3} \doteq 1.047198$		

Example 3. $N = 200$, $\sigma^2 = 0.02$.

j	1	2	3	4
a_j	1	1	1	10
ω_j	$\frac{\pi}{2} \doteq 1.570796$	$\frac{\pi}{3} \doteq 1.047198$	$\frac{\pi}{6} \doteq .523599$	$\frac{3\pi}{4} \doteq 2.356194$

For each of the three examples, the reflection coefficients δ_k and sums of squares of residuals E_k have been computed using Levinson's algorithm (3.29) (see Table 1). In each example it can be seen that E_k decreases as k increases. The rate of decrease of E_k is high for small k. A large value of δ_k generally coincides with a large jump from E_{k-1} to E_k. Zeros $z_j^{(k)}$ of the Szegö polynomials $\rho_k(z)$ are given in Tables 3, 4 and 5, respectively, for Examples 1, 2 and 3. We have included only the zeros that are very near to the unit circle $|z| = 1$; that is, the zeros that provide approximations of the frequencies $\omega_j \approx \operatorname{Arg} z_j^{(k)}$. It can be seen that the approximations have about three significant digits in all cases provided k is sufficiently large. For Examples 1, 2 and 3, it suffices to choose $k = 12, 24$ and 24, respectively. From Table 1 we see that, for higher values of k, E_k decreases very slowly. This concludes our discussion of the frequency analysis problem.

TABLE 1. Reflection coefficients δ_k and sums of squares of residuals E_k.

	Example 1		Example 2		Example 3	
k	δ_k	E_k	δ_k	E_k	δ_k	E_k
0	1.00000	206.44	1.00000	399.04	1.00000	20651.19
1	-.68110	110.67	-.59189	259.24	.67266	11306.98
2	.85320	30.10	.90686	46.08	.83772	3371.93
3	.66065	16.96	.16329	44.81	-.59796	2166.27
4	.33795	15.02	-.07624	44.55	.63988	1279.28
5	.05890	14.97	.32520	39.83	-.26756	1187.69
6	-.23779	14.12	.43363	32.34	.27821	1095.76
7	-.17296	13.70	.33325	28.75	.31942	983.96
8	-.16160	13.34	-.14114	28.28	.00186	983.96
9	-.06837	13.28	-.34191	24.88	.33702	872.19
10	.07683	13.20	-.27696	22.97	.60999	547.66
11	.12469	13.00	-.09115	22.78	.27864	505.13
12	-.05560	12.96	.06167	22.70	-.40173	423.61
13	.09100	12.85	.34276	20.03	-.50309	316.39
14	.20358	12.85	.13461	19.67	-.12329	311.58
15	.00699	12.84	-.05064	19.62	.19343	299.92
16	.00684	12.84	-.11123	19.37	.10820	296.41
17	.00505	12.84	-.07340	19.27	.01145	296.37
18	.01479	12.84	-.00159	19.27	.10034	293.39
19	-.03620	12.82	.01854	19.26	.09033	290.99
20	.11318	12.66	.15901	18.77	-.05774	290.02
30	.06109	12.27	.05124	18.38	.07574	271.70
40	-.00374	12.06	.00510	17.98	-.01212	267.27
49	.04136	11.96	.01360	17.86	.05280	264.61

TABLE 2. Zero $z_1^{(k)}$ of $\rho_k(z)$ for Example 1 giving the approximation $\operatorname{Arg} z_1^{(k)} \approx \omega_1 := \pi/4 \doteq .785398$.

k	$\operatorname{Re} z_1^{(k)}$	$\operatorname{Im} z_1^{(k)}$	$\lvert z_1^{(k)} \rvert$	$\operatorname{Arg} z_1^{(k)}$
4	.700 946	.699 025	.98993	$.\underline{7}84\ 026$
8	.704 172	.704 374	.99599	$.\underline{78}5\ 541$
12	.704 617	.705 289	.99695	$.\underline{78}5\ 874$
16	.705 058	.704 785	.99691	$.\underline{785}\ 204$
20	.705 524	.704 923	.99735	$.\underline{784}\ 959$

TABLE 3. Zeros $z_1^{(k)}$ and $z_2^{(k)}$ of $\rho_k(z)$ for Example 2 giving the approximations

$$\operatorname{Arg} z_1^{(k)} \approx \omega_1 := \frac{\pi}{4} \doteq .785398$$

and

$$\operatorname{Arg} z_2^{(k)} \approx \omega_2 := \frac{\pi}{3} \doteq 1.047198.$$

	k	$\operatorname{Re} z_j^{(k)}$	$\operatorname{Im} z_j^{(k)}$	$\lvert z_j^{(k)} \rvert$	$\operatorname{Arg} z_j^{(k)}$
$j=1$	8	.682 672	.669 799	.95638	$.\underline{7}75\ 880$
	12	.702 575	.696 048	.98898	$.\underline{7}80\ 731$
	16	.705 565	.703 016	.99601	$.\underline{7}83\ 588$
	20	.705 901	.704 518	.99731	$.\underline{7}84\ 417$
	24	.706 041	.704 820	.99763	$.\underline{784}\ 532$
	28	.705 748	.705 112	.99762	$.\underline{784}\ 947$
$j=2$	8	.475 995	.835 592	.96165	$\underline{1}.052\ 991$
	12	.491 088	859 168	.98961	$\underline{1.05}1\ 531$
	16	.496 624	.863 971	.99653	$\underline{1.04}9\ 100$
	20	.498 059	.864 287	.99752	$\underline{1.04}8\ 011$
	24	.498 362	.864 099	.99749	$\underline{1.047}\ 685$
	28	.498 654	.864 058	.99762	$\underline{1.047}\ 379$

TABLE 4. Zeros $z_j^{(k)}$ of $\rho_k(z)$ for Example 3 giving the approximations

$$\operatorname{Arg} z_1^{(k)} \approx \omega_1 := \frac{\pi}{2} \doteq 1.570796, \quad \operatorname{Arg} z_2^{(k)} \approx \omega_2 := \frac{\pi}{3} \doteq 1.047198,$$
$$\operatorname{Arg} z_3^{(k)} \approx \omega_3 := \frac{\pi}{6} \doteq .523599, \quad \operatorname{Arg} z_4^{(k)} \approx \omega_4 := \frac{3\pi}{4} \doteq 2.356194.$$

	k	$\operatorname{Re} z_j^{(k)}$	$\operatorname{Im} z_j^{(k)}$	$\lvert z_j^{(k)}\rvert$	$\operatorname{Arg} z_j^{(k)}$
$j=1$	12	-.009 829	.982 445	.98249	1.580 801
	16	.005 964	.993 505	.99352	1.564 792
	20	.004 095	.994 613	.99462	1.566 678
	24	.002 071	.996 296	.99629	1.568 717
	28	.002 152	.997 496	.99749	1.568 638
$j=2$	12	.487 094	.850 849	.98041	1.050 857
	16	.498 136	.862 708	.99619	1.047 153
	20	.498 729	.861 227	.99521	1.045 892
	24	.499 070	.862 992	.99690	1.046 484
	28	.498 713	.863 708	.99734	1.047 153
$j=3$	12	.858 126	.485 824	.98610	.515 154
	16	.861 865	.496 758	.99477	.522 867
	20	.862 835	.497 871	.99617	.523 349
	24	.863 656	.497 546	.99672	.522 655
	28	.863 437	.497 930	.99672	.523 098
$j=4$	12	-.703 958	.707 034	.99772	2.354 014
	16	-.705 469	.705 293	.99755	2.356 318
	20	-.704 985	.705 663	.99747	2.355 713
	24	-.705 793	.704 758	.99741	2.356 928
	28	-.705 158	.705 448	.99744	2.355 988

4. Szegö's condition and H₂-functions. In this section we give a brief exposition of some aspects of the behavior of Szegö polynomials and their reciprocals under special conditions. For the general content of this section, we refer to [**11–13**]. For the theory of boundary behavior of analytic functions and harmonic functions and the theory of H_p-spaces, we refer to [**10, 16, 25** and **26**]. These results are applied in later sections to problems concerning stochastic processes and construction of digital filters.

For later use we introduce the notation

$$D := [z \in \mathbf{C} : |z| < 1], \qquad \bar{D} := [z \in \mathbf{C} : |z| \leq 1],$$
$$\partial D := [z \in \mathbf{C} : |z| = 1],$$
$$E := [z \in \hat{\mathbf{C}} : z \notin \bar{D}], \qquad \bar{E} := [z \in \hat{\mathbf{C}} : z \notin D].$$

For convenience, we recall some basic facts about Szegö polynomials (cf., Section 2).

Let a distribution function $\psi(\theta)$ on $[-\pi, \pi]$ be given. The distribution function gives rise to moments μ_n (see (2.9)), monic Szegö polynomials $\rho_n(z)$ and their reciprocal polynomials $\rho_n^*(z) := z^n \overline{\rho_n(1/\bar{z})}$ with norms (see (2.15))

$$\beta_0 := \sqrt{\mu_0} = \sqrt{\delta_0}, \qquad \beta_n := \|\rho_n\|_\psi = \|\rho_n^*\|_\psi, \quad n = 0, 1, 2, \ldots, \tag{4.1}$$

reflection coefficients

$$\delta_n := \rho_n(0), \tag{4.2}$$

and normalized Szegö polynomials $\varphi_n(z)$ (see (2.21)) and their reciprocal polynomials $\varphi_n^*(z) := z^n \overline{\varphi_n(1/\bar{z})}$. Here and in the following $\langle \cdot, \cdot \rangle_\psi$ denotes the inner product, and $\| \cdot \|_\psi$ denotes the norm in the Hilbert space $L_2^\psi[-\pi, \pi]$, i.e., for all $F, G \in L_2^\psi[-\pi, \pi]$, we have

$$\langle F, G \rangle_\psi := \int_{-\pi}^{\pi} F(\theta) \overline{G(\theta)} \, d\psi(\theta) \quad \text{and} \quad \|F\|_\psi := \sqrt{\langle F, F \rangle_\psi}.$$

We also note the close relationship between this inner product and the one defined by (2.15); it is given by

$$(f(z), g(z)) := \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\psi(\theta) =: \langle f(e^{i\theta}), g(e^{i\theta}) \rangle_\psi.$$

We may then write

$$\varphi_n(z) = \beta_n^{-1} z^n + \cdots + b_n, \tag{4.3}$$
$$\varphi_n^*(z) = \bar{b}_n z^n + \cdots + \beta_n^{-1} \tag{4.4}$$

where b_n denotes the constant term of $\varphi_n(z)$. Note that $\beta_n = \alpha_n^{-1}$ where α_n is introduced in (2.21). By using the recurrence relation

$$\rho_n(z) = \delta_n \rho_n^*(z) + (1 - |\delta_n|^2) z \rho_{n-1}(z), \tag{4.5}$$

which follows from (2.3) and (2.16), we see that

$$\begin{aligned} \beta_n^2 = \|\rho_n\|_\psi^2 = (\rho_n, z^n) &= \langle \rho_n(e^{i\theta}), e^{in\theta} \rangle_\psi \\ &= (1 - |\delta_n|^2)(\rho_{n-1}, z^{n-1}) = (1 - |\delta_n|^2)\beta_{n-1}^2, \end{aligned} \tag{4.6}$$

from which we obtain the formula

$$\beta_n^2 = \beta_0^2 \prod_{k=1}^{n} (1 - |\delta_k|^2), \quad n \geq 1, \quad \beta_0 = \sqrt{\mu_0}, \tag{4.7}$$

(cf. (2.7b) and (2.21)). Since $|\delta_n| < 1$ for $n \geq 1$, it follows immediately from (4.7) that the sequence $\{\beta_n\}$ of positive numbers is nonincreasing.

The polynomial $\omega_n(z)$ associated with $\varphi_n(z)$ is defined by

$$\omega_n(z) := \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} [\varphi_n(e^{i\theta}) - \varphi_n(z)]\, d\psi(\theta). \tag{4.8}$$

We recall that, from (2.16) and (2.21),

$$\beta_n \varphi_n(z) = Q_{2n+1}(z), \qquad \beta_n \varphi_n^*(z) = Q_{2n}(z), \tag{4.9}$$

where $Q_n(z)$ is the n-th denominator of the positive PC-fraction associated with the moment sequence $\{\mu_n\}$ and distribution function $\psi(\theta)$ (Theorem 2.2). By using the recurrence relations (2.19), one can easily verify that

$$\beta_n \omega_n(z) = P_{2n+1}(z), \qquad \beta_n \omega_n^*(z) = -P_{2n}(z), \tag{4.10}$$

where $P_n(z)$ is the n-th numerator of the positive PC-fraction. Here $\omega_n^*(z) := z^n \overline{\omega_n(1/\bar{z})}$. For more details about $\omega_n(z)$ and $\omega_n^*(z)$, the reader can refer to [**21**]. By taking into account (4.7), (4.9) and (4.10), the determinant formula for continued fractions [**22**, (2.1.9)] yields

$$\omega_n(z)\varphi_n^*(z) + \varphi_n(z)\omega_n^*(z) = -2z^n. \tag{4.11}$$

From the recurrence relations (2.19) we can deduce the following Christoffel-Darboux type formulas
(4.12)

$$\varphi_n^*(x)\overline{\varphi_n^*(y)} - x\bar{y}\varphi_n(x)\overline{\varphi_n(y)} = (1 - x\bar{y})\sum_{k=0}^{n}\varphi_k(x)\overline{\varphi_k(y)}, \quad x, y \in \mathbf{C}$$

(see [**23**, Section 2]). For completeness, we state and prove the following basic result about the zeros of $\varphi_n(z)$.

Lemma 4.1. *All of the zeros of $\varphi_n(z)$ lie in the unit disk D, and all of the zeros of $\varphi_n^*(z)$ lie outside $\bar{D}$.*

Proof. The two statements are easily seen to be equivalent. Thus, it suffices to prove the second one. By setting $x = y = z$ in (4.12) we obtain

$$|\varphi_n^*(z)|^2 - |z|^2|\varphi_n(z)|^2 = (1 - |z|^2)\sum_{k=0}^{n}|\varphi_k(z)|^2. \tag{4.13}$$

Hence, for fixed $z \in D$, we have

$$|\varphi_n^*(z)|^2 \geq \frac{(1 - |z|^2)}{\beta_0^2} > 0, \tag{4.14}$$

since $\varphi_0(z) = \beta_0^{-1}$. It follows that there are no zeros of $\varphi_n^*(z)$ in D. Now assume that $\varphi_n^*(z_0) = 0$ for some $z_0 \in \partial D$. Then $\varphi_n(z_0) = z_0^n\overline{\varphi_n^*(1/\bar{z}_0)} = z_0^n\overline{\varphi_n^*(z_0)} = 0$. Since this contradicts (4.11), it follows that no zero of $\varphi_n^*(z)$ lies on ∂D. $\square$

The distribution function $\psi(\theta)$ has a nonnegative derivative $\psi'(\theta)$ a.e. (with respect to Lebesgue measure) and

$$\int_{-\pi}^{\pi}\psi'(\theta)\,d\theta \leq \int_{-\pi}^{\pi}d\psi(\theta) < \infty. \tag{4.15}$$

Then, also,

$$\int_{-\pi}^{\pi}\ln\psi'(\theta)\,d\theta \leq \int_{-\pi}^{\pi}\psi'(\theta)\,d\theta < \infty. \tag{4.16}$$

However, both of the cases

$$\int_{-\pi}^{\pi} \ln \psi'(\theta)\, d\theta = -\infty \quad \text{and} \quad \int_{-\pi}^{\pi} \ln \psi'(\theta)\, d\theta > -\infty$$

can occur. The distinction between these two cases is of fundamental importance and is sometimes called *Szegö's alternative*. The condition $\int_{-\pi}^{\pi} \ln \psi'(\theta)\, d\theta > -\infty$ is called *Szegö's condition*.

Theorem 4.2. *Let $\psi(\theta)$ be a distribution function on $[-\pi, \pi]$ and let $\varphi_n(z), \varphi_n^*(z), \delta_n$ and β_n be derived from $\psi(\theta)$ as above. Then the following four statements are equivalent*:

$$\int_{-\pi}^{\pi} \ln \psi'(\theta)\, d\theta > -\infty \qquad (\textit{Szegö's condition}). \tag{4.17}$$

$$\{\varphi_n(e^{i\theta})\}_{n=0}^{\infty} \quad (\textit{equivalently } \{e^{in\theta}\}_{n=0}^{\infty}) \text{ \textit{is not complete in} } L_2^{\psi}[-\pi, \pi]. \tag{4.18}$$

$$\lim_{n\to\infty} \beta_n =: \beta > 0 \tag{4.19}$$

$$\sum_{k=1}^{\infty} |\delta_k|^2 < \infty. \tag{4.20}$$

Proof. It follows immediately from (4.7) that (4.19) and (4.20) are equivalent.

We shall now prove the implication (4.17) $\Rightarrow$ (4.19). Both here and later we make use of the following inequality between weighted geometric and arithmetic means:

$$e^{\frac{1}{P}\int_{-\pi}^{\pi} p(\theta) \ln f(\theta)\, d\theta} \le \frac{1}{P} \int_{-\pi}^{\pi} p(\theta) f(\theta)\, d\theta, \tag{4.21}$$

where $p(\theta)$ is nonnegative and Lebesgue integrable, $f(\theta)$ is nonnegative, and $\int_{-\pi}^{\pi} p(\theta)\, d\theta =: P$ (see, e.g., [**11**, p. 17] and [**25**, p. 7]). By using this inequality with $p(\theta) := 1/(2\pi)$ and $f(\theta) := |\varphi_n^*(e^{i\theta})|^2 \psi'(\theta)$, we obtain

$$\int_{-\pi}^{\pi} |\varphi_n^*(e^{i\theta})|^2 \psi'(\theta)\, d\theta \ge 2\pi e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln[|\varphi_n^*(e^{i\theta})|^2 \psi'(\theta)]\, d\theta}. \tag{4.22}$$

Since $\varphi_n^*(z)$ has no zeros for $z \in \bar{D}$ (cf. Lemma 4.1), the function $\ln(|\varphi_n^*(z)|^2)$ is harmonic for $z \in \bar{D}$, and hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(|\varphi_n^*(e^{i\theta})|^2)\, d\theta = \ln(|\varphi_n^*(0)|^2) = \ln\left(\frac{1}{\beta_n^2}\right) \tag{4.23}$$

(see, e.g., [**25**, p. 15] and [**26**, p. 228]). Combining (4.1), (4.22), (4.23) and the fact that

$$\int_{-\pi}^{\pi} f(\theta)\, d\psi(\theta) \geq \int_{-\pi}^{\pi} f(\theta)\psi'(\theta)\, d\theta \text{ for } f(\theta) \geq 0 \text{ a.e.},$$

we obtain

$$\begin{aligned} 1 &\geq 2\pi e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|\varphi_n^*(e^{i\theta})|^2\, d\theta} \cdot e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\psi'(\theta)\, d\theta} \\ &= \frac{2\pi}{\beta_n^2} e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\psi'(\theta)\, d\theta}. \end{aligned} \tag{4.24}$$

It follows from this and Szegö's condition (4.17) that

$$\beta := \lim_{n\to\infty} \beta_n \geq \sqrt{2\pi} e^{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\psi'(\theta)\, d\theta} > 0. \tag{4.25}$$

Next we prove the implication (4.19) $\Rightarrow$ (4.18). By considering the function $\psi_0(\theta) := e^{-i\theta}$, we can write

$$\left\| \psi_0(\theta) - \sum_{k=0}^{n} a_k e^{ik\theta} \right\|_\psi = \left\| 1 - \sum_{\nu=1}^{n+1} a_{\nu-1} e^{i\nu\theta} \right\|_\psi, \tag{4.26}$$

and by the minimum property of $\rho_{n+1}(z)$ (see, e.g., [**13**, Section 2.2]) we then conclude that

$$\left\| \psi_0(\theta) - \sum_{k=0}^{n} a_k e^{ik\theta} \right\|_\psi \geq \|\rho_{n+1}\|_\psi = \beta_{n+1} \geq \beta > 0 \tag{4.27}$$

for an arbitrary linear combination $\sum_{k=0}^{n} a_k e^{ik\theta}$ with arbitrary n. It follows that the system $\{e^{in\theta}\}_{n=0}^{\infty}$ is not complete in $L_2^{\psi}[-\pi,\pi]$; consequently, the system $\{\varphi_n(e^{in\theta})\}_{n=0}^{\infty}$ is not complete in $L_2^{\psi}[-\pi,\pi]$.

Finally we prove the implication (4.18) $\Rightarrow$ (4.17). Since $\{e^{in\theta}\}_{n=0}^{\infty}$ is not complete in $L_2^{\psi}[-\pi,\pi]$, there exists a nonzero element $\varphi(\theta) \in L_2^{\psi}[-\pi,\pi]$ which is orthogonal to every $e^{in\theta}$, $n=0,1,2,\dots$; that is,

$$\int_{-\pi}^{\pi} \overline{\varphi(\theta)} e^{in\theta}\, d\psi(\theta) = 0, \quad n = 0,1,2,\dots . \tag{4.28}$$

Note that, since $\varphi(\theta) \in L_2^{\psi}[-\pi,\pi]$, we also have

$$\int_{-\pi}^{\pi} |\varphi(\theta)|\, d\psi(\theta) < \infty. \tag{4.29}$$

Multiplication of (4.28) by $z^{-(n+1)}$ and summation over n gives

$$\sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \overline{\varphi(\theta)} \frac{e^{in\theta}}{z^{n+1}}\, d\psi(\theta) = 0. \tag{4.30}$$

The series $\sum_{n=0}^{\infty} e^{in\theta}/z^{n+1}$ converges uniformly in θ, for fixed $z \in E$. Consequently, $\sum_{n=0}^{\infty} \overline{\varphi(\theta)}(e^{in\theta}/z^{n+1})$ converges a.e. (for $z \in E$). Furthermore,

$$\begin{aligned} &\left| \int_{-\pi}^{\pi} \left[\sum_{n=0}^{\infty} \frac{e^{in\theta}}{z^{n+1}} \overline{\varphi(\theta)} - \sum_{n=0}^{N} \frac{e^{in\theta}}{z^{n+1}} \overline{\varphi(\theta)} \right] d\psi(\theta) \right| \\ &\qquad \le \max_{-\pi\le\theta\le\pi} \left| \sum_{n=N+1}^{\infty} \frac{e^{in\theta}}{z^{n+1}} \right| \cdot \int_{-\pi}^{\pi} |\varphi(\theta)|\, d\psi(\theta). \end{aligned} \tag{4.31}$$

From (4.29), (4.30) and the uniform convergence of $\sum_{n=0}^{\infty} e^{in\theta}/z^{n+1}$ we conclude that

$$\int_{-\pi}^{\pi} \left[\sum_{n=0}^{\infty} \frac{e^{in\theta}}{z^{n+1}} \overline{\varphi(\theta)} \right] d\psi(\theta) = \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \frac{e^{in\theta}}{z^{n+1}} \overline{\varphi(\theta)}\, d\psi(\theta) = 0, \quad z \in E. \tag{4.32}$$

By summing the geometric series $\sum_{n=0}^{\infty} e^{in\theta}/z^{n+1}$ we may then conclude that

$$\lambda(z) \equiv 0 \quad \text{for } z \in E \tag{4.33}$$

where

$$\lambda(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\overline{\varphi(\theta)}}{e^{i\theta} - z}\, d\psi(\theta). \tag{4.34}$$

We note that the function $\lambda(z)$ is analytic for $z \in D$ and for $z \in E$. We define the complex distribution function $\tau(\theta)$ by

$$d\tau(\theta) := e^{-i\theta}\overline{\varphi(\theta)}\, d\psi(\theta) \tag{4.35}$$

and may then write

$$\lambda(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}\, d\tau(\theta)}{e^{i\theta} - z}. \tag{4.36}$$

The distribution function $\tau(\theta)$ has bounded variation, since

$$\int_{-\pi}^{\pi} |d\tau(\theta)| = \int_{-\pi}^{\pi} |e^{-i\theta}| \cdot |\varphi(\theta)|\, d\psi(\theta) = \int_{-\pi}^{\pi} |\varphi(\theta)|\, d\psi(\theta) < \infty. \tag{4.37}$$

The expression (4.36) is thus an integral of Cauchy-Stieltjes type. Since $\lambda(z) \equiv 0$ for $z \in E$, the integral is a Cauchy-Stieltjes integral and hence belongs to the Hardy space H_1 (see, e.g., [**25**, pp. 65–68]). Consequently,

$$\int_{-\pi}^{\pi} \ln |\tau'(\theta)|\, d\psi > -\infty \tag{4.38}$$

(see, e.g., [**25**, pp. 54–57]). Now $|\tau'(\theta)|^2 = |\varphi(\theta)|^2 \cdot |\psi'(\theta)|^2$, and therefore

$$\int_{-\pi}^{\pi} \ln \psi'(\theta)\, d\theta + \int_{-\pi}^{\pi} \ln[|\varphi(\theta)|^2 \psi'(\theta)]\, d\theta = 2\int_{-\pi}^{\pi} \ln |\tau'(\theta)|\, d\theta > -\infty. \tag{4.39}$$

Since

$$\int_{-\pi}^{\pi} \ln[|\varphi(\theta)|^2 \psi'(\theta)]\, d\theta \leq \int_{-\pi}^{\pi} |\varphi(\theta)|^2\, d\psi(\theta) < \infty, \tag{4.40}$$

we conclude from (4.39) that the Szegö condition (4.17) is satisfied. □

We recall the definition and basic properties of the Hardy space H_2 (see, e.g., [**10, 16, 25, 26**]). A function $H(z)$, which is analytic for $z \in D$, belongs to H_2 iff

$$\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |H(re^{i\theta})|^2 \, d\theta < \infty \tag{4.41}$$

or, equivalently, iff

$$\sum_{n=0}^{\infty} |h_n|^2 < \infty \tag{4.42}$$

where $\{h_n\}$ is the sequence of the Taylor coefficients for $H(z)$ at $z = 0$. If $H(z) \in H_2$, then the limit

$$H(e^{i\theta}) := \lim_{r \to 1^-} H(re^{i\theta}) \tag{4.43}$$

exists a.e. and

$$\int_{-\pi}^{\pi} |H(e^{i\theta})|^2 \, d\theta < \infty. \tag{4.44}$$

Furthermore,

$$\int_{-\pi}^{\pi} \ln |H(e^{i\theta})| \, d\theta > -\infty \text{ if } H(z) \not\equiv 0. \tag{4.45}$$

Theorem 4.3. *Let $\psi(\theta)$ be a distribution function on $[-\pi, \pi]$ and let $\varphi_n(z)$, $\varphi_n^*(z)$, δ_n and β_n be derived as above. Assume that the (equivalent) conditions (4.17)–(4.20) are satisfied. Then the following hold:*

(A) *The sequence $\{1/(\sqrt{2\pi}\varphi_n^*(z))\}$ converges for $z \in D$ to an analytic function $H_0(z)$.*

(B) *The function $H_0(z)$ belongs to H_2.*

(C) *The function $H_0(z)$ satisfies*

$$|H_0(e^{i\theta})|^2 = \psi'(\theta), \text{ a.e.} \tag{4.46}$$

(D) *The function $H_0(z)$ can be expressed by the formula*

$$H_0(z) = e^{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln \psi'(\theta)\, d\theta}. \tag{4.47}$$

Proof. (A). Let $0 < r < 1$. For $|z| \le r$ we obtain from (4.13) the inequality

$$|\varphi_n^*(z)|^2 \ge (1-r^2) \sum_{k=0}^{n} |\varphi_k(z)|^2 \ge \frac{(1-r^2)}{\beta_0^2}. \tag{4.48}$$

From the theory of normal families of analytic functions (see, e.g., [**14**], [**26**, pp. 271–273], [**25**, p. 18]), by considering the analytic functions $1/\varphi_n^*(z)$, we conclude that there exists a subsequence $\{\varphi_{n(\nu)}^*(z)\}$ which converges for $z \in D$, uniformly on compact subsets, to a function $\Pi(z)$ which is either analytic or identically equal to ∞. Since

$$\varphi_{n(\nu)}^*(0) = \frac{1}{\beta_{n(\nu)}} \le \frac{1}{\beta}, \tag{4.49}$$

it follows from (4.19) that $\Pi(z)$ is an analytic function. By setting $x = z$ and $y = 0$ in (4.12), we obtain

$$\frac{1}{\beta_n}\varphi_n^*(z) = \sum_{k=0}^{n} \overline{\varphi_k(0)}\varphi_k(z). \tag{4.50}$$

The Cauchy-Schwarz inequality (with $n > m$) then gives

$$\left|\frac{1}{\beta_n}\varphi_n^*(z) - \frac{1}{\beta_n}\varphi_m^*(z)\right|^2 \le \sum_{k=m+1}^{n} |\varphi_k(z)|^2 \cdot \sum_{k=m+1}^{n} |\varphi_k(0)|^2. \tag{4.51}$$

From (4.48) and the fact that $\{\varphi_{n(\nu)}^*(z)\}$ converges, it follows that the sequence $\{\sum_{k=0}^{n(\nu)} |\varphi_k(z)|^2\}_{\nu=1}^{\infty}$ converges for $z \in D$. Then, also, the series $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ converges for $z \in D$. From this, (4.19) and (4.51), we conclude that $\{\varphi_n^*(z)\}$ is a Cauchy sequence for $z \in D$. Since $\{\varphi_{n(\nu)}^*(z)\}$ already converges to $\Pi(z)$, we may conclude that

$$\lim_{n\to\infty} \varphi_n^*(z) = \Pi(z) \quad \text{for } z \in D. \tag{4.52}$$

Since $\varphi_n^*(z) \neq 0$ for $z \in D$ by Lemma 4.1 and $\Pi(0) = \lim_{n\to\infty} 1/\beta_n = 1/\beta \neq 0$, it follows from Hurwitz's theorem (see, e.g., [**15**, p. 283]) that $\Pi(z) \neq 0$ for all $z \in D$. We define

$$H_0(z) := \frac{1}{\sqrt{2\pi}\Pi(z)}, \tag{4.53}$$

This function is then analytic for $z \in D$.

(B). For $z \in \partial D$ the determinant formula (4.11) can be written

$$\omega_n^*(e^{i\theta})\overline{\varphi_n^*(e^{i\theta})} + \varphi_n^*(e^{i\theta})\overline{\omega_n^*(e^{i\theta})} = -2. \tag{4.54}$$

Consequently,

$$\begin{aligned}\operatorname{Re}\left[-\frac{\omega_n^*(e^{i\theta})}{\varphi_n^*(e^{i\theta})}\right] &= \frac{-\frac{1}{2}\left[\omega_n^*(e^{i\theta})\overline{\varphi_n^*(e^{i\theta})} + \overline{\omega_n^*(e^{i\theta})}\varphi_n^*(e^{i\theta})\right]}{|\varphi_n^*(e^{i\theta})|^2} \\ &= \frac{1}{|\varphi_n^*(e^{i\theta})|^2}.\end{aligned} \tag{4.55}$$

Since $\varphi_n^*(z)$ has no zeros in $\bar{D}$, there exists a neighborhood of $\bar{D}$ where the Taylor series expansion

$$-\frac{\omega_n^*(z)}{\varphi_n^*(z)} = \mu_0 + 2\sum_{k=1}^{\infty} \mu_k z^k \tag{4.56}$$

is valid (see Theorems 2.1 and 2.2 and note that $-\omega_n^*(z)/\varphi_n^*(z) = P_{2n}(z)/Q_{2n}(z)$ converges to $L_0 = \mu_0 + 2\sum_{k=1}^{\infty} \mu_k z^k$ for $z \in D$). Because of uniform convergence of (4.56) on ∂D, we can then integrate term-by-term and obtain

$$\int_{-\pi}^{\pi} \frac{d\theta}{|\varphi_n^*(e^{i\theta})|^2} = -\operatorname{Re}\int_{-\pi}^{\pi} \frac{\omega_n^*(e^{i\theta})}{\varphi_n^*(e^{i\theta})}\, d\theta = 2\pi\mu_0. \tag{4.57}$$

We define

$$I_r^{(n)} := \int_{-\pi}^{\pi} \frac{d\theta}{|\varphi_n^*(re^{i\theta})|^2}, \quad 0 < r \leq 1,\ n = 1, 2, 3, \ldots. \tag{4.58}$$

Since $1/\varphi_n^*(z)$ is analytic for $z \in \bar{D}$, the function $1/|\varphi_n^*(z)|^2$ is subharmonic and thus the integral $I_r^{(n)}$ is a nondecreasing function of r for

$0 < r \leq 1$ (see, e.g., [**25**, p. 23–26], [**26**, 328–330]). Hence, it follows from (4.57) that

$$I_r^{(n)} \leq 2\pi\mu_0, \quad 0 < r \leq 1,\ n = 1, 2, 3, \dots . \tag{4.59}$$

Then, also, by Fatou's lemma

$$\int_{-\pi}^{\pi} \frac{d\theta}{|\Pi(re^{i\theta})|^2} \leq 2\pi\mu_0, \quad 0 < r < 1. \tag{4.60}$$

This means that $H_0(z) = 1/(\sqrt{2\pi}\Pi(z))$ belongs to H_2.

(C). Let $P_r(t)$ denote the Poisson kernel; that is,

$$P_r(t) := \frac{1 - r^2}{1 - 2r\cos t + r^2} = \operatorname{Re}\left(\frac{1 + re^{it}}{1 - re^{it}}\right). \tag{4.61}$$

The function $\operatorname{Re}\left[\omega_n^*(re^{i\theta})/\varphi_n^*(re^{i\theta})\right]$ is harmonic for $z \in \bar{D}$. Hence, it follows by (4.55) that

$$\operatorname{Re}\left[\frac{-\omega_n^*(re^{i\theta})}{\varphi_n^*(re^{i\theta})}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{P_r(\theta - t)}{|\varphi_n^*(e^{it})|^2}\, dt. \tag{4.62}$$

Also, $\ln\left(1/|\varphi_n^*(re^{i\theta})|^2\right)$ is harmonic for $z \in \bar{D}$ and, therefore,

$$\ln\left(\frac{1}{|\varphi_n^*(re^{i\theta})|^2}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\ln\left(\frac{1}{|\varphi_n^*(e^{it})|^2}\right) dt. \tag{4.63}$$

Setting $p(t) := P_r(\theta - t)$ and $f(t) := 1/|\varphi_n^* e^{it})|^2$ in (4.21) and using (4.62) and (4.63), we obtain

$$\frac{1}{|\varphi_n^*(re^{i\theta})|^2} \leq \operatorname{Re}\left[\frac{-\omega_n^*(re^{i\theta})}{\varphi_n^*(re^{i\theta})}\right], \quad 0 < r < 1. \tag{4.64}$$

We recall from Section 2 (Theorems 2.1 and 2.2) that

$$\lim_{n\to\infty} \frac{P_{2n}(z)}{Q_{2n}(z)} = \lim_{n\to\infty} \frac{-\omega_n^*(z)}{\varphi_n^*(z)} = \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\psi(t), \quad z \in D. \tag{4.65}$$

This together with (4.52) and (4.64) gives

$$\frac{1}{|\Pi(re^{i\theta})|^2} \leq \operatorname{Re}\int_{-\pi}^{\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}\, d\psi(t) = \int_{-\pi}^{\pi} P_r(\theta - t)\, d\psi(t). \tag{4.66}$$

Since

$$\lim_{r\to 1^-}\int_{-\pi}^{\pi} P_r(\theta-t)\,d\psi(t) = 2\pi\psi'(\theta) \text{ a.e.} \tag{4.67}$$

(see, e.g., [**25**, p. 34], [**26**, p. 226], [**16**, p. 34]), it follows from (4.66) that

$$\lim_{r\to 1^-}\frac{1}{|\Pi(re^{i\theta})|^2} \leq 2\pi\psi'(\theta). \tag{4.68}$$

We define

$$\Pi(e^{i\theta}) := \lim_{r\to 1^-}\Pi(re^{i\theta}) \text{ a.e.} \tag{4.69}$$

Assume that $1/|\Pi(e^{i\theta})|^2 < 2\pi\psi'(\theta)$ on a set of positive measure. Then

$$\int_{-\pi}^{\pi}\ln[2\pi\psi'(\theta)]\,d\theta > \int_{-\pi}^{\pi}\ln\frac{1}{|\Pi(e^{i\theta})|^2}\,d\theta = 2\pi\ln\frac{1}{|\Pi(0)|^2} = 2\pi\ln(\beta^2).$$

Hence,

$$\beta^2 < 2\pi e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\psi'(\theta)\,d\theta},$$

which contradicts (4.25). Hence, we may conclude that the above assumption is false and

$$\frac{1}{|\Pi(e^{i\theta})|^2} = \lim_{r\to 1^-}\frac{1}{|\Pi(re^{i\theta})|^2} = 2\pi\psi'(\theta) \text{ a.e.}, \tag{4.70}$$

which is the same as (4.46).

(D). The inequality

$$\beta^2 \leq 2\pi e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\psi'(\theta)\,d\theta} \tag{4.71}$$

(not with strict inequality sign) follows from the argument under (C) above without the assumption $1/|\Pi(e^{i\theta})|^2 < 2\pi\psi'(\theta)$ on a set of positive measure. This together with (4.25) gives

$$\beta = \sqrt{2\pi}e^{\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\psi'(\theta)\,d\theta}. \tag{4.72}$$

Since $H_0(z) \in H_2$ and $H_0(z)$ has no zeros, it follows by (4.46) that we may write

$$H_0(z) = S(z) \cdot e^{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln \psi'(\theta)\, d\theta} \quad \text{for } z \in D, \tag{4.73}$$

where $S(z)$ is a singular inner function (i.e., an inner function without zeros (see, e.g., [**25**, p. 78], [**16**, pp. 67–70], [**26**, p. 338]). Using $H_0(0) = \beta/\sqrt{2\pi}$, we obtain, from (4.73),

$$\frac{\beta}{\sqrt{2\pi}} = S(0) e^{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \psi'(\theta)\, d\theta}. \tag{4.74}$$

From (4.72) we then conclude that $S(0) = 1$. This is possible only if $S(z) \equiv 1$. Formula (4.47) follows then from (4.73). □

Theorem 4.4. *Let $\psi(\theta)$ be a distribution function on $[-\pi, \pi]$ and let $\varphi_n(z)$, $\varphi_n^*(z)$, δ_n and β_n be derived as above. We assume that the equivalent conditions* (4.17)–(4.20) *are not satisfied. That is, we assume that*:

$$\int_{-\pi}^{\pi} \ln \psi'(\theta)\, d\theta = -\infty, \tag{4.75}$$

$$\{\varphi_n(e^{i\theta})\}_{n=0}^{\infty} \quad (\textit{or equivalently}, \ \{e^{in\theta}\}_{n=0}^{\infty}) \ \textit{is complete in} \quad L_2^{\psi}[-\pi, \pi], \tag{4.76}$$

$$\lim_{n\to\infty} \beta_n = 0, \tag{4.77}$$

$$\sum_{k=1}^{\infty} |\delta_k|^2 = \infty. \tag{4.78}$$

Then the following hold:

(E) *There exists no function $H(z) \in H_2$ for which $|H(e^{i\theta})|^2 = \psi'(\theta)$* a.e., *except in the case where $\psi(\theta)$ is singular* (*i.e.*, $\psi'(\theta) = 0$ a.e.).

(F) *The sequence $\{1/\varphi_n^*(z)\}$ converges to* 0 *for all $z \in D$.*

Proof. (E). If $H(z) \in H_2$, $H(z) \not\equiv 0$, then

$$\int_{-\pi}^{\pi} \ln |H(e^{i\theta})|^2\, d\theta > -\infty$$

by (4.45). This shows that $|H(e^{i\theta})|^2$ cannot be equal to $\psi'(\theta)$ a.e. by (4.75).

(F). We assume that $\{1/\varphi_n^*(z)\}$ does not converge to 0 for some $z = z_0 \in D$. Then there exists an $\varepsilon > 0$ and a sequence $\{n(\nu)\}_{\nu=1}^{\infty}$ such that

$$\left|\frac{1}{\varphi_{n(\nu)}^*(z_0)}\right| \geq \varepsilon \quad \text{for all } \nu = 1, 2, 3, \dots . \tag{4.79}$$

The functions $1/\varphi_{n(\nu)}^*(z)$ are analytic and different from zero for $z \in \bar{D}$. By (4.48) (which does not depend on (4.17)–(4.20)), it follows that $\{1/\varphi_{n(\nu)}^*(z)\}$ is uniformly bounded on compact subsets of D. Then, by the theory of normal families of analytic functions (see, e.g., [**14**], [**26**, pp. 271–273], [**25**, p. 18]), there exists a subsequence $\{n(\nu(\lambda))\}$ such that $\{1/\varphi_{n(\nu(\lambda))}^*\}$ converges uniformly on compact subsets of D to a function $f(z)$ which is analytic for $z \in D$. From (4.77) it follows that

$$f(0) = \lim_{\lambda\to\infty} \frac{1}{\varphi_{n(\nu(\lambda))}^*(z)} = \lim_{\lambda\to\infty} \beta_{n(\nu(\lambda))} = 0. \tag{4.80}$$

We may then conclude from Hurwitz's theorem (see, e.g., [**14**, p. 283]) that $f(z)$ is identically zero for $z \in D$. On the other hand, (4.79) implies that $f(z_0) \neq 0$. This contradiction shows that $\lim_{n\to\infty} 1/\varphi_n^*(z) = 0$ for all $z \in D$, which was to be proved. $\square$

5. Weakly stationary stochastic processes. A *measure space* (Ω, P) is a set Ω equipped with a probability measure P. A *stochastic variable* $x(\omega)$ is a measurable function on (Ω, P). A sequence $\{x_n(\omega) : n \in \mathbf{Z}\}$ is called a *stochastic process* if all $x_n(\omega)$ are stochastic variables. For a discussion of basic properties of stochastic processes, the reader can refer to [**13**].

Let $\{x_n(\omega)\}$ denote a given stochastic process such that

$$\int_\Omega |x_n(\omega)|^2 \, dP(\omega) < \infty \quad \text{for all } n = 0, \pm 1, \pm 2, \dots .$$

Let L_2 denote the closure of the linear hull $\{\sum_{n=-m}^{m} c_n x_n : c_n \in \mathbf{C}\}$ in $L_2^P(\Omega)$. Then L_2 is a Hilbert space with inner product

$$\langle u, v\rangle_P := \int_\Omega u(\omega)\overline{v(\omega)}\, dP(\omega), \quad u, v \in L_2. \tag{5.1}$$

The *covariance function* $r_{s,t}$ is defined by

$$r_{s,t} := \int_\Omega x_s(\omega)\overline{x_t(\omega)}\,dP(\omega) =: \langle x_s, x_t\rangle_P, \quad s,t \in \mathbf{Z}\,. \tag{5.2}$$

The stochastic process $\{x_n(\omega)\}$ is called *weakly stationary* if

$$\int_\Omega x_n(\omega)\,dP(\omega) = 0 \quad \text{for all } n \in \mathbf{Z} \tag{5.3}$$

and

$$r_{s,t} = r_{s+m,t+m} \quad \text{for all } s,t,m \in \mathbf{Z}\,. \tag{5.4}$$

In this case we define

$$\mu_m = r_{0,m} \quad \text{for } m \in \mathbf{Z}\,. \tag{5.5}$$

Clearly, we also have

$$\mu_m = r_{s,s+m} \quad \text{for all } m,s \in \mathbf{Z}\,. \tag{5.6}$$

It is readily seen that $\mu_{-m} = \bar{\mu}_m$ for $m \in \mathbf{Z}$ and that the Toeplitz forms

$$\sum_{j,k=0}^{n} c_j\bar{c}_k\mu_{j-k}$$

are all positive semi-definite. Therefore, by the Trigonometric moment theorem (Theorem 2.2), there exists a distribution function $\psi(\theta)$ such that

$$\mu_n = \int_{-\pi}^{\pi} e^{-in\theta}\,d\psi(\theta), \quad n \in \mathbf{Z}\,. \tag{5.7}$$

This distribution function is called the *spectral function for the weakly stationary stochastic process.*

The mapping $x_n(\omega) \leftrightarrow e^{-in\theta}$ establishes an algebraic isomorphism between L_2 and $L_2^\psi[-\pi,\pi]$. Since $\langle\cdot,\cdot\rangle_P$ and $\|\cdot\|_P$ denote inner product and norm, respectively, in L_2 and since $\langle\cdot,\cdot\rangle_\psi$ and $\|\cdot\|_\psi$ denote inner product and norm, respectively, in $L_2^\psi[-\pi,\pi]$, it follows by linearity from (5.7) that

(5.8)
$$\left\langle \sum_{k=M}^{N} c_k x_k(\omega), \sum_{j=M}^{N} d_j x_j(\omega) \right\rangle_P = \left\langle \sum_{k=M}^{N} c_k e^{-ik\theta}, \sum_{j=M}^{N} d_j e^{-ij\theta} \right\rangle_\psi$$

and, in particular,

(5.9)
$$\left\| \sum_{k=M}^{N} c_k x_k(\omega) \right\|_P = \left\| \sum_{k=M}^{N} c_k e^{-ik\theta} \right\|_\psi .$$

Thus, the spaces L_2 and $L_2^\psi[-\pi,\pi]$ are isomorphic as Hilbert spaces.

The stochastic process $\{x_n(\omega)\}$ is said to be *deterministic* if $x_0(\omega)$ can be approximated arbitrarily well in the $\|\cdot\|_P$ norm by finite linear combinations from the set $\{x_t(\omega) : t < 0\}$; that is, if $x_0(\omega)$ can be predicted with arbitrarily small error from knowledge of the past. The process is called *nondeterministic* if such approximation is not possible. It can be verified that, since the process is weakly stationary, the concept of deterministic is independent of the choice of $x_0(\omega)$ as the variable to be approximated by the foregoing variables. The choice of any other fixed $x_k(\omega)$ would lead to the same results.

We define

(5.10)
$$E^{(n)} := \min_{g_k^{(n)} \in \mathbf{R}} \left\| x_0(\omega) - \sum_{k=1}^{n} -g_k^{(n)} x_{-k}(\omega) \right\|_P .$$

Then the process is deterministic iff $E^{(n)}$ can be made arbitrarily small for sufficiently large n; that is, iff $\lim_{n\to\infty} E^{(n)} = 0$.

Let $\{\rho_n\}$ denote the sequence of monic Szegö polynomials with respect to the spectral function $\psi(\theta)$ for the stochastic process, and let $\varphi_n(z), \beta_n$ and δ_n have the same meanings relative to $\psi(\theta)$ as in Section 4. By the well-known minimum property of Szegö polynomials (see Section 2, (2.20)) we have

(5.11)
$$\beta_n = \|\rho_n\|_\psi = \min_{g_k^{(n)} \in \mathbf{R}} \left\| e^{in\theta} + \sum_{k=0}^{n-1} g_k^{(n)} e^{ik\theta} \right\|_\psi = \min_{g_k^{(n)} \in \mathbf{R}} \left\| 1 + \sum_{k=1}^{n} g_k^{(n)} e^{ik\theta} \right\|_\psi .$$

From the isomorphism between $L_2^{\psi}[-\pi,\pi]$ and L_2 given by (5.9), it follows that

$$\left\| x_0(\omega) + \sum_{k=1}^{n} g_k^{(n)} x_{-k}(\omega) \right\|_P = \left\| 1 + \sum_{k=1}^{n} g_k e^{ik\theta} \right\|_{\psi}. \tag{5.12}$$

From (5.11) and (5.12), we conclude that

$$E^{(n)} = \beta_n, \tag{5.13}$$

and it is clear from (3.20), (5.11) and (5.13) that $E^{(n)} = E_n$. The following result then follows directly from Theorem 4.2.

Theorem 5.1. *Let $\{x_t(\omega)\}$ be a weakly stationary stochastic process with spectral function $\psi(\theta)$, and let $\varphi_n(z)$, $\varphi_n^*(z)$, δ_n and β_n be derived from $\psi(\theta)$ as in Section* 4. *Then the following statements are equivalent*:

(A) *The process $\{x_t(\omega)\}$ is nondeterministic.*

(B) *The equivalent conditions* (4.17)–(4.20) *are satisfied.*

6. Design of digital filters. For general information concerning the problem treated in this section, we refer to [**7**] and [**9**]. A signal $\{u(n)\}_{m=-\infty}^{\infty}$ is said to have *finite energy* if $\sum_{n=-\infty}^{\infty} |u(m)|^2 < \infty$. We shall say that a digital filter T with transfer function $K(z)$ has *finite energy* if the unit pulse response $\{k(m)\}_{m=-\infty}^{\infty}$ has finite energy. A digital filter T is said to be *causal* if $u(m) = 0$, for $m < m_0$, implies $(Tu)(m) = 0$, for $m < m_0$. An equivalent condition is easily seen to be that $k(m) = 0$ for $m < 0$ (cf., Section 2). We recall that the transfer function $K(z)$ is given by

$$K(z) = \sum_{m=-\infty}^{\infty} k(m) z^{-m}. \tag{6.1}$$

The causality condition requires that $k(m) = 0$ for $m < 0$; if, in addition, T has finite energy, then $\sum_{m=0}^{\infty} |k(m)|^2 < \infty$.

A function $K(z)$ which is analytic in $E := [z \in \hat{\mathbf{C}} : |z| > 1]$ is said to belong to K_2 if

$$\sum_{m=0}^{\infty} |k(m)|^2 < \infty, \tag{6.2}$$

where $\{k(m)\}$ is the sequence of coefficients in the Laurent expansion at $z = \infty$. Thus, the transfer function of any causal filter with finite energy belongs to K_2.

To every function $H(z) \in H_2$ we define a function $H^{\#}(z)$ by

$$H^{\#}(z) := \overline{H(1/\bar{z})}, \quad z \in E, \tag{6.3}$$

and to every function $K(z) \in K_2$ we define a function $K^{\#}(z)$ by

$$K^{\#}(z) := \overline{K(1/\bar{z})}, \quad z \in D := [z \in \mathbf{C} : |z| < 1]. \tag{6.4}$$

It follows immediately that $H^{\#}(z) \in K_2$, $K^{\#}(z) \in H_2$ and

$$H^{\#\#}(z) = H(z) \quad \text{and} \quad K^{\#\#}(z) = K(z). \tag{6.5}$$

Furthermore,

$$\lim_{\rho \to 1^+} K(\rho e^{i\theta}) = \lim_{r \to 1^-} \overline{H(re^{i\theta})}. \tag{6.6}$$

We see from this and the properties of H_2 that, if $K(z) \in K_2$, then the limit

$$K(e^{i\theta}) = \lim_{\rho \to 1^+} K(\rho e^{i\theta}) \tag{6.7}$$

exists a.e. and

$$\int_{-\pi}^{\pi} |K(e^{i\theta})|^2 \, d\theta < \infty. \tag{6.8}$$

In this section our interest is primarily in the following problem, which arises naturally in many situations. Let $\Phi(\theta)$ be a given nonnegative function on $[-\pi, \pi]$. We wish to construct a causal filter T with finite energy whose magnitude response function $|K(e^{i\theta})|$ equals $\Phi(\theta)$ a.e. We shall consider only the situation in which $\Phi(\theta)$ is an even function. It can be seen that this is the case iff the unit pulse response $\{k(m)\}$ is a sequence of real numbers. In the special case that $\Phi(\theta) \equiv 1$, a solution to the problem is called an *all-pass filter*.

We see from the discussion following (6.1) that a causal filter with finite energy has a transfer function which belongs to K_2. Hence, by (6.8) it is necessary to have

$$\int_{-\pi}^{\pi} [\Phi(\theta)]^2 \, d\theta < \infty \tag{6.9}$$

if the magnitude response of the filter equals $\Phi(\theta)$. Property (6.9) is, therefore, a necessary condition for the filter construction problem to have a solution.

By the aid of Theorems 4.3 and 4.4 we can now obtain

Theorem 6.1. *Let $\Phi(\theta)$ be a given real-valued function on $[-\pi, \pi]$, satisfying the following conditions*:

$$\Phi(\theta) \geq 0 \text{ for } -\pi \leq \theta \leq \pi, \tag{6.10}$$

$$\Phi(-\theta) = \Phi(\theta) \text{ for } -\pi \leq \theta \leq \pi, \tag{6.11}$$

$$\int_{-\pi}^{\pi} [\Phi(\theta)]^2 \, d\theta < \infty. \tag{6.12}$$

Let the distribution function $\psi(\theta)$ be defined by

$$\psi(\theta) := \int_{-\pi}^{\theta} [\Phi(t)]^2 \, dt + \sigma(\theta), \tag{6.13}$$

where $\sigma(\theta)$ is an arbitrary singular distribution function (*in particular, $\sigma(\theta)$ may be identically zero*). *Let $\varphi_n(z), \varphi_n^*(z), \delta_n$ and β_n be derived from $\psi(\theta)$ as in Section* 4. *Then the following hold*:

(A) *If the* (*equivalent*) *conditions* (4.17)–(4.20) *are satisfied, then the function*

(6.14)

$$K_0(z) := H_0^{\#}(z) := \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}\,\overline{\varphi_n^*(1/\bar{z})}} = e^{-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln \Phi(\theta)\, d\theta},$$

is defined for $z \in E$ and is the transfer function of a causal filter T with finite energy, satisfying

$$\lim_{\rho\to 1^+} |K_0(\rho e^{i\theta})| = \Phi(\theta) \text{ a.e.}$$

(B) *If the (equivalent) conditions* (4.17)–(4.20) *are satisfied and if* $K(z)$ *is the transfer function of a causal filter with finite energy satisfying*

$$\lim_{\rho\to 1^+} |K(\rho e^{i\theta})| = \Phi(\theta) \text{ a.e.,} \tag{6.15}$$

then there exists a function $J(z)$ *which is the transfer function of an all-pass filter (i.e.,* $J^{\#}(z)$ *is an inner function) such that*

$$K(z) = J(z)K_0(z). \tag{6.16}$$

(C) *If the equivalent conditions* (4.17)–(4.20) *are not satisfied, then there exists no causal filter with finite energy such that the transfer function* $K(z)$ *satisfies* $\lim_{\rho\to 1^+} |K(\rho e^{i\theta})| = \Phi(\theta)$ a.e., *except in the case where* $\Phi(\theta) = 0$ a.e.

Proof. Note that $\psi(\theta)$ is a distribution function because of (6.12). We also note that $\psi'(\theta) = [\Phi(\theta)]^2$ a.e.

(A). From Theorem 4.3, it follows that the function

$$H_0(z) := \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}\varphi_n^*(z)} = e^{-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln \Phi(\theta)\, d\theta} \tag{6.17}$$

belongs to H_2 and satisfies

$$\lim_{r\to 1^-} |H_0(re^{i\theta})|^2 = [\Phi(\theta)]^2 \quad \text{a.e.} \tag{6.18}$$

Then $K_0(z)$ belongs to K_2 and, by (6.6), $K_0(z)$ satisfies (6.15). From considerations preceding the statement of the theorem, it follows that $K_0(z)$ is the transfer function of a causal filter with finite energy.

(B). Let $K(z)$ be an arbitrary function with the desired properties and set $H(z) := K^{\#}(z)$. Then $H(z) \in H_2$ and $\lim_{r\to 1^-} |H(re^{i\theta})| = \Phi(\theta)$ a.e. by (6.6).

Consequently, we may write

$$H(z) = I(z)H_0(z) \tag{6.19}$$

where $I(z)$ is an inner function (see, e.g., [**16**, p. 67], [**26**, pp. 336–338], [**25**, p. 78]). Then $K(z) := H^{\#}(z)$ may be written in the form (6.16) with $J(z) := I^{\#}(z)$. Clearly, every function of the form (6.16) is a solution of the problem.

(C). If the conditions (4.17)–(4.20) are not satisfied and if $\Phi(\theta)$ is not zero a.e., then by Theorem 4.4 there exists no function $H(z)$ in H_2 such that $\lim_{r\to 1^-} |H(re^{i\theta})| = \Phi(\theta)$ a.e.; hence, no function $K(z)$ exists in K_2 such that $\lim_{\rho\to 1^+} |K(\rho e^{i\theta})| = \Phi(\theta)$ a.e. This means that there exists no causal filter with finite energy whose transfer function satisfies

$$\lim_{\rho\to 1^+} |K(\rho e^{i\theta})| = \Phi(\theta) \text{ a.e.} \quad \square$$

We conclude this paper with a brief summary of the procedure and examples for constructing approximations $A_n(z)$ of the transfer function $K_0(z)$ of a causal filter T with finite energy satisfying a given amplitude response condition (6.15). For illustration, we present some numerical and graphical results for particular examples.

Suppose that a function $\Phi(\theta)$ satisfying the conditions of Theorem 6.1 is given. Our first step is to compute moments

$$\mu_n := \int_{-\pi}^{\pi} e^{-in\theta}\, d\psi(\theta) = \int_{-\pi}^{\pi} e^{-in\theta}[\Phi(\theta)]^2\, d\theta, \tag{6.20}$$

where $\psi(\theta)$ is defined by (6.13). Next we apply Levinson's algorithm (3.29) to compute the reflection coefficients δ_n and the coefficients $q_j^{(n)}$ for the Szegö reciprocal polynomials $\rho_n^*(z) = \sum_{j=0}^{n} q_{n-j}^{(n)} z^j$, $q_n^{(n)} := 1$. We then set

$$A_n(z) := \frac{1}{\sqrt{2\pi}\,\overline{(\varphi_n^*(1/\bar{z}))}} = \frac{\beta_n}{\sqrt{2\pi}\,\overline{(\rho_n^*(1/\bar{z}))}}, \tag{6.21}$$

where the β_n can be computed by (4.7). The filter T_n with transfer function $A_n(z)$ then has magnitude response function

$$G_n(\theta) := |A_n(e^{i\theta})| = \frac{1}{\sqrt{2\pi}|\varphi_n^*(e^{i\theta})|} = \frac{\beta_n}{\sqrt{2\pi}|\rho_n^*(e^{i\theta})|}, \tag{6.22}$$

which approximates the given $\Phi(\theta)$.

For illustration we consider functions $\Phi_\varepsilon(\theta)$ of the form

$$\Phi_\varepsilon(\theta) = \begin{cases} 1, & \text{if } 0 \le |\theta| < \frac{\pi}{2} \\ \varepsilon, & \text{if } \frac{\pi}{2} < |\theta| < \pi. \end{cases} \tag{6.23}$$

The moments

$$\mu_n := \int_{-\pi}^{\pi} e^{-in\theta} [\Phi_\varepsilon(\theta)]^2 \, d\theta, \quad n = 0, 1, 2, \dots \tag{6.24}$$

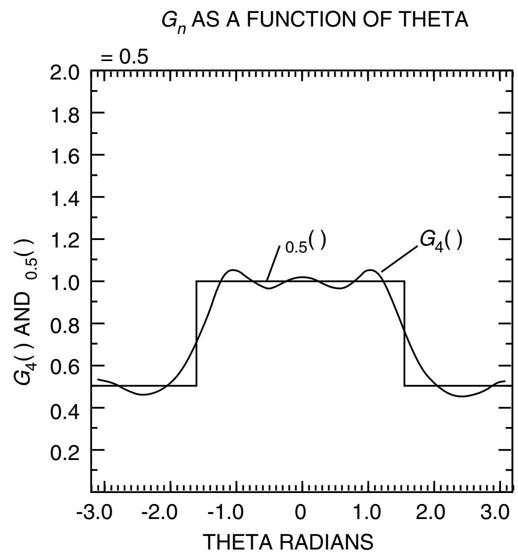
are then given by $\mu_0 = \pi(1+\varepsilon^2)$, $\mu_{2m} = 0$ for $m \ge 1$, and

$$\mu_{2m+1} = \frac{2(-1)^m(1-\varepsilon^2)}{2m+1}, \quad m = 0, 1, 2, \dots .$$

Example 1. In (6.23) and (6.24) we choose $\varepsilon = 0.5$. Figure 1 shows graphs of $G_4(\theta)$ and $G_{20}(\theta)$ superimposed to $\Phi_{0.5}(\theta)$. Clearly, $G_n(\theta)$ appears to converge to $\Phi_{0.5}(\theta)$ as predicted by theory. However, there exists a "Gibbs phenomenon" near the discontinuities at $\theta = \pm\pi/2$, also expected.

Example 2. Consider the case with $\varepsilon = 0.1$ (see Figure 2 for graphs of $G_4(\theta)$ and $G_{20}(\theta)$ superimposed to $\Phi_{0.1}(\theta)$). The large oscillations of $G_n(\theta)$ near the discontinuities of $\Phi_{0.1}(\theta)$ are even more pronounced. It is believed that these oscillations could be significantly reduced by altering the definition of $\Phi_\varepsilon(\theta)$ to eliminate the discontinuity and perhaps choose a $\Phi(\theta)$ (for example, using spline functions) to be smooth. Considerations of this type will be dealt with in future studies.

Acknowledgment. The authors gratefully acknowledge the able assistance of Kevin D. Jones for the computations and computer graphics used for illustrations.


G_n AS A FUNCTION OF THETA
= 0.5
G_4( ) AND $_{0.5}$( )
$_{0.5}$( )
G_4( )
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
-3.0 -2.0 -1.0 0 1.0 2.0 3.0
THETA RADIANS

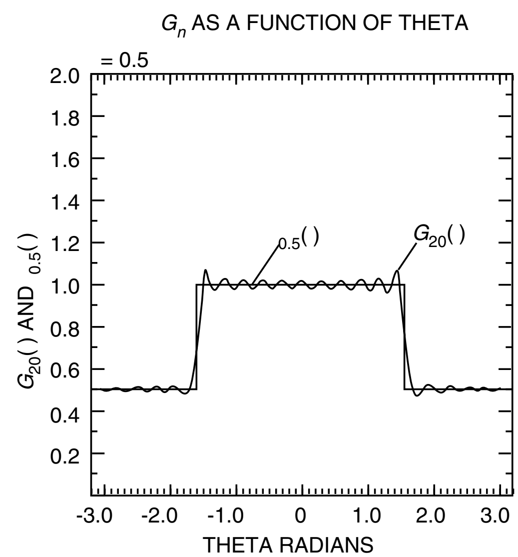
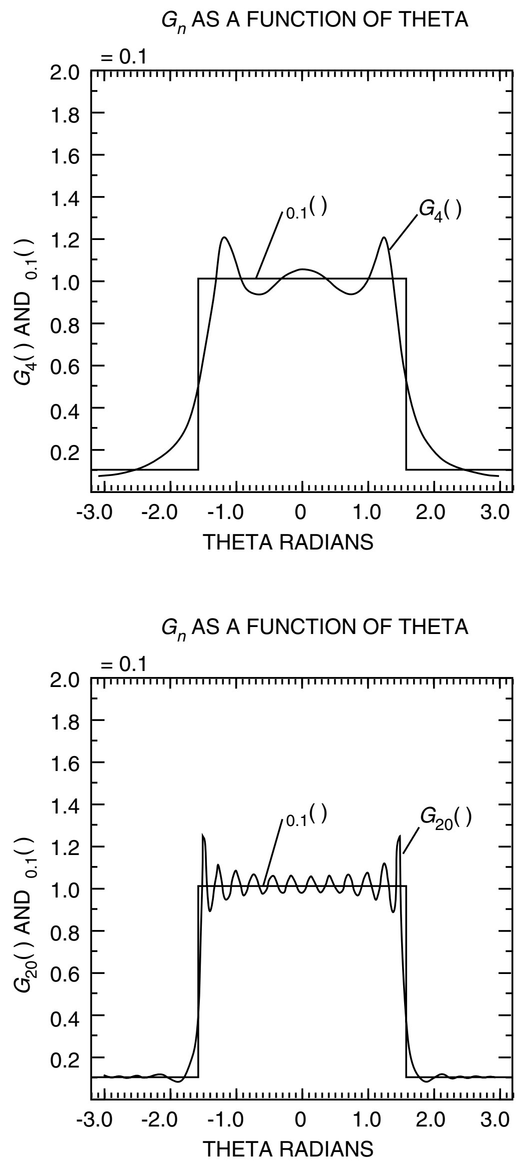

G_n AS A FUNCTION OF THETA
= 0.5
G_{20}( ) AND $_{0.5}$( )
$_{0.5}$( )
G_{20}( )
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
-3.0 -2.0 -1.0 0 1.0 2.0 3.0
THETA RADIANS

FIGURE 1.



G_n AS A FUNCTION OF THETA
= 0.1
G_4( ) AND $_{0.1}$( )
$_{0.1}$( )
G_4( )
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
-3.0
-2.0
-1.0
0
1.0
2.0
3.0
THETA RADIANS
G_n AS A FUNCTION OF THETA
= 0.1
G_{20}( ) AND $_{0.1}$( )
$_{0.1}$( )
G_{20}( )
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
-3.0
-2.0
-1.0
0
1.0
2.0
3.0
THETA RADIANS

FIGURE 2.

REFERENCES

1. M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs and mathematical tables*, National Bureau of Standards, Appl. Math. Ser. 55, U.S. Govt. Printing Office, Washington, D.C., 1964.

2. N.I. Akhiezer, *The classical moment problem and some related questions in analysis*, (Translated by N. Kemmer), Hafner, New York, 1965.

3. Gregory S. Ammar and William B. Gragg, *The implementation and use of the generalized Schur algorithm*, in *Computation and combinatorial methods in systems theory* (C.I. Byrnes and A. Lindquist, eds.), Elsevier Science Publishers B.V., North-Holland, 1986, 265–279.

4. ———, *The generalized Schur algorithm for the superfast solution of Toeplitz systems*, in *Rational approximation and its applications in mathematics and physics* (J. Gilewicz, M. Pindor and W. Siemaszko, eds.), Lecture Notes in Mathematics **1237**, Springer, Berlin, 1987, 315 –330.

5. ———, *Superfast solution of real positive definite Toeplitz systems*, SIAM J. Matrix Anal. Appl., **9**, No. 1 (Jan. 1988), 61–76.

6. R.R. Bitmead and B.D.O. Anderson, *Asymptotically fast solution of Toeplitz and related systems of linear equations*, Linear Algebra Appl. **34** (1980), 103–116.

7. Adhemar Bultheel, *Algorithms to compute the reflection coefficients of digital filters*, *Numerical methods of approximation theory*, vol. 7, (L. Collatz, G. Meinardus, H. Werner, eds.) Birkhäuser Verlag, Basel (1984), 33–50.

8. George Cybenko, *The numerical stability of the Levinson-Durbin algorithm for Toeplitz systems of equations*, SIAM J. Sci. Statist. Comput. **1** (1980), 303–319.

9. P. Dewilde and H. Dym, *Schur recursions, error formulas, and convergence of rational estimators for stationary stochastic sequences*, IEEE Trans. Inform. Theory, **IT-27** (1981), 446–461.

10. Peter L. Duren, *Theory of H^P Spaces*, Academic Press, New York, 1970.

11. Ya. L. Geronimus, *Polynomials orthogonal on a circle and their applications*, Amer. Math. Soc. Trans. **104** (1954).

12. ———, *Orthogonal polynomials*, Consultants Bureau, New York, 1961.

13. Ulf Grenander and Gabor Szegö, *Toeplitz forms and their applications*, University of California Press, Berkeley, 1958.

14. P. Henrici, *Applied and computational complex analysis*, vol. 1, Power Series, Integration, Conformal Mapping and Location of Zeros, Wiley, New York, 1974.

15. ———, *Applied and computational complex analysis*, vol. 2, *Special functions, integral transforms, asymptotics and continued fractions*, John Wiley and Sons, New York, 1977.

16. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

17. K. Hoffman and R. Kunze, *Linear algebra* (Second Edition), Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

18. William B. Jones, Olav Njåstad and W.J. Thron, *Schur fractions, Perron-Carathéodory fractions and Szegö polynomials, a survey*, in *Analytic theory of continued fractions* II (W.J. Thron, ed.), Lecture Notes in Mathematics **1199**, Springer-Verlag, New York, 1986, 127–158.

19. ———, *Continued fractions associated with the trigonometric and other strong moment problems*, Constr. Approx. **2** (1986), 197–211.

20. ———, *Perron-Carathéodory continued fractions*, in *Rational approximation and its applications to mathematics and physics* (J. Gilewicz, M. Pindor, W. Siemaszko, eds.), Lecture Notes in Mathematics **1237**, Springer-Verlag, New York, 1987, 188–206.

21. ———, *Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle*, Bull. London Math. Soc., **21** (1989), 113–152.

22. William B. Jones and W.J. Thron, *Continued fractions: analytic theory and applications*, Encyclopedia of Mathematics and its Applications 11, Addison-Wesley Publishing Company, Reading, MA, 1980, distributed now by Cambridge University Press, New York.

23. ———, *A constructive proof of convergence of the even approximants of positive PC-fractions*, Rocky Mountain J. Math., **19** No. 1 (Winter 1989), 199–210.

24. Norman Levinson, *The Wiener RMS* (*root mean square*)*, error criterion in filter design and prediction*, J. Math and Physics **25** (1947), 261–278.

25. I.I. Priwalow, *Randeigenschaften analytischer funktionen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.

26. W. Rudin, *Real and complex analysis*, Second Edition, McGraw-Hill Book Company, New York, 1974.

27. Andrew E. Yagle and Bernard C. Levy, *The Schur algorithm and its applications*, Acta Applicandae Mathematicae **3**, D. Reidel Pub. Co., 1985, 255–284.

28. Norbert Wiener, *Extrapolation, interpolation and smoothing of stationary time series*, Published jointly by The Technology Press of the Massachusetts Institute of Technology, and John Wiley and Sons, Inc., New York, 1949.

Department of Mathematics, University of Colorado, Boulder, CO 80309-0426

Department of Mathematics, University of Trondheim (NTH), 7034 Trondheim, Norway