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ON THE VORTEX SOLUTIONS OF SOME NONLINEAR SCALAR FIELD EQUATIONS*

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1. Introduction. Complex scalar nonlinear evolution equations have been used to model the dynamics of quantum structures. An example is the dynamics of quantum vortices in the theory of superfluids [**2, 1**]. In [**3**] a study is made of the effective dynamics of interacting vortices given nonlinear Schroedinger (NLS), Klein-Gordon (NLKG), and heat equations (NLH) as exact dynamics. The equations of evolution are

$$\begin{aligned} -i\psi_t &= \Delta\psi + (1-|\psi|^2)\psi && \text{(NLS)}\\ \psi_{tt} &= \Delta\psi + (1-|\psi|^2)\psi && \text{(NLKG)}\\ \psi_t &= \Delta\psi + (1-|\psi|^2)\psi && \text{(NLH)} \end{aligned}$$

where $\psi : \mathbf{R}^2 \times \mathbf{R}_+ \to \mathbf{C}$.

Vortex solutions of these equations are obtained in the form:

$$\psi_n(r,\theta) = U_n(r)e^{in\theta} \tag{1}$$

where (r,θ) denotes polar coordinates in $\mathbf{R}^2$, and $U_n(r)$ satisfies:

$$-\Delta_r U + n^2/r^2 U - (1-|U|^2)U = 0 \tag{2}$$

$$U(0) = 0 \quad \text{and} \quad U(r) \to 1 \quad \text{as } r \to \infty. \tag{3}$$

Here, $\Delta_r = \partial_{rr} + 1/r\partial_r$ is the radial Laplacian in $\mathbf{R}^2$. It can be shown [**3**] that a solution of (2)–(3) has the asymptotic behavior

$$U_n(r) \approx ar^n[1 - r^2/4(n+1)] \quad \text{as } r \to 0 \tag{4a}$$

$$1 - n^2/2r^2 \quad \text{as } r \to \infty. \tag{4b}$$

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The solutions $\psi_n(r,\theta)$ for $|n| \geq 1$ are vortices in the sense that the vector field defined by its real and imaginary parts can be thought of as a flow field with *circulation* about the zero of $\psi = \psi_n$ given by $2\pi n = \int \nabla\varphi \cdot dl$, where $\varphi = \theta$ denotes the phase of ψ. A construction of the functions $U_n(r)$ is presented in [**3**]. The approach is to convert (2) to an equivalent integral equation and to solve by iteration. While this technique constructs the isolated n-vortex ψ_n, it does not give information which can be used toward a stability analysis of the solution $\psi_n(r,\theta)$ relative to the underlying dynamics. The dynamic stability question is fundamental, for the effective dynamics are viewed as that of a set of isolated yet interacting vortices in the limit of small ratio of vortex core size to inter-vortex distance [**1,3**]. Our goal here is to present a *variational* characterization of these vortex solutions as *minimizers* of a natural *relative* energy of the system. A relative energy functional is employed due to the infinite energy of the vortex solutions (logarithmic divergence). This relative energy functional is natural from the dynamic standpoint and provides some information about the stability properties of the n-vortex.

2. A family of approximate problems. We fix n and let U^R solve

$$-\Delta_r U^R + n^2/r^2 U^R - (1-|U^R|^2)U^R = 0 \tag{5R}$$

$$U^R(0) = 0 \quad \text{and} \quad U^R(R) = 1. \tag{6R}$$

We shall construct a solution of (5R)–(6R) by solving a minimization problem. Let

$$j(f) = \frac{1}{2}\left(|f_r|^2 + \frac{n^2}{r^2}|f|^2\right) + \frac{1}{4}(1-|f|^2)^2 \tag{7}$$

and define

$$J^R[f] = \int_0^R j(f) r\,dr. \tag{8}$$

Clearly, $J^R(f) \geq 0$.

Proposition 1. Min$\{J^R[f] : f(0) = 0, f(R) = 1\}$ *is attained at a* C^∞ *function* U^R *satisfying* (5R)–(6R).

Proof. Let $\{f_j\}$ denote a minimizing sequence. Then,

$$\int_0^R [|\partial_r f_j|^2 + n^2 r^{-2}|f_j|^2 r\,dr \le C_R \tag{9}$$

and we have that $\{f_j\}$ is bounded in H^1. Therefore, there is a subsequence with a weak H^1 limit, and which converges strongly in L^p, for any $p > 2$. We denote this limit by U^R. The foregoing and the weak lower semi-continuity of the first two terms of J^R imply that $J^R[f_*] = \inf J^R[f]$. Computing the Euler-Lagrange equation of J^R about the minimizer f_* gives (5R)–(6R). That $U^R \in C^\infty$ follows from elliptic regularity techniques. □

To pass to the limit as $R \to \infty$, we require á priori estimates on the sequence $\{U^R(\cdot, n)\}$, which are independent of R.

Proposition 2. *Let* $U^R(r)$ *solve* (5R)–(6R). *Then,*

$$\partial_r U^R(r) > 0 \quad \text{for } r \in (0,R) \quad \text{and} \quad 0 < U^R(r) < 1 \tag{10a}$$

(10b)
$$r^2 U^R(r)^2 + \int_0^r [1 - U^R(s)^2]^2 s\,ds = n^2 U^R(r)^2 + r^2[1 - U^R(r)^2]^2$$

$$[\partial_r U^R(R)]^2 \le \frac{n^2}{R^2} \tag{10c}$$

$$\int_0^R [1 - U^R(s)^2]^2 s\,ds \le n^2 \tag{10d}$$

$$[1 - U^R(r)^2]^2 \le \frac{2n^2}{r}. \tag{10e}$$

Proof. (10a): Let w denote $\partial_r U^R$. Then w satisfies the equation $(L^R + r^{-2})w = 2n^2 r^{-3} U^R$, where $L^R = -\Delta - 1 + 3(U^R)^2 + n^2 r^{-2}$ is a nonnegative operator, being the second variation of J^R at its minimum. We claim that w is positive since the right hand side of the equation for w is pointwise positive. To see this, observe that $\exp(-Ht)$, where $H = L^R + r^{-2}$, is a positivity preserving semigroup and that $H^{-1} = \int_0^\infty \exp(-Ht)\,dt$ is then also positivity preserving (see [**4**]).

(10b): Multiply (5R) by $r^2\partial_r U$, integrate from r to s, and then integrate by parts.

(10c), (10d): Set $r = R$ in (10b).

(10e):

$$
\begin{aligned}
[1 - U^R(r)^2]^2 &= -\int_r^R [1 - U^R(s)^2] U(s)\partial_s U(s)\, ds \\
&\le \left[\int_r^R s[1 - U^2(s)]^2\, ds\right]^{1/2} \\
&\quad \cdot \left[\int_r^R s^{-3} U^2(s)\{n^2 U^2(s) + s^2(1 - U^2(s))^2/2\}\, ds\right]^{1/2} \\
&\le n\left[n^2 \int_r^R s^{-3} ds + r^{-2} \int_r^R (1 - U^2(s))^2 s\, ds/2\right]^{1/2} \\
&\le n^2/r.
\end{aligned}
$$

To get the first inequality we used (10b) and to get the second we used (10a) and (10d). This completes the proof of Proposition 2. □

Using the estimates (10), we can pass to the limit in the weak form of (2)–(3) to obtain a weak solution of (2)–(3). Elliptic theory can then be used to conclude that the limiting solution, $U(\cdot, n)$, is a classical solution.

We also have uniqueness by the following argument communicated to me by P.L. Lions. Let $\theta > 1$, and define $g = \theta U$, where U solves (2)–(3). Let v be any other solution of (2)–(3) and define $d = g - v$. Since $\theta > 1$, we have that for r larger than some R_0, $d > 0$. We now show that $d \ge 0$ for $r \le R_0$.

First, note that d satisfies the equation $-\Delta d - d + n^2/r^2 d + g^3 - v^3 = (1 - \theta^{-2})g^3$. By the convexity of the function $f(s) = s^3$, $g^3 - v^3 \le 3g^2(g - v)$, and, therefore, $L(\theta)d \ge (1 - \theta^{-2})g^3 > 0$. Here, $L(\theta) = -\Delta - 1 + 3\theta^2 U^2 + n^2 r^{-2} \ge L(1)$, since $\theta > 1$. Now $L(1)$ is the second variation of the functional $\mathcal{E}_n[\cdot]$ (see section 3) for which $U(\cdot, n)$ is a minimizer and, therefore, $L(1) \ge 0$ and $L(\theta) > 0$ for $\theta > 1$. It follows that $d \ge 0$ or $g \ge v$. Letting $\theta \to 1$, we have $U(\cdot, n) \ge v$. Similarly, by interchanging the roles of U and v we get $v \ge U$, implying $U = v$.

3. $\mathbf{U_n(\cdot, n)}$ as a minimizer of the relative energy. Let $e(\psi) = 1/2|\nabla\psi|^2 + 1/4\ (1 - |\psi|^2)^2$. It is easily checked formally that $\int e(\psi)\, d^2x$ is a conserved integral for NLS, is decreasing for NLH and that $\int |\psi_t|^2 + e(\psi)\, d^2x$ is conserved for NLKG.

A possible approach to the stability of the n-vortex relative to some underlying dynamics is the Lyapunov method (see, for example, [**5, 7**] for an application to the stability of solitary waves of NLS). This method attempts to exploit the property of an equilibrium being a (possibly constrained) minimum of a suitable energy functional. An immediate difficulty is that $\psi_n(r, n)$ has infinite energy due to a logarithmic divergence. This can be seen using the asymptotic relation (4b). The situation can be remedied by introducing a *relative* energy functional [**3**]:

$$\mathcal{E}_n[f] = \int [e(f) - e(U(\cdot, n))]r\, dr. \tag{11}$$

Theorem 2. $\mathrm{Min}\{\mathcal{E}_n[f] : f(0) = 0,\ f(\infty)e^{-in\theta} = 1\} = 0$, *and the minimum is attained at* ψ_n.

Outline of the Proof. First by the analysis of Section 2, $\mathcal{E}_n[g] = \int_{|x|<R}[e(g) - e(U(\cdot, n))]r\, dr \geq 0$. The result can be obtained introducing for each admissible f a sequence $\{f^R\}$, where $f^R(0) = 0$, $f^R(r) = 1$ for $r \geq R$ and $f^R \to f$ as $R \to \infty$. $e(f^R)$ is then added and subtracted in the integrand of (11). The result is then obtained by breaking up the integral into integrals over different ranges of r and then using the estimates (10) to pass to the limit. $\square$

4. Remarks on well-posedness and stability. Here we discuss some questions concerning the dynamics of NLS with the boundary condition $\psi \to \exp(in\theta)$ as $|x| \to \infty$ which appear to be open.

(1) Although it is straightforward to prove a local existence result for NLS in some space with large $|x|$ asymptotics given by $e^{in\theta}$, continuing to a global solution is nontrivial. For example, in a neighborhood of a solution ψ_n, it is natural to use the *relative* energy functional to obtain a priori bounds. Since it merely defines a semi-norm, it appears that more information is required to obtain a global solution.

(2) The variational analysis of sections 3 and 4 suggests some sort of stability or metastability relative to data of the form $\psi(r,t=0) = (U(r,n)+\varphi_0(r))\exp(in\theta)$, where φ_0 is small. In this case, the linearized system for $\varphi(r,t) = u+iv$ is $\partial_t w = JLw$, where $w = (u,v)^t$, J is the unit symplectic matrix and L is the diagonal and self-adjoint matrix operator with diagonal entries l_{11} and l_{22} having the large r behavior: $l_{11} \sim -\Delta$ and $l_{22} \sim \Delta - 2$. Therefore, the continuous spectrum of L is the entire imaginary axis (an application of Weyl's theorem on essential spectrum). This, together with our variational analysis, suggests that $\psi_n(r,\theta)$ may not be nonlinearly stable and that the nature of the instability may be of a slow sub-exponential type. This is in contrast to the study of solitary wave stability, where zero is an isolated point in the spectrum of JL and possible secular growth could be eliminated by *modulation of symmetries* [**5, 6, 7, 8**].

5. Another question of interest concerns the stability or metastability properties of ψ_n, relative to *arbitrary* initial perturbations which preserve the boundary condition at ∞. Are the $|n| > 1$ vortices unstable?

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