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CLOSED SUBALGEBRAS OF THE BANACH ALGEBRA OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS ON AN INTERVAL

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ABSTRACT. Several classes of closed subalgebras of $D^1(I)$ are studied in this paper. A number of results on singly-generated subalgebras are given, including the result that such subalgebras are regular (in the sense of Shilov) if and only if the range of the generator does not separate the plane. Other conditions sufficient for closed subalgebras to be regular are also given. For instance, closed separating subalgebras are shown to be regular and singly-generated. The paper closes with a characterization of those closed separating subalgebras over which $D^1(I)$ is integral.

Introduction. In this paper, we are concerned with closed subalgebras of the Banach algebra $D^1(I)$ of continuously differentiable complex-valued functions on a closed interval $I = [a, b]$ of real numbers. The norm $\|\cdot\|_1$ on $D^1(I)$ is defined by $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$, $f \in D^1(I)$, where $\|\cdot\|_\infty$ is the sup norm on I. We shall always assume that our subalgebras contain the constant functions.

Sections one and two are concerned primarily with the singly-generated closed subalgebras of $D^1(I)$. For $f \in D^1(I)$ we use A_f to denote the closed subalgebra generated by f and let S_f be its set of critical points. Section 1 is preliminary in nature, and contains definitions and technical results used later in the paper. In Section 2, we are interested in identifying the functions $g \in D^1(I)$ which must belong to A_f. Obviously, any characterization of such functions must involve the derivative of g. For the case where the range $f(I)$ of f does not separate the plane, we show (Theorem 2.1) that $g \in A_f$ if and only if $f(x) = f(y)$ implies $g(x) = g(y)$ and $g'(x)f'(y) = g'(y)f'(x)$, and $f'(x) = 0$ implies $g'(x) = 0$. Also in Section 2, we refine our results for the case where f is real-valued.

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Sections 3 and 4 are concerned with conditions on a closed subalgebra B of $D^1(I)$ sufficient for B to be regular (in the sense of Shilov). Our main result (Theorem 3.4) states that B is regular if and only if B is inverse-closed on I. It follows from this theorem that A_f is regular if and only if $f(I)$ does not separate the plane and that any B which is conjugate-closed is also regular. In Section 4, our attention is focused on closed subalgebras B which are separating on I. Such algebras are shown to be conjugate-closed, regular, and to have the form A_f for some real-valued $f \in D^1(I)$.

In Section 5, we consider conditions on a closed subalgebra B over which $D^1(I)$ is integral. For the case where B is separating on I, we give a necessary and sufficient condition for $D^1(I)$ to be integral over B. The condition is topological and is stated in terms of the n^{th} derived set of the set of common critical points of the functions in B. The appendix contains two examples.

We should point out that some of our results are analogous to certain results concerning $C(I)$, the uniform algebra of continuous complex-valued functions on I. We cite two instances. It follows from a theorem of J. Walsh (see [**10**, p. 444]) that if $f \in C(I)$ is one-to-one on I, then f is a generator for $C(I)$. In our context, if $f \in D^1(I)$ is one-to-one, then the analogous statement would be that f is a generator for the closed subalgebra of functions $g \in D^1(I)$ such that $g'(x) = 0$ whenever $f'(x) = 0$. That this is true is shown in Proposition 1.3. The other instance is a theorem of G. Stolzenberg (see [**9**, p. 186]) which states that if a set $\Lambda \subset D^1(I)$ separates the points of I, then $C(I)$ is the smallest uniformly closed subalgebra containing Λ. For the closure B in $D^1(I)$ of the subalgebra generated by Λ, the analogous result would be that B coincides with the closed subalgebra of $D^1(I)$ of functions g such that $S \subset S_g$, where $S = \cap_{f\in B} S_f$. This is shown in Section 4.

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1. Preliminaries. If A is a commutative Banach algebra, then $\sum_A$ will denote the carrier space of A, that is, the space of non-zero complex homomorphisms on A to $\mathbf{C}$ with the usual Gelfand topology. $C(X)$ will denote the set of all continuous, complex-valued functions on X, a compact metric space. For $B \subset C(I), \overline{B}^\infty$ will denote the uniform closure of B in $C(I)$. If $B \subseteq D^1(I)$, then the set of derivatives of the functions in B will be denoted by B'. When B is a closed subalgebra, B' will be a closed linear subspace of $C(I)$ since subalgebras are assumed to contain the constant functions. Furthermore, since the spectral radius of any $g \in D^1(I)$ is $\|g\|_\infty$, the uniform norm over I, we have that $\sum_B = \sum_{\overline{B}^\infty}$. (In general, if B_1 is a subalgebra of B_2 and if the restriction mapping $\pi_{B_1}^{B_2} : \sum_{B_2} \to \sum_{B_1}$ is $1-1$ and onto, then we will write $\sum_{B_1} = \sum_{B_2}$.) An algebra A of complex-valued functions on a non-empty set X is called inverse-closed on X if $1/f \in A$ whenever $f \in A$ and $f(x) \neq 0$ for all $x \in X$.

There are two natural subalgebras associated with each $f \in D^1(I)$. A_f is the closed subalgebra of $D^1(I)$ generated by f, that is, the closure in $D^1(I)$ of the set of polynomials in f. Q_f is the closure in $D^1(I)$ of the set of functions of the form g/h, where $g, h \in A_f$ and h is non-vanishing on I. Clearly, $A_f \subseteq Q_f$ and Q_f is inverse-closed on I. $\overline{A_f}^\infty$ and $\overline{Q_f}^\infty$ can be viewed as subalgebras of $C(f(I))$ as follows. Let $g \in \overline{Q_f}^\infty$. Since $f(x) = f(y)$ implies that $g(x) = g(y), \widehat{g}(f(x)) = g(x), x \in I$, defines a function on $f(I)$ which is easily shown to be continuous. If we set $\psi(g) = \widehat{g}$, then ψ is an isometric isomorphism of $\overline{Q_f}^\infty$ into $C(f(I))$. Let $\mathbf{R}(f(I))$ and $\mathbf{P}(f(I))$ denote respectively the uniform closures in $C(f(I))$ of the set of rational functions with poles off $f(I)$ and the set of polynomials in z. Then $\psi(\overline{Q_f}^\infty) = \mathbf{R}(f(I))$ and $\psi(\overline{A_f}^\infty) = \mathbf{P}(f(I))$. It follows that $\sum_{Q_f} = \sum_{\overline{Q_f}^\infty} = f(I), \sum_{A_f} = \sum_{\overline{A_f}^\infty} = \widehat{f(I)}$, where $\widehat{f(I)}$ denotes the polynomial convex hull of $f(I)$, and the Shilov boundary ∂A_f of A_f is bdy $(f(I))$ (see [**3**; Theorems 1.3 and 1.4, p. 27]). Note that if $g \in Q_f$, then the function $\widehat{g} = \psi(g)$ is the Gelfand transform of g (when $\sum_{Q_f}$ is identified with $f(I)$). If $g \in A_f$, then $\widehat{g}$ denotes either the Gelfand transform (on $\widehat{f(I)}$) of g or $\psi(g)$. The context will make it clear which we mean.

As an immediate consequence of the above, we have that A_f is inverse-closed on I if and only if $\mathbf{P}(f(I))$ is inverse-closed on $f(I)$. Now, it is easily seen that $\mathbf{P}(f(I))$ is inverse-closed on $f(I)$ if and only if $\widehat{f(I)} = f(I)$, that is, $f(I)$ does not separate the plane. Moreover, since the planar Lebesgue measure of $f(I)$ is zero by Sard's Lemma (see [**8**, Theorem 3.14, p. 72]), it follows from Lavrentiev's Theorem (see [**3**, p. 48]) that $\widehat{f(I)} = f(I)$ if and only if $\mathbf{P}(f(I)) = C(f(I))$. Thus, we have

Lemma 1.1. *Let $f \in D^1(I)$. Then the following are equivalent*:

(i) *A_f is inverse-closed on I.*

(ii) *$f(I)$ does not separate the plane.*

(iii) $\psi(\overline{A}_f^\infty) = C(f(I))$.

The next proposition is the key to our description of A_f. For subsets A_1 and A_2 of $C(I)$, $A_1 \cdot A_2$ will denote the set of functions of the form $f_1 f_2$, where $f_i \in A_i, i = 1, 2$.

Proposition 1.2. *Let $f \in D^1(I)$. Then $\overline{A}_f^\infty \cdot A'_f = A'_f$. Moreover, if $f(I)$ does not separate the plane, then $\phi(f)f' \in A'_f$ for each $\phi \in C(f(I))$.*

Proof. Since $1 \in A_f$, we immediately have $A'_f \subseteq \overline{A}_f^\infty \cdot A'_f$. To establish the reverse inclusion, let $g \in \overline{A}_f^\infty$ and $h \in A'_f$. Pick sequences $\{p_n(f)\}_{n=1}^\infty$ and $\{q_n(f)\}_{n=1}^\infty$ of polynomials in f such that $p_n(f) \to g$ and $q_n(f)f' \to h$ uniformly on I. Consequently, $p_n(f)q_n(f)f' \in A'_f$ for each $n \geq 1$ and $p_n(f)q_n(f)f' \to gh$ uniformly on I. By our earlier remarks, A'_f is uniformly closed so that $gh \in A'_f$, and it follows that $\overline{A}_f^\infty \cdot A'_f = A'_f$. The second assertion of the proposition now follows from the above and Lemma 1.1. $\square$

Recall that the set of zeros of $f', f \in D^1(I)$, is denoted by S_f. If B is a closed subalgebra of $D^1(I)$, then set $S(B) = \cap_{f \in B} S_f$. For a closed subset S of I, let $A_S = \{g \in D^1(I) : S \subset S_g\}$ and $k_f(S) = \{g \in \overline{A}_f^\infty : g(x) = 0 \text{ for } x \in S\}$. Note that A_S is a conjugate-closed, closed subalgebra of $D^1(I)$.

Proposition 1.3. *If $f \in D^1(I)$ is either one-to-one on I or real-valued and monotone on I, then $A_f = A_{S_f}$. If, in addition, S_f is empty, then $A_f = D^1(I)$. Conversely, if S is a closed subset of I, then A_S is singly-generated by a non-negative function $g \in D^1(I)$ and $A_S = A_{S_g}$.*

Proof. If f is one-to-one on I, then $\overline{A}_f^{\infty} = C(f(I))$ by a Theorem of Walsh (see [**10**, p. 444]). On the other hand, if f is real-valued on I, then it follows from the Stone-Weierstrass Theorem that $\psi(\overline{A}_f^{\infty}) = C(f(I))$ so that $\overline{A}_f^{\infty} = \{g \in C(I) : f(x) = f(y) \text{ implies } g(x) = g(y)\}$. If, in addition to being real-valued, f is monotone on I, then $f(x) = f(y), x < y$, implies that f is constant on $[x, y]$, so that $A'_f \subseteq \overline{A}_f^{\infty}$. Thus, under either hypothesis on $f, A'_f \subseteq \overline{A}_f^{\infty}$ and $\overline{A}_f^{\infty}$ is a conjugate-closed, closed subalgebra of $C(I)$. It follows from Proposition 1.2 that A'_f is an ideal in $\overline{A}_f^{\infty}$. Hence, $A'_f = k_f(S_f)$ since A'_f is uniformly closed in $\overline{A}_f^{\infty}$. Thus, $A_f = A_{S_f}$ since $A'_{S_f} = k_f(S_f)$ and both subalgebras contain the constant functions. It now follows that if $S_f = \varnothing$, then $A_f = D^1(I)$.

Conversely, if S is a closed subset of I, then let $f \in D^1(I)$ be defined by $f(x) = \int_a^x \delta(t)dt$, where $x \in I = [a, b]$ and, for $t \in I, \delta(t)$ is the distance from t to S. Then f is real-valued and monotone (increasing) on I, and $S_f = S$. By the first part of the proof, $A_f = A_{S_f} = A_S$. □

For a given equivalence relation R on I, we set $A^R = \{f \in D^1(I) : xRy \text{ implies } f(x) = f(y)\}$. Clearly A^R is a conjugate-closed, closed subalgebra of $D^1(I)$ which is also inverse-closed on I. If B is a closed subalgebra of $D^1(I)$, then $R(B)$ denotes the equivalence relation on I induced by the functions in B, that is, $xR(B)y$ if and only if $g(x) = g(y)$ holds for all $g \in B$. When $B = A_f$, we simply write R_f for $R(B)$. B_f will denote the set of functions in $A^{R_f} \cap A_{S_f}$ which satisfy the additional condition that $g'(x)f'(y) = g'(y)f'(x)$ whenever $f(x) = f(y)$.

Proposition 1.4. *Let $f \in D^1(I)$. Then the following hold*:

(i) *B_f is a closed subalgebra of $D^1(I)$ containing A_f; and*

(ii) *B_f is inverse-closed on I.*

Proof. The verification of (i) is straightforward and hence omitted. To show (ii), let $g \in B_f, g(x) \neq 0$ for all $x \in I$. Then $1/g \in A^{R_f}$ since A^{R^f} is inverse-closed on I. From the formula $(1/g)'(x) = -g'(x)/g(x)^2, x \in I$, we see that $1/g \in A_{S_f}$. Thus, $1/g \in A^{R_f} \cap A_{S_f}$. Now suppose that $f(x) = f(y)$. Then $g(x) = g(y)$ and $g'(x)f'(y) = g'(y)f'(x)$ both hold so that $(-g'(x)/g(x)^2)f'(y) = (-g'(y)/g(y)^2)f'(x)$. Thus, $(1/g)'(x)f'(y) = (1/g)'(y)f'(x)$ holds whenever $f(x) = f(y)$. Hence, $1/g \in B_f$ and therefore, B_f is inverse-closed on I. □

2. Singly-generated subalgebras. In this section, we continue our study of singly-generated closed subalgebras. From Section 1, we know that $A_f \subseteq B_f \subseteq A^{R_f} \cap A_{S_f}$. The subalgebras B_f and $A^{R_f} \cap A_{S_f}$ are somewhat easier to describe than A_f since their definition does not involve the norm on $D^1(I)$. Thus, it is of interest to know the exact relationship between A_f, B_f and $A^{R_f} \cap A_{S_f}$. We begin with Theorem 2.1, which can be viewed as an approximation theorem.

Theorem 2.1. *Let $f \in D^1(I)$. Then $A_f = B_f$ if and only if $f(I)$ does not separate the plane.*

Proof. Suppose that $A_f = B_f$. By Proposition 1.4, B_f is inverse-closed on I so that A_f is inverse-closed on I. That $f(I)$ does not separate the plane now follows from Lemma 1.1.

Conversely, suppose $f(I)$ does not separate the plane. From Section 1, we know that $A_f \subseteq B_f$ so we need only show the reverse inclusion. Let $g \in B_f$ and let $z \in f(I)$. We first show that there exists $h_z \in A_f$ such that $h_z' \equiv g'$ on $f^{-1}(z)$. If $f^{-1}(z) \subset S_f$, then we take $h_z \equiv 0$. Then $h_z' \equiv g'$ on $f^{-1}(z)$ since $g \in B_f$ implies g' is zero whenever f' is zero. If $f^{-1}(z) \not\subset S_f$, then let $x \in f^{-1}(z) \backslash S_f$. Then $f'(x) \neq 0$. Define h_z by $h_z' = (g'(x)/f'(x))f'$. Clearly $h_z \in A_f$. If $y \in f^{-1}(z)$, then $f(x) = f(y)$ so that $g'(x)f'(y) = g'(y)f'(x)$. Since $f'(x) \neq 0, g'(y) = (g'(x)/f'(x))f'(y) = h_z'(y)$. Thus, for all $z \in f(I)$, there exists $h_z \in A_f$ such that $h_z' \equiv g'$ on $f^{-1}(z)$. Let $\epsilon > 0$ be given and let W_z be the open set (in I) where $|g' - h_z'| < \epsilon$. Then $f^{-1}(z) \subset W_z$ and hence there exists an open neighborhood V_z (in $f(I)$) of z such that $f^{-1}(V_z) \subset W_z$. Since $f(I)$ is compact, there are points $z_1, \ldots, z_n \in f(I)$ such that $\cup_{i=1}^n V_{z_i} = f(I)$. Let $u_1, \ldots, u_n \in C(f(I))$

be a partition of unity such that $\sum_{i=1}^n u_i \equiv 1, 0 \leq u_i \leq 1$ and $u_i \equiv 0$ on $f(I)\backslash V_{z_i}, i = 1, 2, \ldots, n$, and let h be defined by $h' = \sum_{i=1}^n u_i(f)h'_{z_i}$. By Proposition 1.2, $h' \in A'_f$. On the other hand, if $x \in I$, then

$$\begin{aligned} |h'(x) - g'(x)| &= \left| \sum_{i=1}^n u_i(f(x))[h'_{z_i}(x) - g'(x)] \right| \\ &\leq \sum_{i=1}^n [u_i(f(x))|h'_{z_i}(x) - g'(x)|] \\ &\leq \left[\sum_{i=1}^n u_i(f(x)) \right] \cdot \epsilon = \epsilon, \end{aligned}$$

since $|h'_{z_i} - g'| < \epsilon$ on f^{-1} (supp u_i), $i = 1, 2, \ldots, n$, implies that $0 \leq u_i(f)|h'_{z_i} - g'| \leq u_i(f) \cdot \epsilon$. It follows that $g' \in A'_f$ since A'_f is uniformly closed. But this means that $g \in A_f$ and, hence, $A_f = B_f$. □

Next we give two consequences of the theorem. The first gives a necessary and sufficient condition for A_f to be conjugate-closed, while the second gives the somewhat surprising result that when $A^{R_f} \cap A_{S_f}$ is singly-generated, then it is also generated by f.

Corollary 2.2. *A_f is conjugate-closed if and only if $f(I)$ does not separate the plane and $f(x) = f(y)$ implies $u'(x)v'(y) = u'(y)v'(x)$, where $u = \mathrm{Re}(f), v = \mathrm{Im}(f)$.*

Proof. First note that $u'(x)v'(y) = u'(y)v'(x)$ holds if and only if $u'(x)f'(y) = u'(y)f'(x)$ holds. Now, suppose that A_f is conjugate-closed. Then $\overline{A}_f^\infty$ is conjugate-closed so that $\mathbf{P}\,(f(I))$ is also conjugate-closed. Thus, $\mathbf{P}\,(f(I)) = C(f(I))$ by the Stone-Weierstrass Theorem, and it follows from Lemma 1.1 that $f(I)$ does not separate the plane. Since $u \in A_f$, we have by the above observation that $f(x) = f(y)$ implies $u'(x)v'(y) = u'(y)v'(x)$.

Conversely, suppose that $f(I)$ does not separate the plane and that $f(x) = f(y)$ implies $u'(x)v'(y) = u'(y)v'(x)$. Then $A_f = B_f$ and $f(x) = f(y)$ implies that $u'(x)f'(y) = u'(y)f'(x)$. Thus, $u \in B_f$ and hence $u \in A_f$. But then v is also in A_f. If $A_{u,v}$ is the closed subalgebra generated by u and v, then $A_f \subseteq A_{u,v} \subseteq A_f$ so that $A_f = A_{u,v}$. But since u, v are real-valued, $A_{u,v}$ is conjugate-closed, and we can conclude that A_f is conjugate-closed. □

It is not difficult to give an example where $f(I)$ does not separate the plane and A_f is not conjugate closed. On the other hand, if f is real-valued, then A_f is conjugate-closed. Moreover, there are examples where A_f is conjugate-closed and $f(I)$ cannot be embedded in the real line. Thus, such an A_f can not be generated by a real-valued function.

Corollary 2.3. *If $A^{R_f} \cap A_{S_f}$ is singly-generated, then $A_f = A^{R_f} \cap A_{S_f}$.*

Proof. Suppose that $A_g = A^{R_f} \cap A_{S_f}$. Then $R_g = R_f$ and $S_g = S_f$ since $A_f \subseteq A^{R_f} \cap A_{S_f} = A_g \subseteq A^{R_g} \cap A_{S_g}$. Now, $R_g = R_f$ implies that $f(I)$ and $g(I)$ are homeomorphic. Since $A_g = A^{R_f} \cap A_{S_f}, A_g$ is conjugate-closed. By the previous corollary, $g(I)$ does not separate the plane. Thus, it follows that $f(I)$ does not separate the plane (see [4; Corollary 1, p. 101]). Hence, by the theorem, $A_f = B_f$. Since $f \in A_g, g(x) = g(y)$ implies that $f'(x)g'(y) = f'(y)g'(x)$. But $f(x) = f(y)$ is equivalent to $g(x) = g(y)$ so that $f(x) = f(y)$ implies that $g'(x)f'(y) = g'(y)f'(x)$. Since $g \in A^{R_f} \cap A_{S_f}$, we have that $g \in B_f$, and, hence, $g \in A_f$. Thus, $A_f = A_g$, that is $A_f = A^{R_f} \cap A_{S_f}$. □

As we will see later in this section, when f is real-valued and $A^{R_f} \cap A_{S_f}$ is not singly-generated, then it is doubly-generated by f and $|f - f(a)|$, where $I = [a, b]$.

Unless stated to the contrary, in the remainder of this section we will assume that f is real-valued and with no loss of generality that $I = [0, 1]$. Therefore, $f(I)$ does not separate the plane and, consequently, $A_f = B_f$. The next two lemmas are technical in nature and will be used later in this section. For a subset $S \subset \mathbf{R}$, let S^0 denote the interior of S.

Lemma 2.4. *Let $f \in D^1(I)$ be real-valued and $f(x_1) = f(x_2)$,*

$x_1, x_2 \in I$. If $x_1 \in I$ and $x_2 \in I^0$ or if $I = [x_1, x_2]$ and $f'(x_1)f'(x_2) \leq$

0, then the equation

$$(*) \qquad g'(x_1)f'(x_2) = g'(x_2)f'(x_1)$$

holds for all $g \in A^{R_f} \cap A_{S_f}$.

Proof. First observe that if either $f'(x_1)$ or $f'(x_2)$ is zero (without any further restrictions on x_1 or x_2), then $(*)$ follows from the inclusion $S_f \subset S_g$ for all $g \in A^{R_f} \cap A_{S_f}$. Thus, assume that $f'(x_1)$ and $f'(x_2)$ are both non-zero. Consequently, under either hypothesis of the lemma, there are closed intervals N_1 and N_2 such that x_i is an endpoint of N_i and f' is non-vanishing on $N_i, i = 1, 2$, and $f(N_1) = f(N_2)$. Set $N = f(N_1)$ and $f_i = f|N_i, i = 1, 2$. Then each f_i has a differentiable inverse $f_i^{-1} : N \to N_i$. Let $\psi(x) = f_2^{-1}(f_1(x)), x \in N_1$ Then ψ is differentiable on $N_1, \psi(x_1) = x_2, \psi'(x_1) \neq 0$, and $f(\psi(x)) = f(x)$ for $x \in N_1$. Now, suppose $g \in A^{R_f} \cap A_{S_f}$. Then $g(\psi(x)) = g(x)$ must hold for $x \in N_1$. Consequently, $f'(x_1) = f'(\psi(x_1))\psi'(x_1) = f'(x_2)\psi'(x_1)$ and $g'(x_1) = g'(\psi(x_1))\psi'(x_1) = g'(x_2)\psi'(x_1)$. Since $\psi'(x_1) \neq 0$, we have that $g'(x_1) = 0$ if and only if $g'(x_2) = 0$, in which case $(*)$ holds. If $g'(x_1)$ and $g'(x_2)$ are non-zero, then combining $f'(x_1) = f'(x_2)\psi'(x_1)$ and $g'(x_1) = g'(x_2)\psi'(x_1)$ gives $(*)$. □

If $f(0) = f(1)$ and $g \in A^{R_f} \cap A_{S_f}$, f real-valued, then $g'(0)f'(1)$ is not necessarily equal to $g'(1)f'(0)$. Thus such a g is not in B_f. On the other hand, if we assume in addition that $g'(0) = g'(1) = 0$, then $g'(0)f'(1) = g'(1)f'(0)$ must hold. Combining this observation with Lemma 2.4 we have

Lemma 2.5. *If* $f \in D^1(I)$ *is real-valued, then* $A^{R_f} \cap A_{S_f \cup \{0,1\}} \subseteq B_f$.

Theorem 2.6. *Let* $f \in D^1(I)$ *be real-valued. Then* $A_f = A^{R_f} \cap A_{S_f}$ *if any one of the following conditions hold:*

(i) $f(0) \neq f(1)$;

(ii) $f(0) = f(1)$ and $f'(0)f'(1) \leq 0$;

(iii) $f(0) = f(1), f'(0)f'(1) > 0$ *and there exists* $x_0 \in I^0$ *such that* $f(x_0) = f(0)$ *and* $f'(x_0) \neq 0$.

Conversely, if f does not satisfy any of the above three conditions, then $A_f \subsetneqq A^{R_f} \cap A_{S_f}$.

Proof. Since f is real-valued, to show $A_f = A^{R_f} \cap A_{S_f}$ it suffices to show that $A^{R_f} \cap A_{S_f} \subseteq B_f$. Therefore, suppose $f(x) = f(y), x, y \in I$, and $g \in A^{R_f} \cap A_{S_f}$. We need only show that the condition $g'(x)f'(y) = g'(y)f'(x)$ holds when $f(x) = f(y)$. If $x \in I, y \in I^0$, the condition holds by Lemma 2.4. Thus if (i) holds, we immediately have $g \in B_f$. It remains only to verify the condition with $x = 0$ and $y = 1$ when either (ii) or (iii) holds. By Lemma 2.4 this is the case when (ii) holds. Suppose, finally, that (iii) holds and that $x_0 \in I^0, f'(x_0) \neq 0$ and $f(x_0) = f(0)$. By applying Lemma 2.4 to the pairs $\{0, x_0\}$ and $\{x_0, 1\}$ we have $g'(0)/f'(0) = g'(x_0)/f'(x_0)$ and $g'(x_0)/f'(x_0) = g'(1)/f'(1)$ so that $g'(0)f'(1) = g'(1)f'(0)$. Thus, in all cases, $g \in B_f$. Therefore $A_f = A^{R_f} \cap A_{S_f}$.

Now, suppose that none of the above conditions on f hold, that is, $f(0) = f(1), f'(0)f'(1) > 0$ and if $f(x_0) = f(0), x_0 \in I^0$, then $f'(x_0) = 0$. Without loss of generality, we assume that $f'(0)$ and $f'(1)$ are both positive. We will show that $g = |f - f(0)|$ is in $A^{R_f} \cap A_{S_f}$ but not in A_f. Let V be an open set in I. If $f(t) \geq f(0)$ holds for all $t \in V$, then $g \equiv f - f(0)$ on V so that $g' \equiv f'$ on V. Similarly, if $f(t) \leq f(0)$ holds for all $t \in V$, then $g' \equiv -f'$ on V. In either case, g is continuously differentiable on V. Moreover, since $f'(0)$ and $f'(1)$ are both positive, we have that g is continuously differentiable on an open set containing the endpoints of I and $g'(0) = f'(0), g'(1) = -f'(1)$. To show that g is differentiable at $x_0 \in I^0$, where $f(x_0) = f(0)$ and hence $f'(x_0) = 0$ by hypothesis, consider

$$\left|\frac{g(x_0 + h) - g(x_0)}{h}\right| = \left|\frac{f(x_0 + h) - f(x_0)}{h}\right| \to |f'(x_0)| = 0.$$

as $h \to 0$. Thus, $g'(x_0)$ exists and is equal to zero. Consequently, g is differentiable and $|g'| \equiv |f'|$ on I. It remains to show that g' is continuous at x_0. Since $|g'| \equiv |f'|$ on I, we have that $|g'(x)| = |f'(x)| \to |f'(x_0)| = 0 = |g'(x_0)|$. Thus, g' is continuous at x_0 and hence $g \in D^1(I)$.

To complete the proof, note that $f(x) = f(y)$ implies $g(x) = g(y)$ and $|g'| \equiv |f'|$ implies $S_g = S_f$. Thus, $g \in A^{R_f} \cap A_{S_f}$. Since

$g'(0) = f'(0), g'(1) = -f'(1)$ and $f'(0), f'(1)$ are both positive, it follows that $g'(0)f'(1) > 0 > g'(1)f'(0)$. Thus, $g \notin B_f$ and, hence, $g \notin A_f$. □

Corollary 2.7. *If $f \in D^1(I)$ is real-valued and if $A_f \subsetneqq A^{R_f} \cap A_{S_f}$, then $A^{R_f} \cap A_{S_f} = A_f \oplus \mathbf{C} \cdot |f - f(0)|$. Moreover, there exists a continuous point derivation d on $A^{R_f} \cap A_{S_f}$ such that $d^{-1}(0) = A_f$.*

Proof. Without loss of generality, assume that $f(0) = 0$, and that $f'(0)$ and $f'(1)$ are both positive. We already know that $|f| \in A^{R_f} \cap A_{S_f} \backslash A_f$ so that $A_f + \mathbf{C}\,|f|$ is actually a (topological) direct sum. Now, set $h_0 = f + |f|$ and $h_1 = f - |f|$. Then $h_0, h_1 \in A_f \oplus \mathbf{C}\,|f|$. Moreover, $h_0'(0) = 2f'(0), h_0'(1) = 0, h_1'(0) = 0$ and $h_1'(1) = 2f'(1)$. If $g \in A^{R_f} \cap A_{S_f}$, then set $h = g - (g'(0)/2f'(0))h_0 - (g'(1)/2f'(1))h_1$. It is easily seen that $h'(0) = h'(1) = 0$. Thus, $h \in A^{R_f} \cap A_{S_f \cup \{0,1\}}$. By Lemma 2.5, $h \in A_f$. Therefore $g \in A_f \oplus \mathbf{C}\,|f|$ and it follows that $A^{R_f} \cap A_{S_f} = A_f \oplus \mathbf{C}\,|f|$.

To prove the second assertion of the corollary, let $d(g) = f'(0)g'(1) - f'(1)g'(0)$ for $g \in A^{R_f} \cap A_{S_f}$. Then d is a continuous point derivation on $A^{R_f} \cap A_{S_f}$ at the complex homomorphism ϕ_0, where $\phi_0(g) = g(0), g \in A^{R_f} \cap A_{S_f}$. Moreover, $d(f) = 0$ and $d(|f|) = -2f'(0)f'(1)$ so that $A_f \subseteq d^{-1}(0)$ and d is not identically zero on $A^{R_f} \cap A_{S_f}$. By the first part of the proof, we know that A_f has codimension 1 in $A^{R_f} \cap A_{S_f}$, so it follows that $d^{-1}(0) = A_f$. □

In closing this section, we give applications of Theorems 2.1 and 2.6. Let $I = [-1, 1]$ for convenience. Suppose that $f \in D^1(I)$ is real-valued and even, and that $S_f = \{0\}$. Then A_f is the set of even functions in $D^1(I)$. To see this, let $g \in D^1(I)$ be even. Thus, $f(x) = f(y)$ implies $g(x) = g(y)$, and $g'(0) = 0$. Since $f'(x) = 0$ only for $x = 0, g \in A^{R_f} \cap A_{S_f}$. This shows that $A^{R_f} \cap A_{S_f}$ is the set of even functions on I. On the other hand, $f'(-1) = -f'(1)$ so that $f'(-1)f'(1) < 0$ and hence, $A_f = A^{R_f} \cap A_{S_f}$. Since $A^{R_f} \cap A_{S_f}$ is the set of even functions, as shown above, the conclusion follows. If we replace the hypothesis that f is real-valued with the hypothesis that f is one-to-one on $[-1, 0]$ and retain the hypotheses that f is even and that $S_f = \{0\}$, then A_f is still the subalgebra of even functions in $D^1(I)$. This follows from Theorem 2.1.

3. Regular closed subalgebras. Let us recall that a commutative Banach algebra A is called regular (in the sense of Shilov) if every set $F \subset \sum_A$ which is closed in the Gelfand topology is a hull, that is, closed in the hull-kernel topology on $\sum_A$. It is well-known that $D^1(I)$ is regular. The main theorem (3.3) of this section gives a necessary and sufficient condition that a closed subalgebra of $D^1(I)$ be regular. It is convenient to first prove two propositions that are of interest in themselves.

Proposition 3.1. *Suppose that A is commutative Banach algebra with identity and that J is a closed, totally-disconnected subset of $\sum_A$. If J is a hull and if $k(J) = \{a \in A : \widehat{a} \equiv 0$ on $J\}$ is regular, then A is regular.*

Proof. Since J is a hull, $\sum_{k(J)} = \sum_A \backslash J$. Note that if F is closed, then $F \cup J$ is a hull since $k(J)$ is regular. Now, suppose $\phi \in \sum_A \backslash F$. If $\phi \notin J$, then there is an $a \in A$ such that $\widehat{a}(\phi) \neq 0$ and $\widehat{a}$ is identically zero on $J \cup F$, and hence, identically zero on F. Now, suppose that $\phi \in J, \phi \in \sum_A \backslash F$. Since J is totally-disconnected and closed, there exists an open set V in $\sum_A$ such that $\phi \in V, F \subset V^c$, and $V \cap J$ is compact and (relatively) open in J. Set $F_1 = V \cap J$ and $F_2 = V^c \cap (J \cup F)$. Then F_1 and F_2 are compact, $F_1 \cap F_2 = \varnothing, F \subset F_2$ and $J \subset F_1 \cup F_2$. Hence, $F_1 \cup F_2$ is a hull. Since $F_1 \cap F_2 = \varnothing$ and F_1, F_2 are compact, F_1 and F_2 are hulls (see [**6**; Corollary 3.64, p. 169]). Thus, there exists an $a \in A$ such that $\widehat{a}(\phi) \neq 0$ and $\widehat{a}|F_2 \equiv 0$. Thus, $\widehat{a}|F \equiv 0$, and we can conclude that A is regular. □

For the next proposition, we need the following discussion. Let $f \in D^1(I)$. If $z \in f(I)\backslash f(S_f \cup \{a,b\})$, where $I = [a,b]$, then we say that z is a regular point if there is a neighborhood V(in $f(I)$) of z such that V is an arc. Since we are assuming that $z \notin f(S_f \cup \{a,b\})$, there are closed intervals $I_1, I_2, \dots, I_n$ in the interior I^0 of I such that f is one-to-one and f' is nonvanishing on each $I_j, f(I_j) = f(I_1)$ for $1 \leq j \leq n, f^{-1}(f(I_1)) = \cup_{j=1}^n I_j$ and $f(I_1)$ is a neighborhood of z.

Clearly $f(I_1)$ is also an arc and $f(I_1^0)$ is open in $f(I)$. If J denotes $\{z \in f(I)\backslash f(S_f \cup \{a,b\}) : z \text{ non-regular}\} \cup f(S_f \cup \{a,b\})$, then J is a closed totally-disconnected subset of $f(I)$ (see [**1**; Lemma 6, p. 132]).

Proposition 3.2. *Let $f \in D^1(I)$. Then Q_f is regular.*

Proof. Let $J \subset f(I)$ be the set of non-regular points. We will show that J is a hull and that $k(J)$ is regular. To this end, let $z_0 \in f(I)\backslash J$, and let $V_0 \subset f(I)\backslash J$ be an arbitrary open neighborhood in $f(I)$ of z_0. Let $I_1, \dots, I_n$ be the closed intervals in the preceding discussion. It can easily be arranged that $f(I_1) \subset V_0$. Let $[s_1, s_2] = I_1$ and let t_0 be the point in (s_1, s_2) such that $f(t_0) = z_0$. Define $\delta(t+is) = f(t) + s\lambda, \lambda \in \mathbf{C}$. Since $f'(t_0) \neq 0$, we can choose λ in such a way that the Jacobian of δ is positive at t_0. By the Inverse Function Theorem (see [**8**, p. 35]), there exists an open disk $\Delta(t_0)$ in $\mathbf{C}$ such that the Jacobian of δ is positive and δ is one-to-one on $\Delta(t_0), W = \delta(\Delta(t_0))$ is open in $\mathbf{C}$ and δ^{-1} is continuously differentiable on W. By suitably contracting the radius of $\Delta(t_0)$, we can assume that $W \subset \tilde{V}$ and $\Delta(t_0) \cap \mathbf{R} \subset (s_1, s_2) = I_1^0$, where $\tilde{V}$ is open in $\mathbf{C}$ and $\tilde{V} \cap f(I) = f(I_1^0)$. Thus, $W \cap f(I) \subseteq f(I_1^0)$.

Let Γ be a circle in $\Delta(t_0)$, centered at t_0, let $h(t+is)$ be a continuously differentiable function on $\Delta(t_0)$ whose closed support is interior to Γ, and let $t_1, t_2 = \Gamma \cap \mathbf{R}$. We further require that h satisfy

$$(*) \qquad \int_{t_1}^{t_2} h(t)f'(t)dt = 0 \quad \text{and} \quad \int_{t_1}^{t_0} h(t)f'(t)dt \neq 0.$$

Define $\tilde{h}(z) = h(\delta^{-1}(z))$ for $z \in W$ and $\tilde{h}(z) = 0$ for $z \in \mathbf{C}\backslash W$. Then the closed support of $\tilde{h}$ is interior to $\gamma = \delta(\Gamma)$, and hence $\tilde{h}$ is continuously differentiable on $\mathbf{C}$. Now, set $X = f(I) \cup \gamma$. Then X is a compact set in $\mathbf{C}$ and has zero planar measure by Sard's Lemma. We will show that $\tilde{h}$ can be uniformly approximated on X by a rational function whose poles lie off X but interior to γ. (To do this, we use the outline in [**2**] for a constructive proof of the Hartog's-Rosenthal Theorem—see page 161.) For each $k \in \mathbf{Z}^+$, let G_k be an open set containing X with smooth orientable boundary $\text{Bdy}(G_k)$ such that

$$\left|\tilde{h}(z) - \frac{1}{2\pi i}\int_{\text{Bdy}(G_k)} \frac{\tilde{h}(\zeta)}{\zeta - z}d\zeta\right| < (2k)^{-1}$$

for all $z \in X$ (see loc. cit., pages 151 and 162). Since X is compact and $X \subset G_k$, we can find a set $\{\zeta_0^{(k)}, \zeta_1^{(k)}, \dots, \zeta_{N_k}^{(k)}\} \subset \mathrm{Bdy}(G_k)$ such that

$$\left| \frac{1}{2\pi i} \int_{\mathrm{Bdy}(G_k)} \frac{h(\zeta)}{\zeta - z} d\zeta - \sigma_k(z) \right| < (2k)^{-1},$$

for all $z \in X$, where

$$\sigma_k(z) = \frac{1}{2\pi i} \sum_{j=1}^{N_k} \frac{\tilde{h}(\zeta_j^{(k)})(\zeta_j^{(k)} - \zeta_{j-1}^{(k)})}{\zeta_j^{(k)} - z}.$$

Since the closed support of $\tilde{h}$ is interior to $\gamma, \sigma_k(z)$ is a rational function whose only poles are interior to γ and off X, that is, lie in the bounded components of $\mathbf{C} \setminus (\gamma \cup f([t_1, t_2]))$. Furthermore, $\sigma_k \to \tilde{h}$ uniformly on X.

Since $\mathbf{C} \setminus (\Gamma \cup [t_1, t_2])$ has exactly two bounded components, $\mathbf{C} \setminus (\gamma \cup f([t_1, t_2])$ also has exactly two bounded components. Call them Ω_1 and Ω_2. Then the poles of σ_k lie in $\Omega_1 \cup \Omega_2$. Let $\alpha_i \in \Omega_i, i = 1, 2$. Since Ω_1 and Ω_2 are both connected, we can replace each σ_k (if necessary) with a rational function τ_k such that the poles of τ_k lie in the set $\{\alpha_1, \alpha_2\}$ and $\tau_k \to \tilde{h}$ uniformly on X.

Now, let $\Gamma_1 = \{z \in \Gamma : \mathrm{Im}\, z \geq 0\}$ and $\Gamma_2 = \{z \in \Gamma : \mathrm{Im}\, z \leq 0\}$. If $\gamma_j = \delta(\Gamma_j) \cup f([t_1, t_2])$, then γ_j is a simple closed curve (with the usual positive orientation) and has winding number $\frac{1}{2\pi i} \int_{\gamma_j} \frac{d\zeta}{\zeta - \alpha_j} = 1$ for $j = 1, 2$ and $\frac{1}{2\pi i} \int_{\gamma_j} \frac{d\zeta}{\zeta - \alpha_l} = 0$ for $j \neq l$. Thus,

$$\frac{1}{2\pi i} \int_{\gamma_j} \tau_k(\zeta) d\zeta = \mathrm{Res}(\tau_k, \alpha_j), \qquad j = 1, 2.$$

It follows from condition $(*)$ that

$$\int_{\gamma_j} \tilde{h}(\zeta) d\zeta = 0, \qquad j = 1, 2.$$

To see this, first recall that $\tilde{h} \equiv 0$ on $\delta(\Gamma_j)$ and that $\gamma_j = \delta(\Gamma_j) \cup f([t_1, t_2])$. If we parameterize $f([t_1, t_2])$ by $\zeta(t) = f(t)$, then the above integral is equal to

$$\int_{t_1}^{t_2} h(t) f'(t) dt,$$

which is zero by hypothesis. Now, recall that $\tau_k \to \tilde{h}$ uniformly on $X = f(I) \cup \delta([])$ so that $\tau_k \to \tilde{h}$ uniformly on $\gamma_j, j = 1, 2$. Hence,

$$\int_{\gamma_j} \tau_k(\zeta) d\zeta \to 0, \qquad j = 1, 2,$$

so that Res $(\tau_k, \alpha_j) \to 0, j = 1, 2$, as $k \to +\infty$. Thus, if we let $\eta_k(z) = \tau_k(z) - \sum_{j=1}^{2} (\mathrm{Res}(\tau_k, \alpha_j))/(z - \alpha_j)$, then $\eta_k \to \tilde{h}$ uniformly on X and has zero residue at each pole (if any) which must lie in $\{\alpha_1, \alpha_2\}$. It now follows that $\eta_k(z)$ is the derivative of a rational function $\omega_k(z)$ whose only poles are in the set $\{\alpha_1, \alpha_2\}$. Thus, $\omega_k(f) \in Q_f$ and $\omega_k(f)' = \eta_k(f)f'$. Since $\eta_k \to \tilde{h}$ uniformly and since Q'_f is uniformly closed, we have that $\tilde{h}(f)f' \in Q'_f$. Let $H \in Q_f$ satisfy $H(a) = 0$ and $H' = \tilde{h}(f)f'$. We next show that $\widehat{H}(z_0) \neq 0$ and $\widehat{H}(z) = 0$ for $z \in f(I_1)$.

Let $I_j = [s_1^{(j)}, s_2^{(j)}], j = 1, 2, \dots, n$. Note that $s_m^{(1)} = s_m, m = 1, 2$. Without loss of generality, we will assume that $s_2 < s_1^{(j)}$ for $j = 2, 3, \dots, n$. For $x \in I \setminus \cup_{j=1}^{n} I_j$, we have

$$\begin{aligned} H(x) = \int_a^x \tilde{h}(f(t))f'(t)dt &= \sum_{s_2^{(j)} < x} \int_{s_1^{(j)}}^{s_2^{(j)}} \tilde{h}(f(t))f'(t)dt \\ &= \sum_{s_2^{(j)} < x} (H(s_2^{(j)}) - H(s_1^{(j)})), \end{aligned}$$

since $\tilde{h}(f)$ is zero off $\cup_{j=1}^{n} I_j$.

Since $f(I_j) = f(I_1)$ and f is one-to-one on each I_j, either $f(s_1^{(j)}) = f(s_1)$ and $f(s_2^{(j)}) = f(s_2)$ or $f(s_1^{(j)}) = f(s_2)$ and $f(s_2^{(j)}) = f(s_1)$. Thus, the same pair of equations hold for H. Furthermore, since $\tilde{h}(f) \equiv 0$ on $[s_1, t_1] \cup [t_2, s_2]$, we have that

$$H(s_2) - H(s_1) = \int_{s_1}^{s_2} \tilde{h}(f(t))f'(t)dt = \int_{t_1}^{t_2} \tilde{h}(f(t))f'(t)dt = 0.$$

Thus, $H(s_2^{(j)}) - H(s_1^{(j)}) = \pm(H(s_2) - H(s_1)) = 0$ and we can conclude that $H(x) = 0$ for all $x \in I \setminus \cup_{j=1}^{n} I_j$, that is, $\widehat{H}(z) = 0$ if $z \notin f(I_1)$.

We next show that $H(t_0) \neq 0$. First $H(s_1) = 0$ since $H(x) = 0$ for all $x \in [a, s_1)$. Thus,

$$H(t_0) = \int_{s_1}^{t_0} \tilde{h}(f(t))f'(t)dt = \int_{t_1}^{t_0} \tilde{h}(f(t))f'(t)dt$$
$$= \int_{t_1}^{t_0} h(t)f'(t)dt$$

since $\tilde{h}(f) \equiv 0$ on $[s_1, t_1]$. It follows from condition $(*)$ that $H(t_0) \neq 0$. Thus $\widehat{H}(z_0) \neq 0$, where $z_0 = f(t_0)$, and $\widehat{H}(z) = 0$ for all $z \notin f(I_1)$. This means that $\widehat{H}(z) = 0$ off V_0 and, in particular, $\widehat{H}(z) = 0$ for all $z \in J$. This shows that J is a hull and that $k(J)$ is regular. It now follows from Proposition 3.1 that Q_f is regular. □

For the next theorem, which is the main result of this section, and one of its corollaries, we need the following easily proved lemma. (For similar results and their proofs, see [**5**; Theorem 4.1, p. 234] or [**11**; Theorem 9.28, p. 9–16].)

Lemma 3.3. *Let B be a commutative Banach algebra with identity and let Λ be a set of generators of B. If each $a \in \Lambda$ is contained in a regular, closed subalgebra of B, then B is regular.*

Theorem 3.4. *Let B be a closed subalgebra of $D^1(I)$. Then B is regular if and only if B is inverse-closed on I.*

Proof. Suppose that B is regular. Since any subalgebra of a semi-simple Banach algebra is also semi-simple, it follows that every complex homomorphism on B extends to $D^1(I)$ (see [**6**, p. 175]). If $g \in B$ and $g(x) \neq 0$ for all $x \in I$, then $\widehat{g}$ is non-vanishing on $\sum_B$ so that $g^{-1} \in B$. Hence, B is inverse-closed on I.

Conversely, suppose that B is inverse-closed on I. Then $Q_f \subset B$ for any $f \in B$. Since Q_f is regular for any $f \in B$, it follows from the above lemma that B is regular. □

Corollary 3.5. *Let $f \in D^1(I)$. Then A_f is regular if and only if $f(I)$ does not separate the plane.*

The corollary follows immediately from the Theorem since A_f is inverse-closed on I if and only if $f(I)$ does not separate the plane.

COROLLARY **3.6.** *Suppose B is a closed subalgebra of $D^1(I)$. If B is conjugate-closed, then B is regular. In particular, A^R, A_S and $A^R \cap A_S$ are regular for any relation R on I and closed set $S \subset I$.*

Proof. Since B is conjugate-closed, the real-valued functions $f \in B$ generate B. By Corollary 3.5, A_f is regular. That B is regular now follows from Lemma 3.3. Since A^R, A_S and $A^R \cap A_S$ are conjugate-closed, they are regular by the first part of the proof. □

It is not true in general that a regular, closed subalgebra of $D^1(I)$ is necessarily conjugate-closed (see Corollary 2.2 and the remarks following it, and Corollary 3.5). On the other hand, closed separating subalgebras are both regular and conjugate-closed as will be shown in next section.

The final result of this section is an extension of Proposition 3.2. Let A be a closed subalgebra of $D^1(I)$ and let $H_A = \{f \in A : f(x) \neq 0$ for all $x \in I\}$. Then each $f \in H_A$ has an inverse in $D^1(I)$ so that f is not a topological divisor of zero in $D^1(I)$, and, hence, not a topological divisor of zero in A. (Note: The set of non-topological divisors of zero in A can be larger than H_A. We give an example in the Appendix.) Let $[A, H_A]$ denote the set of functions of the form fh^{-1} where $f \in A$ and $h \in H_A$, and let Q_A denote the closure of $[A, H_A]$ in $D^1(I)$. Clearly, Q_A is a subalgebra of $D^1(I)$, and $Q_f = Q_{A_f}$ for each $f \in D^1(I)$. For a closed subalgebra A, let $\pi_A = \pi_A^{D^1(I)}$.

Proposition 3.7. *Let A be a closed subalgebra of $D^1(I)$. Then Q_A is regular and $\sum_{Q_A} = \pi_A(I)$. Furthermore, if B is a regular, closed subalgebra of $D^1(I)$ containing A, then $Q_A \subseteq B$.*

Proof. In view of Theorem 3.4, it suffices to show that Q_A is inverse-closed on I. Let $f \in H_{Q_A}$. We will show $f^{-1} \in Q_A$. There exists $f_n g_n^{-1} \in [A, H_A]$ such that $\|f_n g_n^{-1} - f\|_1 \to 0$ as $n \to \infty$. Since f is non-vanishing on I, there exists $N \in \mathbf{Z}^+$ such that $f_n \in H_A$ for

$n \geq N$. Thus, $n \geq N$ implies that $(f_n g_n^{-1})^{-1} = f_n^{-1} h_n \in [A, H_A]$, that is, $n \geq N$ implies $(f_n g_n^{-1})^{-1} \in Q_A$. It follows that either f is a topological divisor of zero in Q_A or $f^{-1} \in Q_A$ (see [**6**; Theorem 1.5.4, p. 22]). But the former is impossible by our earlier comments since $f \in H_{Q_A}$ so that Q_A is inverse-closed on I. By Theorem 3.4, Q_A is regular.

Now, suppose that $\phi \in \pi_A(I)$. Then ϕ extends to a complex homomorphism ϕ_x on $D^1(I)$, where $x \in I$, and $\phi_x(g) = g(x), g \in D^1(I)$. Clearly, $\phi_x(fg^{-1}) = f(x)/g(x), fg^{-1} \in [A, H_A]$, defines a continuous extension $\tilde{\phi}$ of ϕ to Q_A. Thus, the range of the restriction mapping $\pi_A^{Q_A}$ of $\sum_{Q_A}$ into $\sum_A$ must contain $\pi_A(I)$. Furthermore, the extension $\tilde{\phi}$ of ϕ is easily seen to be unique so that $\pi_A^{Q_A}$ is also one-to-one. Now, let $\tilde{\phi} \in \sum_{Q_A}$. Since Q_A is regular and semi-simple, $\tilde{\phi}$ extends to a complex homomorphism $\tilde{\phi}_x$ on $D^1(I)$ (see loc. cit., Theorem 3.7.5, p. 175). Then $\phi = \tilde{\phi}|A = \tilde{\phi}_x|A = \pi_A(x)$ so that $\pi_A^{Q_A}(\sum_{Q_A}) = \pi_A(I)$. Thus, $\sum_{Q_A}$ is (identifiable with) $\pi_A(I)$.

Suppose, finally, that B is a regular, closed subalgebra of $D^1(I)$ containing A. To show $Q_A \subseteq B$, it suffices to show that if $f \in H_A$, then $f^{-1} \in B$. For such an f, we have $f(x) \neq 0$ for all $x \in I$. Since B is regular, it is inverse-closed on I by Theorem 3.4. Hence, $f^{-1} \in B$ for any $f \in H_A$. Thus, it follows that $Q_A \subseteq B$. □

In closing this section, we note that there can be many regular, closed subalgebras B of $D^1(I)$ which contain A and for which $\sum_B = \pi_A(I)$ (identification being made by the restriction mapping π_A^B). If $R = R(A)$, the relation induced on I by A, then clearly such an algebra B must be contained in A^R. We next show that $\sum_{A^R} = \pi_A(I)$. It suffices to show $\sum_{A^R} = \sum_{Q_A}$. By Corollary 3.6, we know that A^R is regular. Since Q_A is regular and contained in $A^R, \pi_{Q_A}^{A^R}(\sum_{A^R}) = \sum_{Q_A}$. To see that $\pi_{Q_A}^{A^R}$ is $1-1$, let $\phi_i = \pi_{A^R}(x_i), i = 1, 2$, satisfy $\phi_1|A = \phi_2|A$. Then $f(x_1) = f(x_2)$ holds for all $f \in A$ so that $x_1 R x_2$. Hence, $\phi_1 = \phi_2$ and thus the mapping $\pi_{Q_A}^{A^R}$ is one-to-one and $\sum_{A^R} = \pi_A(I)$. Thus, A^R is the largest regular, closed subalgebra with carrier space $\pi_A(I)$ and containing A. Of course, by the last proposition, Q_A is the smallest regular closed subalgebra containing A and having carrier space $\pi_A(I)$.

4. Closed separating subalgebras. Suppose that S is a closed subset of I. It is easily seen that A_S will separate the points of I if and only if S^0, the interior of S, is empty. The principal result of this section is that if B is a separating, closed subalgebra of $D^1(I)$, then B must have the form $A_S, S = S(B)$. It then follows from Proposition 1.3 that B is singly-generated with a real-valued generator. We begin with a needed lemma. For a closed set $S \subset I$, let $J(S)$ be the ideal formed by taking the closure of the set of functions in $D^1(I)$ which vanish on a neighborhood of S, the neighborhood depending on the function. It is well-known that $J(S) = \{g \in D^1(I) : g|S \equiv g'|S \equiv 0\}$ (see [**6**, p. 301–2]). From this characterization of $J(S)$, it easily follows that $J(S) \subseteq A_S$ and that $A_S/J(S)$ is semi-simple.

Lemma 4.1. *Let $S \subset I$ be closed. Then $A_S/J(S)$ is generated by its idempotents.*

Proof. Let $g \in A_S$. Then $g' \equiv 0$ on S so there exists a sequence $\{h_n\}_{n=1}^{\infty} \subset C(I)$ such that $h_n \to g'$ (uniformly on I) and $h_n \equiv 0$ on a neighborhood V_n of S. For each $n \geq 1$, set $g_n(x) = \int_0^x h_n(t)dt + g(0)$. Then $g_n \in A_S$ and $g_n \to g$ (in norm on $D^1(I)$). Moreover, g_n is locally constant on V_n so that $(g_n + J/(S))^\wedge$ assumes only a finite number of distinct values on $S = \sum_{A_S/J(S)}$. Thus, $(g_n + J(S))^\wedge$ is a (finite) linear combination of idempotents in $A_S/J(S)^\wedge$. Since $A_S/J(S)$ is semi-simple, $g_n + J(S)$ is a linear combination of idempotents, and since $g_n + J(S) \to g + J(S)$ (in the quotient norm), the conclusion of the lemma follows. □

Theorem 4.2. *If B is a closed separating subalgebra of $D^1(I)$, then B is regular and, moreover, $B = A_{S(B)}$.*

Proof. Since B is separating, $\overline{B}^{\infty} = C(I)$ (see [**9**, p. 186]). Hence, $\sum_B = I$ so that B is inverse-closed on I and we can conclude that B is regular. We next show that $J(S(B)) \subset B$. Since B is closed, it suffices to show that if $g \in D^1(I)$ and $g \equiv 0$ on a neighborhood of $S(B)$, then $g \in B$. Let g be such a function and let $x \in I \backslash S(B)$. Then there is an $f \in B$ such that $f'(x) \neq 0$. Let N be a closed interval such that N is a closed neighborhood (in I) of x, $N \cap S_f = \varnothing$, and f is $1-1$ on N. Then, by Proposition 1.3, $f_N = f|N$ is a generator for $D^1(N)$ since f_N is $1-1$ and has a non-vanishing derivative on N. Since $f_N \in B|N$, we have that $B|N$ is dense in $D^1(N)$. Thus, there exists a sequence $\{g_n\}_1^\infty$ in B such that $g_n|N \to g|N$ (in the norm on $D^1(N)$), that is $g_n|N \to g|N$ and $g_n'|N \to g'|N$ uniformly on N. Since B is regular and $\sum_B = I$, there exists $u \in B$ such that $u \equiv 1$ on a neighborhood V in I of x and $u \equiv 0$ off N. Then $u' \equiv 0$ off N so that we can conclude that $ug_n \to ug$ and $(ug_n)' \to (ug)'$ uniformly on I, that is, $ug_n \to ug$ (in the norm on $D^1(I)$). But B closed means that $ug \in B$. Thus, g belongs locally to B at x since $u \equiv 1$ in some neighborhood of x, so that g belongs locally to B at each $x \in I\backslash S(B)$. Since $g \equiv 0$ on a neighborhood of $S(B)$, we have that g belongs locally to B at each point of $S(B)$. Combining the above assertions, we have that g belongs locally to B at all points of I. Since B is regular and $\sum_B = I, g \in B$. By the definition of $J(S(B))$ we conclude that $J(S(B)) \subseteq B$.

To show that $B = A_{S(B)}$, first note that $B \subset A_{S(B)}$. Since $J(S(B)) \subseteq B, B/J(S(B)) \subseteq A_{S(B)}/J(S(B))$; in fact, $B/J(S(B))$ is a closed subalgebra of $A_{S(B)}/J(S(B))$ with respect to the quotient norm on the latter. Furthermore, $\sum_{B/J(S(B))} = S(B) = \sum_{A_{S(B)}/J(S(B))}$ because both B and $A_{S(B)}$ are regular. This implies that every idempotent in $A_{S(B)}/J(S(B))$ is automatically in $B/J(S(B))$, so that $B/J(S(B))$ is dense in $A_{S(B)}/J(S(B))$ by Lemma 4.1. Recalling our earlier observation that $B/J(S(B))$ is closed in $A_{S(B)}/J(S(B))$, we can conclude that $B/J(S(B)) = A_{S(B)}/J(S(B))$. Thus, $B = A_{S(B)}$ since $J(S(B)) \subseteq B \subseteq A_{S(B)}$. □

From the above theorem, we can conclude that a closed separating subalgebra of $D^1(I)$ is always conjugate-closed.

Combining Proposition 1.3 and the theorem, we have

Corollary 4.3. *If B is a closed separating subalgebra of $D^1(I)$, then B is singly-generated (with a real generator). Furthermore, if $S(B) = \varnothing$, then $B = D^1(I)$.*

The next corollary extends the part of Proposition 1.3 which states that if f is one-to-one on I, then $A_f = A_{S_f}$. For $f_1, \dots, f_n \in D^1(I)$, let $A_{f_1,f_2,\dots,f_n}$ denote the closed subalgebra of $D^1(I)$ generated by $f_1,\dots,f_n$.

Corollary 4.4. *If $f_1,\dots,f_n \in D^1(I)$ separate the points of I, then $A_{f_1,\dots,f_n} = A_S, S = \cap_{i=1}^n S_{f_i}$. Thus, if $f_1',\dots,f_n'$ have no common zeros, then $A_{f_1,\dots,f_n} = D^1(I)$.*

We next give a characterization of $D^1(I)$ amongst the closed separating subalgebras of $D^1(I)$.

Corollary 4.5. *Suppose B is a closed separating subalgebra of $D^1(I)$. Then $B = D^1(I)$ if and only if, at each $x \in I$, there is a non-trivial continuous point derivation.*

Proof. The necessity of the condition is obvious. Thus, suppose that, for each $x \in I$, there exists a non-trivial continuous point derivation $\zeta_x : B \to \mathbf{C}$. Let $J_B(x)$ be the minimal closed ideal in B with hull $\{x\}$. If $g \in J_B(x)$ then $\zeta_x(g) = 0$, since $J_B(x) \subseteq \overline{M_x^2} \subseteq \zeta_x^{-1}(0)$, where M_x is the maximal ideal in B at $x \in I$. Thus, for any $x \in S(B), J(S(B)) \subseteq J_B(x) \subseteq \zeta_x^{-1}(0)$, so that $\tilde{\zeta}_x(g + J(S(B))) = \tilde{\zeta}_x(g)$ is a well-defined, continuous point derivation on $B/J(S(B))$. Since $B = A_{S(B)}, B/J(S(B))$ is generated by its idempotents. But $\tilde{\zeta}_x(u) = 0$ for all idempotents $u \in B/J(S(B))$ so that $\tilde{\zeta}_x \equiv 0$. Hence, $\zeta_x \equiv 0$ for each $x \in S_B$, a contradiction. Thus, $S(B)$ must be empty and, by Corollary 4.3, $B = D^1(I)$. □

5. Integral dependence over closed subalgebras. In this section, we are concerned with closed subalgebras B over which $D^1(I)$ is integral; that is, every $f \in D^1(I)$ satisfies a monic polynomial with coefficients in B. For example, if B has finite codimension in $D^1(I)$, then it is easily seen that $D^1(I)$ is integral over B. However, as Corollary 5.3 shows, this is not a necessary condition; indeed, there are closed countable subsets S of I such that $D^1(I)$ is integral over A_S. In this case, A_S has infinite codimension in $D^1(I)$ and A_S is separating on I. On the other hand, as the corollary also shows, $D^1(I)$ need not be integral over A_S even if the latter is separating. In general, for $D^1(I)$ to be integral over B it is necessary for B to be regular (see

[**5**; Theorem 4.3, p. 236]). The above remarks show that this is not a sufficient condition. A necessary and sufficient condition is given in the final result of this section. Let $S \subset I$ be a closed subset of I. We will denote the set of limit points of S by S'. For a non-negative integer n, the n^{th} derived set S_n of S is defined to be S'_{n-1} for $n \geq 2, S_1 = S'$ and $S_0 = S$.

Lemma 5.1. *Let S be a closed subset of $I = [a,b]$ and $n \geq 0$. If $g \in A_{S_{n+1}}$, then there exists $f \in A_S$ such that $f \equiv g$ on S_n.*

Proof. We first prove the lemma for $n = 0$. If $S = \{s_1, \dots, s_k\}$, then $S' = \varnothing$ and $A_{S'} = D^1(I)$. Let $g \in D^1(I)$. Since S finite implies A_S is separating, there exists $f_i \in A_S$ such that $f_i(s_j) = \delta_{ij}, i, j = 1, 2, \dots, k$. Then $f = \sum_{i=1}^k g(s_i) f_i$ is in A_S and $f \equiv g$ on S.

Suppose next that S is infinite, or, equivalently, that S' is not empty. Let $g \in A_{S'}$. It suffices to assume that g is real-valued since $A_{S'}$ is conjugate-closed. Let $I \backslash S = \cup I_n$, where I_n are the components of $I \backslash S$. It is easily seen that the I_n's are mutually disjoint and are either open intervals or half-open intervals. Let $\overline{I}_n = [a_n, b_n], n \geq 1$, and let E denote the union $\cup(\overline{I}_n \backslash I_n)$. Then $E \subset S$ and, hence, $E' \subseteq S'$. It is easily seen that there are functions $h_n \in C(I)$ such that for each $n \geq 1$,

(a) $h_n \equiv 0$ on $I \backslash I_n$;

(b) $\int_{a_n}^{b_n} h_n(t)dt = g(b_n) - g(a_n)$; and

(c) $\|h_n\|_\infty \leq 2\|g'\|_{\overline{I}_n}$, where $\|g'\|_{\overline{I}_n} = \max_{x \in \overline{I}_n} |g'(x)|$.

We will show that $\sum h_n$ converges uniformly on I. (Of course, if $I \backslash S$ has only finitely many components, then there is nothing to prove.) Let $m > n \geq 1$. Then

$$\|\sum_{i=n}^m h_i\|_\infty = \max_{n \leq i \leq m} \|h_i\|_\infty \leq 2 \max_{n \leq i \leq m} \|g'\|_{\overline{I}_i}.$$

Now, let $\epsilon > 0$ be given. Since g' is uniformly continuous on I, there exists $\delta > 0$ such that $t, s \in I, |t - s| < \delta$ imply that $|g'(t) - g'(s)| < \epsilon$. Since $\sum_{n=1}^\infty (b_n - a_n) \leq b - a$, there exists N such that $n \geq N$ implies $b_n - a_n < \delta$. Thus, for $n \geq N$, $\|g'\|_{\overline{I}_n} < |g'(a_n)| + \epsilon$, and it follows that, for $m > n \geq N$,

$$\|\sum_{i=n}^{m} h_i\|_\infty \leq 2 \max_{n\leq i\leq m} (|g'(a_i)| + \epsilon).$$

Now, $g'(a_i) \to 0$ as $i \to +\infty$. If not, then there is a subsequence $\{a_{i_k}\}$ of $\{a_i\}$ and a $\lambda > 0$ such that $|g'(a_{i_k})| \geq \lambda$ for $k \geq 1$. Without loss of generality, assume that $a_{i_k} \to c$. Since all the a_i's are distinct, $c \in E'$. Since $E' \subset S', c \in S'$ and therefore $g'(a_{i_k}) \to g'(c) = 0$, a contradiction. Thus, $|g'(a_i)| \to 0$ as $i \to \infty$. It now follows that there exists $N_1 > N$ such that $m > n \geq N_1$ implies $\| \sum_{i=n}^{m} h_i\|_\infty < 4\epsilon$. Thus, $\sum h_n$ converges uniformly, say to $h \in C(I)$. Now, let f be defined by $f(x) = g(a) + \int_a^x h(t)dt, x \in I$. Clearly $f \in A_S$.

To conclude the proof, we show that $f(s) = g(s)$ for $s \in S$. If $s = a$, then $f(a) = g(a)$ certainly holds. For $s > a$, let $J = [a, s]$, and $E_0 = J \cap (S\backslash S')$. Then $J = (J \cap S') \cup E_0 \cup (\cup_{b_n \leq s} I_n^0)$. Let μ be Lebesgue measure on I. Then $\mu(E_0) = 0$ since E_0 is at most countable. Since $h|S' \equiv g'|S' \equiv 0$,

$$\begin{aligned} g(s) - g(a) &= \int_a^s g'(t)dt = \int_J g' d\mu = \sum_{b_n\leq s} \int_{I_n^0} g' d\mu \\ &= \sum_{b_n\leq s} \int_{I_n^0} h d\mu = \int_J h d\mu = \int_a^s h(t)dt = f(s) - g(a). \end{aligned}$$

Thus, $g(s) = f(s)$ for all $s \in S$. The general statement now follows from the above by mathematical induction. □

If $f \in D^1(I)$ is integral over the subalgebra B, then $\mathcal{I}(f, B)$ will denote the smallest positive integer n for which there is a monic polynomial $\beta(x)$ of degree n with coefficients in B such that $\beta(f) = 0$. Let $\mathcal{I}(B) = \sup_{f\in D^1(I)} \mathcal{I}(f, B)$ when $D^1(I)$ is integral over B.

Theorem 5.2. *Suppose that S is a non-empty closed subset of I. Then the following hold:*

(a) *If $S_n = \varnothing$ for some $n \geq 1$, then $D^1(I)$ is integral over A_S and $\mathcal{I}(A_S) \leq n + 1$; and*

(b) *If* $f \in D^1(I)$ *is integral over* A_S *and* $S_f \cap S = \varnothing$, *then* $S_n = \varnothing$, *where* $n = \mathcal{I}(f, A_S) - 1$. *Moreover,* $D^1(I)$ *is integral over* A_S *and* $\mathcal{I}(A_S) = n + 1$.

Proof. (a). Let $f \in D^1(I)$. By Lemma 5.1, there exists $\alpha_0 \in A_S$ such that $g_0 = (n+1)!f + n!\alpha_0$ is identically zero on S_{n-1} since $S_n = \varnothing$. Set $h_0 = ((n+1)!/2!)f^2 + n!\alpha_0 f$. Then $h_0' = g_0 f' + n! f \alpha_0'$ is identically zero on S_{n-1} since $g_0 \equiv 0$ on S_{n-1} and $\alpha_0 \in A_S \subseteq A_{S_{n-1}}$. Hence, $h_0 \in A_{S_{n-1}}$.

Now, assume that there are functions $\alpha_0, \dots, \alpha_k \in A_S$ such that

$$g_k = \frac{(n+1)!}{(k+1)!} f^{k+1} + \frac{n!}{k!}\alpha_0 f^k + \cdots + (n-k)!\alpha_k$$

is identically zero on S_{n-k-1}. It follows that

$$h_k = \frac{(n+1)!}{(k+2)!} f^{k+2} + \frac{n!}{(k+1)!}\alpha_0 f^{k+1} + \cdots + (n-k)!\alpha_k f$$

belongs to $A_{S_{n-k-1}}$ since $g_k \equiv 0$ on S_{n-k-1} and $\alpha_0, \dots, \alpha_k \in A_S \subseteq A_{S_{n-k-1}}$ together imply that

$$h_k' = g_k f' + \frac{n! f^{k+1}}{(k+1)!}\alpha_0' + \cdots + (n-k)! f \alpha_k'$$

is identically zero on S_{n-k-1}. By Lemma 5.1, there exists $\alpha_{k+1} \in A_S$ such that $g_{k+1} = h_k + (n-k-1)!\alpha_{k+1}$ is identically zero on S_{n-k-2}. Hence, by mathematical induction, there are functions $\alpha_0, \dots, \alpha_{n-1} \in A_S$ such that $g_{n-1} = (n+1)f^n + n\alpha_0 f^{n-1} + \cdots + \alpha_{n-1}$ is identically zero on $S_{n-(n-1)-1} = S_0 = S$. Consequently, $\alpha_n = -f^{n+1} - \alpha_0 f^n - \cdots - \alpha_{n-1} f$ belongs to A_S since $\alpha_n' = -g_{n-1} f' - \alpha_0' f^n - \cdots - \alpha_{n-1}' f$. Thus, $D^1(I)$ is integral over A_S and $\mathcal{I}(A_S) \le n+1$.

(b). Suppose that $\beta(x) = x^{n+1} + \beta_n x^n + \cdots + \beta_0, \beta_i \in A_S, i = 0, \dots, n$, and that $f \in D^1(I)$ satisfies $\beta(f) \equiv 0$ and $S_f \cap S = \varnothing$. Assume that $n + 1 = \mathcal{I}(f, A_S)$. Let $\beta^{(j)}(x)$ denote the formal j^{th} derivative of $\beta(x)$. Since $\beta(f) \equiv 0$ on I, we have that $\beta(f)' = \beta'(f)f' + \sum_{i=0}^n \beta_i' f^i$ is identically zero on I. Then $\beta'(f) \equiv 0$ on S since $S_f \cap S = \varnothing$ and $\beta_i \in A_S, i = 1, 2, \dots, n$. Now, suppose $\beta^{(k)}(f) \equiv 0$ on S_{k-1}. Then $\beta^{(k)}(f)' = \beta^{(k+1)}(f)f' + \sum_{i=k}^n \frac{i!}{(i-k)!}\beta_i' f^{i-k}$ is identically zero on S_k

since $S_k = (S_{k-1})'$, and it follows that $\beta^{(k+1)}(f) \equiv 0$ on S_k. Thus, by induction, $\beta^{(n+1)}(f) \equiv 0$ on S_n. But $\beta^{(n+1)}(f) = (n+1)!$ so we can conclude that $S_n = \varnothing$. From part (a), we have that $D^1(I)$ is integral over A_S and that $\mathcal{I}(A_S) \le n+1$. But $n+1 = \mathcal{I}(f, A_S) \le \mathcal{I}(A_S) \le n+1$ so that $\mathcal{I}(A_S) = n+1$. □

Corollary 5.3. *Let S be a non-empty closed subset of I. Then the following are equivalent:*

(a) *$S_n = \varnothing$ for some $n \ge 1$;*

(b) *$D^1(I)$ is integral over A_S; and*

(c) *there are functions $f_1, \dots, f_m \in D^1(I)$ which are integral over A_S and $S \cap (\cap_{i=1}^m S_{f_i}) = \varnothing$.*

Proof. That (a) implies (b) follows from part (a) of the theorem. Now, suppose (b) holds. Then $f(t) = t, t \in I$, satisfies condition (c). Finally, suppose that (c) holds. For $s \in S$, there is a neighborhood N_s of s and an f_i such that f_i' is non-vanishing on N_s. Without loss of generality, we can assume that N_s is closed. Then f_i integral over A_S implies that f_i is integral over $A_{N_s \cap S}$ since $A_{N_s \cap S} \supset A_S$. Hence, by the theorem, there exists a positive integer $n(s)$ such that $(N_s \cap S)_{n(s)} = \varnothing$. Now, since S is compact, there are points $s_1, \dots, s_k \in S$ such that $\cup_{i=1}^k N_{s_i} \supset S$. Let $n = \max_{1 \le i \le k} n(s_i)$. Then $S_n = \cup_{i=1}^k (N_{s_i} \cap S)_n = \varnothing$. Thus, (c) implies (a). □

It follows from the corollary that if S is a closed subset of I such that $S_n \ne \varnothing$ for all $n \ge 1$, then $D^1(I)$ is not integral over A_S. If, in addition, $S^0 = \varnothing$, then A_S is also separating on I. Thus, A_S separating on I is not sufficient for $D^1(I)$ to be integral over A_S. On the other hand, $D^1(I)$ can be integral over A_S without A_S having finite codimension in $D^1(I)$. For example, if S has a unique limit point, then A_S has infinite codimension in $D^1(I)$. Since $S_2 = \varnothing, D^1(I)$ is integral over A_S and A_S is separating on I.

We conclude this section with a necessary and sufficient condition that $D^1(I)$ be integral over a closed subalgebra. **Proposition 5.4.** *Let B be a closed subalgebra of $D^1(I)$. Then $D^1(I)$ is integral over B if and only if $S(B)_n = \varnothing$ for some $n \ge 0$ and $A_{S(B)}$ is integral over B.*

Proof. If $D^1(I)$ is integral over B, then $D^1(I)$ is integral over $A_{S(B)}$ and $A_{S(B)}$ is integral over B. By Theorem 5.2, $S(B)_n = \varnothing$ for some $n \geq 0$. Conversely, if $S(B)_n = \varnothing$, then $D^1(I)$ is integral over $A_{S(B)}$, and if $A_{S(B)}$ is integral over B, then $D^1(I)$ is integral over B (see [**12**; Theorem 2, p. 256]). $\square$

The hypothesis that $A_{S(B)}$ be integral over B is satisfied if B has finite codimension in $A_{S(B)}$. But this condition is not necessary. For example, if B is the subalgebra of even functions in $D^1([-1,1])$, then $D^1(I)$ is integral over B but B does not have finite codimension in $A_{S(B)} = A_{\{0\}}$.

Appendix

We begin with an example of a function f such that f is not a topological divisor of zero in A_f and is not contained in H_{A_f}; that is, there is an $x \in I$ such that $f(x) = 0$. The function f is defined by $f(x) = e^{2\pi i x}, x \in [0,1]$, and $f(x) = 1/2 + 1/2e^{4\pi i x}, x \in [1, 3/2]$. Set $I = [0, 3/2]$. The range of f consists of two circles which are tangent at $z = 1 : f(I) = \{|z| = 1\} \cup \{|z - 1/2| = 1/2\}$. The carrier space $\sum_{A_f}$ of A_f is precisely $\Delta = \{|z| \leq 1\}$ and $\partial A = \{|z| = 1\}$. Now, note that $f(5/4) = 0$ so that $0 \in f(I)$. Thus f is not in H_{A_f}. On the other hand, f is not a topological divisor of zero in A_f. Suppose $g_k \in A_f$ and $\|g_k f\|_1 = \|g_k f\|_\infty + \|g_k' f + g_k f'\|_\infty \to 0$. Then $\|g_k f\|_\infty \to 0$. But $\|g_k f\|_\infty = \|\widehat{g}_k \widehat{f}\|_\Delta = \|\widehat{g}_k \widehat{f}\|_{\partial A}$. Since $|\widehat{f}| \equiv 1$ on ∂A, we have that $\|\widehat{g}_k\|_{\partial A} = \|g_k\|_\infty \to 0$. Thus, since $\|g_k' f + g_k f'\|_\infty \to 0$, we have that $\|g_k' f\|_\infty \to 0$. Now, without loss of generality, assume $g_k = p_k(f)$, a polynomial in f. Then $\|p_k'(f) f' f\|_\infty \to 0$. Since $f'(x) \neq 0$ for all $x \in I$, we have $\|p_k'(f) f\|_\infty \to 0$. Repeating the argument which showed $\|g_k\|_\infty \to 0$, one can show that $\|p_k'(f)\|_\infty \to 0$. Thus, $\|g_k\|_1 \to 0$ and, hence, f is not a topological divisor of zero in A_f. It is one, however, in $D^1(I)$. We conclude this appendix with an example of a regular closed subalgebra of $D^1(I)$ which is not finitely-generated. In view of Corollary 4.4 and of Corollary 2.5 combined with Corollary 3.6, we might be tempted to conclude that regular closed subalgebras of $D^1(I)$ are necessarily finitely-generated. That this is not the case is shown by the following example. Let R be the equivalence relation on $I = [0,1]$ defined as follows: xRy if and only if $x = y$

or $x, y \in \{1/2^n : n \geq 1\} \cup \{0\}$. Then A^R is regular by Corollary 3.6. To show that A^R is not finitely-generated, we use a result of D. Sherbert (see [**7**; Proposition 8.3, p. 261]) which states that if A is a semi-simple commutative Banach algebra with n generators, then the linear space of continuous point derivations at any $\psi \in \sum_A$ must have dimension at most n. For our example, let $d_n(f) = f'(1/2^n), n \geq 1$. Then $d_n : A^R \to \mathbf{C}$ is a continuous point derivation at $\psi \in \sum_{A^R}$, where $\psi(f) = f(0), f \in A^R$. It is easily shown that the D_n's are linearly independent (over $\mathbf{C}$) so that, by Sherbert's result, A^R cannot be finitely-generated.

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