ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 2, Spring 1991

LIMITING BEHAVIOR OF SOLUTIONS OF $\mathbf{u_t = \Delta u^m}$ as $\mathbf{m \to \infty}$

PAUL E. SACKS

Consider the Cauchy problem for the porous medium equation

$$u_t = \Delta(|u|^{m-1}u) \qquad x \in \mathbf{R}^N, \quad t > 0 \tag{0.1}$$

$$u(x,0) = f(x) \qquad x \in \mathbf{R}^N. \tag{0.2}$$

We are interested in the behavior of the solution u as $m \to \infty$, for a fixed initial function f. Some study of this question was first carried out by Elliott, Herrero, King and Ockendon [**4**].

Under various conditions on f we will see that for fixed $t > 0$

$$u_m(\cdot, t) \to u_\infty \quad \text{as } m \to \infty \tag{0.3}$$

where u_m denotes the solution of (0.1)–(0.2), and $u_\infty = u_\infty(x)$ satisfies the "differential inclusion"

$$u_\infty - \Delta\varphi_\infty(u_\infty) \ni f. \tag{0.4}$$

Here φ_∞ is the maximal monotone graph

$$\varphi_\infty(s) = \begin{cases} 0, & |s| < 1 \\ \pm[0,\infty), & s = \pm 1 \\ \varnothing, & |s| > 1 \end{cases} \tag{0.5}$$

and the meaning of (0.4) is that there exists a function $w = w(x)$ such that

$$w(x) \in \varphi_\infty(u_\infty(x)) \quad \text{a.e.} \quad \text{and} \quad u_\infty - \Delta w = f \quad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{0.6}$$

Received by the editors on March 28, 1988, and in revised form on May 29, 1989.
This research was supported by the Air Force Office of Scientific Research under Grant No. AFOSR 88-0031. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation therein.

The existence of a solution u_∞ of (0.4), for any $f \in L^1(\mathbf{R}^N)$, is demonstrated in [**1**].

Formally, a solution of (0.4) should satisfy

$$\begin{cases} u_\infty = f & |u_\infty| < 1 \\ u_\infty = \pm 1 & \text{otherwise.} \end{cases} \tag{0.7}$$

Because of this characteristic shape, the term "mesa problem" was used in [**4**]. If $f \geq f_0 > -1$, the problem (0.4) is actually equivalent to an obstacle problem on $\mathbf{R}^N$, namely, if

$$\Delta \psi = f - 1 \qquad x \in \mathbf{R}^N,$$

then $v = w + \psi$ satisfies

$$v \geq \psi, \qquad -\Delta v \geq 0, \qquad (v - \psi)\Delta v = 0 \tag{0.8}$$

which is the complementary form of the obstacle problem [**8**, p. 79]. The set $\{x \in \mathbf{R}^N : u_\infty(x) = 1\}$ (the collection of mesas) is the same as the noncoincidence set $\{x \in \mathbf{R}^N : v(x) > \psi(x)\}$ for the obstacle problem. Thus, there is a large literature which may be consulted concerning the regularity of u_∞, w and the free boundary $\partial\{u_\infty = 1\}$.

1. In this section we describe some formal calculations leading to the convergence result (0.3). Precise theorems will be given in the next section.

For $f \in L^1(\mathbf{R}^N)$ and $m > 0$ a solution of (0.1)–(0.2) may be obtained via nonlinear semigroup theory [**5**]. We define a nonlinear operator

$$A_m : D(A_m) \subset L^1(\mathbf{R}^N) \to L^1(\mathbf{R}^N) \tag{1.1}$$

by the formula

$$A_m u = -\Delta(|u|^{m-1}u) \qquad u \in D(A_m). \tag{1.2}$$

The exact definition of A_m is given in [**2**] based on the results in [**1**]. The operator A_m is m-accretive in $L^1(\mathbf{R}^N)$, that is, $(I + A_m)^{-1}$ is defined and nonexpansive on $L^1(\mathbf{R}^N)$. By the Crandall-Liggett theorem, the limit

$$S_m(t)f = \lim_{n\to\infty} \left(I + \frac{t}{n} A_m \right)^{-n} f \tag{1.3}$$

exists for $f \in L^1(\mathbf{R}^N)$, $u_m(x,t) = (S_m(t)f)(x)$ belongs to $C([0,T]; L^1(\mathbf{R}^N))$ for any $T < \infty$ and u_m is the weak solution of (0.1)–(0.2).

Now as $m \to \infty$ the nonlinearities $|u|^{m-1}u$ converge in the sense of graphs to the maximal monotone graph φ_∞ defined in (0.5). According to the general continuous dependence result [**2**], it follows that $A_m \to A_\infty$ in the sense of m-accretive operators, where $A_\infty u =$ "$-\Delta\varphi_\infty(u)$" is a multivalued m-accretive operator on $L^1(\mathbf{R}^N)$, defined in [**2**]. The precise meaning of this statement is that for $\lambda > 0$ and $f \in L^1(\mathbf{R}^N)$,

$$(I + \lambda A_m)^{-1} f \to (I + \lambda A_\infty)^{-1} f \quad \text{in } L^1(\mathbf{R}^N) \tag{1.4}$$

and $v = (I + \lambda A_\infty)^{-1} f$ means that there exists $w = w(x)$ such that

$$w(x) \in \varphi_\infty(v(x)) \quad \text{a.e.} \quad \text{and} \quad v - \lambda\Delta w = f \quad \text{in } \mathcal{D}'(\mathbf{R}^N) \tag{1.5}$$

(i.e. (0.4) holds with u_∞ replaced by v and φ_∞ replaced by $\lambda\varphi_\infty$).

With all this in mind, we can make the following heuristic argument for the convergence result (0.3). By an exchange of limits, we may expect that u_m converges to u^* where

$$\begin{aligned} u^*(\cdot,t) &= \lim_{n\to\infty} \lim_{m\to\infty} \left(I + \frac{t}{n} A_m\right)^{-n} f \\ &= \lim_{n\to\infty} \left(I + \frac{t}{n} A_\infty\right)^{-n} f \end{aligned} \tag{1.6}$$

provided this makes sense. Of course, it is not yet clear why u^* should be defined, or why the exchange of limits should be legitimate.

Let us check here that u^* is well defined and, in fact, $u^*(\cdot,t) = u_\infty$ defined in (0.4). First observe that $\lambda\varphi_\infty = \varphi_\infty$ for any $\lambda > 0$ and so $(I + \frac{t}{n}A_\infty)^{-n} f = (I + A_\infty)^{-n} f$ for any t, $n \geq 0$. Next, if $|f| \leq 1$, then $(I + A_\infty)^{-1} f = f$, and, furthermore, for any $f \in L^1(\mathbf{R}^N)$, if $v = (I + A_\infty)^{-1} f$, then $v \in D(A_\infty)$, so $|v(x)| \leq 1$. Hence, $(I + \frac{t}{n}A_\infty)^{-n} f = (I + A_\infty)^{-1} f$ for any n, t, which means that $u^*(\cdot,t)$ is defined and $u^*(\cdot,t) = u_\infty$ for any $t > 0$.

2. The argument of the previous section leads us to conjecture that (0.3) holds for any $f \in L^1(\mathbf{R}^N)$. We know of no counterexamples to this, and we now describe conditions on f for which (0.3) has been proven. In each case the convergence takes place at least in $C([t_0,T];L^1(\mathbf{R}^N))$ for any $0 < t_0 < T < \infty$. In general, $u_m(x,0) = f(x) \neq u_\infty(x)$ so we cannot let $t_0 = 0$. If $f \in L^p(\mathbf{R}^N)$, then $u_m \to u_\infty$ in $L^p((0,T) \times \mathbf{R}^N)$ for any $T < \infty$ and $p < \infty$. Since u_∞ will be discontinuous in general (see examples below) we cannot expect convergence in L^∞, no matter how smooth f is.

Under any of the following conditions on f, the convergence result (0.3) is valid.

(i) (Special case of results in [**2**]) $f \in L^1(\mathbf{R}^N)$, $||f||_{L^\infty(\mathbf{R}^N)} \leq 1$. In this case $u_\infty = f \in \overline{D(A_\infty)}$ so this is a regular perturbation problem. The convergence will take place in $C([0,T];L^1(\mathbf{R}^N))$. If $||f||_{L^\infty(\mathbf{R}^N)} < 1$, then (0.3) is especially simple to prove, see, e.g., [**3**], Section 3.

(ii) (See [**4**]) $f(x) = M\delta(x-x_0)$, a Dirac delta function. In this case it is not hard to show that the solution of (0.4) exists and is given by

$$u_\infty = \begin{cases} 1, & |x-x_0| < \left(\frac{M}{\omega_N}\right)^{1/N} \\ 0, & |x-x_0| > \left(\frac{M}{\omega_N}\right)^{1/N} \end{cases} \tag{2.1}$$

($\omega_N =$ volume of unit ball in $\mathbf{R}^N$). On the other hand, the solution of (0.1)–(0.2) is the well-known Barenblatt-Pattle solution,

$$u_m(x,t) = \frac{1}{t^k}\left(a^2 - \frac{C|x-x_0|^2}{t^{2k}}\right)_+^{\frac{1}{m-1}} \tag{2.2}$$

with $k = (m-1+2/N)^{-1}$, $C = k(m-1)/2m$, and a is chosen so that $\int_{\mathbf{R}^N} u_m(x,t)\,dx = M$ for $t > 0$ (which is possible because $\int_{\mathbf{R}^N} u_m(x,t)\,dx$ is independent of time). Thus, by direct calculation one can verify that $u_m(x,t) \to u_\infty(x)$. The sphere $\left\{x : |x-x_0| = (M/\omega_N)^{1/N}\right\}$ represents the limiting position of the free boundary for the solution (2.2) as $m \to \infty$.

(iii) [**3**] $f \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $f(x) \geq 0$, $f \in C^1(\operatorname{supp} f)$, $f_r < 0$ in $\mathbf{R}^N \backslash \{0\} \cap \operatorname{supp} f$, $f_{r_{x_0}} \leq 0$ in $\mathbf{R}^N \backslash B(0,1) \cap \operatorname{supp} f$ for all $x_0 \in B(0,\varepsilon_0)$, some $\varepsilon_0 > 0$, where $r_{x_0} = |x-x_0|$ and $B(0,r) = \{x : |x| < r\}$.

(iv) [**10**] $f \in L^1(\mathbf{R}^N)$, $f(x) \geq 0$, $f(x) = f(|x|)$.

(v) [**10**] $N = 1$, $f \in L^1(\mathbf{R})$, $f(x) \geq 0$.

Before [**10**], it was also proved in [**3**] that (0.3) holds in the case $N = 1$, $f \in L^1(\mathbf{R})$, $f(x) \geq 0$, $f'(x)$ piecewise continuous on $\mathbf{R}$, $f(x)$ changes monotonicity k times and the set $\{f > 1\}$ consists of k disjoint intervals.

We conclude this section by mentioning a class of examples for which the limit can be computed explicitly. Suppose $f \in L^1(\mathbf{R}^N)$, $f(x) \geq 0$, $f(x) = f(|x|)$, $f(0) > 1$ and $f_r \leq 0$. Then there exists a unique $R \in (0, \infty)$ such that

$$\frac{1}{\omega_N R^N} \int_{|x|<R} f(x)\,dx = 1. \tag{2.3}$$

It is not hard to check that $\int u_\infty\,dx = \int f\,dx$; hence, using the fact that u_∞ must be radially symmetric and decreasing we find that

$$u_\infty(x) = \begin{cases} 1 & |x| < R \\ f(x) & |x| > R. \end{cases} \tag{2.4}$$

With these conditions on f, we may conclude from case (iv) above that $u_m(\cdot, t) \to u_\infty$ as $m \to \infty$.

3. In this section we describe some elements of the proof in case (v) above; see [**10**] for complete details. Some of the arguments in the proof are adapted from those in [**3**].

To begin with, we have the standard L^1 estimate for the Cauchy problem (0.1)–(0.2) ([**2,5**])

$$||u_m(\cdot, t) - \hat{u}_m(\cdot, t)||_{L^1(\mathbf{R})} \leq ||f - \hat{f}||_{L^1(\mathbf{R})} \tag{3.1}$$

where $\hat{u}_m$ denotes the solution with initial value $\hat{f}$. Similarly for the solutions of (0.4) we have [**1**]

$$||u_\infty - \hat{u}_\infty||_{L^1(\mathbf{R})} \leq ||f - \hat{f}||_{L^1(\mathbf{R})}. \tag{3.2}$$

In particular, it follows that we need only prove the convergence on a dense subset of $L^1(\mathbf{R})$; assume, therefore, that $f \in C_0^\infty(\mathbf{R})$.

Now fix $T > 0$. From (3.1) we get

$$\int_{\mathbf{R}} |u_m(x,t)|\,dx \le \int_{\mathbf{R}} |f(x)|\,dx \tag{3.3}$$

and

$$\int_{\mathbf{R}} |u_m(x+h,t) - u_m(x,t)|\,dx \le \int_{\mathbf{R}} |f(x+h) - f(x)|\,dx \tag{3.4}$$

from which it follows that $\{u_m(\cdot,T)\}$ is precompact in $L^1_{\text{loc}}(\mathbf{R})$. We can also show that $\operatorname{supp} u_m(\cdot,T)$ is bounded independently of m, hence $\{u_m(\cdot,T)\}$ is precompact in $L^1(\mathbf{R})$.

The next claim is that for any $\delta > 0$ there exists m_0 such that for $m \ge m_0$

$$0 \le u_m(x,T) \le 1 + \delta \qquad x \in \mathbf{R}. \tag{3.5}$$

This may be proved in several ways, the easiest perhaps being a comparison argument using the explicit solution (2.2) with t replaced by $t + t_0$, t_0 suitably chosen. Thus, we can find a subsequence $m_k \to \infty$ and a limit function u_T^* such that $0 \le u_T^*(x) \le 1$ and $u_{m_k}(\cdot,T) \to u_T^*$ in $L^1(\mathbf{R})$.

By the continuous dependence result [**2**] for $t > T$

$$u_{m_k}(\cdot,t) = S_{m_k}(t-T)u_{m_k}(\cdot,T) \to S_\infty(t-T)u_T^* = u_T^* \tag{3.6}$$

where S_∞ is the semigroup generated by A_∞. Now by a diagonalization argument, there is a further subsequence, again denoted m_k, and a function u^*, $|u^*(x)| \le 1$ such that

$$u_{m_k}(\cdot,t) \to u^* \quad \text{in} \quad C([t_0,T];L^1(\mathbf{R})) \tag{3.7}$$

for any $0 < t_0 < T < \infty$. The proof will be completed once we show that $u^* = u_\infty$, i.e., that u^* satisfies (0.4).

Integration of (0.1) with respect to t from 0 to 1 gives

$$u_{m_k}(\cdot,1) - w''_{m_k} = f \quad \text{in} \quad \mathcal{D}'(\mathbf{R}) \tag{3.8}$$

where $w_{m_k} = \int_0^1 u_{m_k}^{m_k}(x,s)\, ds$. For a further such subsequence $m_k \to \infty$ we may obtain a limit function $w^* = w^*(x)$ that $w_{m_k} \to w^*$, and so from (2.8),

$$u^* - (w^*)'' = f \quad \text{in} \quad \mathcal{D}'(\mathbf{R}). \tag{3.9}$$

The conclusion $u^* = u^\infty$ follows provided we show that $w^* \in \varphi_\infty(u^*)$ a.e., that is to say, $w^*(x) = 0$ a.e. on the set where $u^*(x) < 1$.

To prove this last property we show that for a.e. such x,(i)$u_{m_k}^{m_k}(x,s) \to 0$ for all $s \in (0,1]$ and (ii) there is a uniform bound for $u_{m_k}^{m_k}(x,s)$. The conclusion that $w^*(x) = 0$ then follows from the definition of w_{m_k} and the dominated convergence theorem. We remark that it is only in this last step that the one dimensionality is used in an essential way.

4. To conclude we describe some generalizations of the results which have been described here.

(i) The space $L^1(\mathbf{R})$ can be replaced by $\mathcal{M}^+(\mathbf{R})$, the space of nonnegative finite Radon measures. See [**9**] for an existence theory for (0.1)–(0.2) in this case (for $m > 1$) and [**11**] for study of (0.4) when $f \in \mathcal{M}^+(\mathbf{R})$.

(ii) [**7**] the nonlinearities $|u|^{m-1}u$ can be replaced by a more general sequence of functions converging to φ_∞ in the sense of graphs. In particular, the convergence results discussed in Section 2 all remain valid with the same proofs, if $|u|^{m-1}u$ is replaced by $(|u|^{m-1}u)/m$, which is the case actually considered in [**4**].

(iii) [**6**] Limits of the type $\lim_{m\to\infty} u_m(\cdot, t_m)$ can be studied where $t_m \to 0$ or $t_m \to \infty$ as $m \to \infty$.

(iv) Analogous problems in bounded domains can be considered, say with Dirichlet boundary conditions. This is done in [**10**] for the case when the domain is an interval in $\mathbf{R}$.

(v) Study of the analogous hyperbolic problem in which Δu^m is replaced by $(u^m)_x$ has been carried out [**12**].

Note added in proof. Since this article was written, the convergence result (0.3) has been proved for all nonnegative $f \in L^1(\mathbf{R}^n)$. See *On the limit of solutions of* $u_t = \Delta u^m$ *as* $m \to \infty$ by P. Bénilan,

L. Boccardo and M. Herrero, in *Some topics in nonlinear P.D.E.'s*, Proceedings Int. Conf. Torino 1989 (M. Bertsch, ed.).

REFERENCES

1. Ph. Benilan, H. Brezis and M.G. Crandall, *A semilinear elliptic equation in* $L^1(\mathbf{R}^N)$, Ann. Scuola Norm. Sup. Pisa, Serie **V** (1979, vol 2), 523–555.

2. Ph. Benilan and M.G. Crandall, *The continuous dependence on* φ *of solutions of* $u_t - \Delta\varphi(u) = 0$, Indiana Univ. Math. J. **30** (1981), 161–177.

3. L.A. Caffarelli and A. Friedman, *Asymptotic behavior of solutions of* $u_t = \Delta u^m$ *as* $m \to \infty$, Indiana Univ. Math. J. **36** (1987), 711–728.

4. C.M. Elliott, M. Herrero, J.R. King and J.R. Ockendon, *The mesa problem: Diffusion patterns for* $u_t = \nabla \cdot (u^m \nabla u)$ *as* $m \to \infty$, IMA J. Appl. Math. **37** (1986), 147–154.

5. L.C. Evans, *Applications of nonlinear semigroup theory to certain partial differential equations*, in *Nonlinear evolution equations* (M.G. Crandall, ed.), Academic Press, New York, 1978.

6. A. Friedman and K. Höllig, *On the mesa problem*, J. Math. Anal. Appl. **123** (1987), 564–571.

7. A. Friedman and S. Huang, *Asymptotic behavior of solutions of* $u_t = \Delta\varphi_m(u)$ *as* $m \to \infty$ *with inconsistent initial values*, Analyse Mathematique et applications, Gauthier-Villars, Paris, 1988.

8. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

9. M. Pierre, *Uniqueness of the solution of* $u_t - \Delta\varphi(u) = 0$ *with initial datum a measure*, Nonlinear Analysis, T.M.A. **6** (1982), 175–187.

10. P. Sacks, *A singular limit problem for the porous medium equation*, J. Math. Anal. Appl. **140** (**2**) (1989), 456–466.

11. J.L. Vazquez, *On a semilinear equation in* $\mathbf{R}^2$ *involving bounded measures*, Proc. Roy. Soc. Edinburgh Sect. A **95** (1983), 181–202.

12. X. Xu, *Asymptotic behavior of solutions of hyperbolic conservation laws* $u_t + (u^m)_x = 0$ *as* $m \to \infty$ *with inconsistent initial values*, Proc. Roy. Soc. Edinburgh Sect. A **113** (1989), 61–71.

Department of Mathematics, Iowa State University, Ames, Iowa 50011