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# ALMOST PERIODICITY AND DEGENERATE PARABOLIC EQUATIONS

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**1. Introduction and results.** In the present paper we are interested in the almost periodic solutions to degenerate parabolic problems with special emphasis on the case of Stefan or porous media equations. We recall that the case of nondegenerate parabolic problems has been studied in the framework of nonlinear semigroups theory in [**3**] (for the case of linear or nonlinear variational inequalities see also [**2**]).

The central tool in this paper is Haraux's theorem on the existence of almost periodic solutions to abstract first order (in $t$) equations with $m$-monotone operators, which are not uniformly monotone.

We give now the precise framework of this paper.

For the definition of (weakly) almost periodic functions in abstract spaces, we refer to [**1**, p. 1]. Let *$f(t)$ be a function in $L^p_{\text{loc}}(R;X)$*, $p \geq 1$, and *consider the function*

$$g(t) = \int_0^1 f(t+s)\,ds$$

*from $R$ to $L^p(0,1;X)$*; we say that $f$ is *$S^p$-(weakly) almost periodic (bounded) in $X$ if $g(t)$ is (weakly) almost periodic (bounded) in $L^p(0,1;X)$.*

Let $\Omega \subset R^N$ be a bounded open set with smooth boundary $\Gamma$; consider the degenerate parabolic problem

$$D_t u - \Delta\beta(u) \ni f, \qquad \beta(u)_{|\Gamma} = 0 \tag{1.1}$$

where $\beta$ is an increasing locally Lipschitz continuous function on $R$ with $\beta(0) = 0$.

The Cauchy problem for (1.1) has been studied by H. Brézis, [**4**], as an application of general results on nonlinear semigroups, choosing as a Hilbert space the space $H^{-1}(\Omega)$.

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We define the functions $\Phi(r)$ and $y(r)$ by the following relations

$$\Phi(r) = \int_0^r \beta(s)\,ds \tag{1.2}$$

$$\gamma(r) = \int_0^r (\beta'(s))^{1/2}\,ds. \tag{1.3}$$

We consider the following assumptions on the function $\beta$:

$$\begin{aligned} &c_1|r|^m - c_2 \leq \Phi(r) \leq c_3|\beta(r)|^2 + c_4 \\ &|\beta(s)| \leq c_5|\beta'(s)| + c_6 \qquad (c_i \geq 0,\ i = 1, \dots, 6; m \geq 2). \end{aligned} \tag{H}$$

**Theorem 1.** *Let $f$ be $S^4$-almost periodic in $L^4(\Omega)$; then there exists a solution $u$ to (1.1) which is weakly almost periodic in $L^m(\Omega)$ and such that the trajectory of $\gamma(u)$ is relatively compact in $L^2(\Omega)$ and $S^2$-bounded in $H^1(\Omega)$.*

We apply the result of Theorem 1 to the cases:

$$\beta(r) = -r^- + (r-1)^+ \quad \text{(Stefan case)} \tag{1.4}$$

$$\beta(r) = |r|^{m-2}r \quad \text{(Porous media case)}. \tag{1.5}$$

**Corollary 1.** *Let $f$ be $S^2$-almost periodic and $\beta$ given by (1.4); there exists a solution to (1.1) which is weakly almost periodic in $L^2(\Omega)$ and such that $\beta(u)$ is almost periodic in $L^2(\Omega)$ and $S^2$-almost periodic in $H^1(\Omega)$.*

*Moreover, if $u$ is a solution of (1.1) bounded in $L^2(\Omega)$ and $\tilde{u}$ is the solution of a Cauchy problem for (1.1) with initial data $u_0$ in $H^{-1}(\Omega)$ with $\beta(u)$ in $L^2(\Omega)$, we have $\lim_{t\to+\infty}(\beta(u) - \beta(\tilde{u})) = 0$ in $L^2(\Omega)$.*

*Then if $u_1, u_2$ are two solutions to (1.1), which are almost periodic in $H^{-1}(\Omega)$, then $\beta(u_1) = \beta(u_2)$ and $(u_1 - u_2)$ is independent of $t$.*

**Remark 1.** We observe that (1.1) can be written as

$$D_t\psi(v) - \Delta v \ni f, \qquad v_{|\Gamma} = 0 \tag{1.1'}$$

where $\psi$ is the inverse graph of $\beta$; from Corollary 1 we obtain the existence of a unique $L^2(\Omega)$-almost periodic solution, which "attracts" all solutions to the Cauchy problem for (1.1′) in $L^2(\Omega)$ as $t \to +\infty$. Finally, we point out that the uniqueness of the $L^2(\Omega)$ almost periodic solution of (1.1′) does not imply the uniqueness of the $L^2(\Omega)$ bounded solution of (1.1′).

**Corollary 2.** *Let $f$ be $S^2$-almost periodic in $L^4(\Omega)$ and $\beta$ given by* (1.5)*; there exists a solution of* (1.1) *which is almost periodic in $L^m(\Omega)$ and we have uniqueness for the solution of* (1.1) *which is almost periodic in $H^{-1}(\Omega)$. Moreover, if $\tilde{u}$ is the solution of a Cauchy problem for* (1.1) *with initial data $u_0$ in $H^{-1}(\Omega)$, we have*

$$\lim_{t\to+\infty} (\tilde{u} - u) = 0 \qquad \text{in } L^m(\Omega).$$

In section 2 of this paper we prove that, if $\tilde{u}$ is the solution of the Cauchy problem for (1.1) with $\tilde{u}(0) = u_0$, $\Phi(u_0) \in L^1(\Omega)$, then $\Phi(\tilde{u})$ is bounded on $R_+$ in $L^1(\Omega)$. This result, together with the one of A. Haraux, recalled in the same section, gives the existence of an $L^m(\Omega)$-weakly almost periodic solution. In section 3 we finish the proof of Theorem 1 and in section 4 we give a proof of Corollaries 1 and 2.

Finally, we observe that the results in this paper can be extended to other cases of boundary conditions.

**2. Existence of a weakly almost periodic solution in $L^m(\Omega)$.** Consider the Cauchy problem

$$\begin{gathered} D_t\tilde{u} - \Delta\beta(\tilde{u}) \ni f \\ \beta(\tilde{u})_{|\Gamma} = 0, \qquad \tilde{u}(0) = u_0 \in H^{-1}(\Omega) \end{gathered} \tag{2.1}$$

where $\beta(u_0)$ is in $H_0^1(\Omega)$ (for the results in this section it is enough to assume $\Phi(u_0) \in L^1(\Omega)$).

From the results of [**4**], (2.1) has a solution in $H^1(0, T; H^{-1}(\Omega))$, with $\beta(\tilde{u})$ in the space $L^2(0, T; H_0^1(\Omega))$, $\forall T > 0$.

We multiply the equation (2.1) by $\beta(\tilde{u})$ and we obtain

$$D_t\left(\int_\Omega \Phi(\tilde{u})\,dx\right) + ||\beta(\tilde{u})||_1^2 \le \int_\Omega f\beta(\tilde{u})\,dx \tag{2.2}$$

($||\cdot||_1$ denotes the usual norm in the space $H_0^1(\Omega)$).

From (2.2) we have

$$D_t\left(\int_\Omega \Phi(\tilde{u})\,dx\right) + \frac{1}{2}||\beta(\tilde{u})||_1^2 \le C_1 \int_\Omega |f|^2\,dx. \tag{2.3}$$

and by (H) we can write (2.3) as

$$D_t\left(\int_\Omega \Phi(\tilde{u})\,dx\right) + C_2 \int_\Omega \Phi(\tilde{u})\,dx \le C_1 \int_\Omega |f|^2\,dx + C_3. \tag{2.4}$$

The boundedness of $\Phi(\tilde{u})$ in $L^1(\Omega)$ on $R_+$ can be proved by (2.4) and the following easy lemma (probably well known).

**Lemma 1.** *Let $\psi(t)$ be a positive absolutely continuous function such that*

$$D_t\psi(t) + \alpha\psi(t) \le \eta(t), \qquad \psi(0) = \tilde{\psi}, \qquad \alpha > 0, \tag{2.5}$$

*where $\eta(t)$ is $S^1$-bounded on $R_+$; then*

$$\psi(t) \le C$$

*where $C$ depends only on $\tilde{\psi}$ and* $\operatorname{Sup}_{R_+} \int_t^{t+1} |\eta(s)|\,ds$.

We omit the proof of the lemma, which is very easy.

From (H) we obtain

$$\int_\Omega |\tilde{u}|^m\,dx \le \int_\Omega \Phi(\tilde{u})\,dx + C_4 \le C \tag{2.6}$$

on $R_+$, where $C$ is a constant independent of $t$ in $R_+$.

From (2.3) we obtain

$$\int_t^{t+1} ||\beta(\tilde{u})||_1^2\,ds \le 4\operatorname{Sup}_{R_+} \int_\Omega \Phi(\tilde{u})\,dx + 2C_1 \int_t^{t+1} \int_\Omega |f|^2\,dx\,ds. \tag{2.7}$$

Then

$$\text{(2.8)}\quad \int_t^{t+1} ||D_t u||_{-1}^2\, ds \\ \le C_5\Big(\operatorname{Sup}_{R_+}\int_\Omega \Phi(\tilde u)\,dx + \operatorname{Sup}_{R_+}\int_t^{t+1}\int_\Omega |f|^2\,dx\,ds\Big).$$

By (2.6), (2.7), (2.8) using classical methods to prove the existence of a bounded solution [**1**] an existence result can be proved for a solution $u$ of (1.1) on $R$ such that

$$\text{(2.9)}\qquad \int_\Omega |u|^m\,dx \le C'$$

$$\text{(2.10)}\qquad \int_t^{t+1} ||\beta(u)||_1^2\,ds \le C'$$

$$\text{(2.11)}\qquad \int_t^{t+1} ||D_t u||_{-1}^2\,ds \le C',$$

where $C'$ is a constant independent of $t$ in $R$; we observe that (2.11) implies that uniform continuity on $R$ of $u$ in $H^{-1}(\Omega)$.

We recall the following result due to A. Haraux, [**5**].

**Theorem 2.** *Let $H$ be a Hilbert space and $A$ a maximal monotone operator in $H$. Consider the equation*

$$\text{(2.12)}\qquad D_t u + Au \ni f$$

*where $f$ is an $S^2$ almost periodic function in $H$. If there exists a solution $\tilde u$ of* (2.12) *which is uniformly continuous in $H$ and with relatively compact trajectory, then there exists a solution of* (2.12) *on $R$ which is almost periodic in $H$.*

From the proof of Theorem 2 we can assume that the range of $u$ is contained in the convex closure of the range of $\tilde u$.

Theorem 2 leaves open the question if all solutions of (2.12) uniformly continuous on $R$ with relatively compact trajectory are almost periodic in $H$. The answer is *positive* if $A$ is a subdifferential [**5**].

We observe now that the operator $v \to (-\Delta\beta(v))$ can be seen as a subdifferential operator in $H^{-1}(\Omega)$; then from the first part of this section we have the existence of a solution $u$ to (1.1) which is almost periodic in $H^{-1}(\Omega)$ and bounded in $L^m(\Omega)$.

From the boundedness in $L^m(\Omega)$ we obtain easily the weak almost periodicity of $u$ in $L^m(\Omega)$.

We recall also that from (2.10) $\beta(u)$ is $S^2$ bounded in $H^1(\Omega)$.

**3. Proof of Theorem 1.** By approximation (formally by multiplication of (1.1) by $D_t\beta(\tilde{u})$) we easily obtain

$$\int_\Omega |D_t\gamma(\tilde{u})|^2\,dx + \frac{1}{2}D_t(||\beta(\tilde{u})||_1^2) \le \int_\Omega f y'(\tilde{u})D_t\gamma(\tilde{u})\,dx. \tag{3.1}$$

From (3.1) (2.10) and the $S^4$-boundedness of $f$ in $L^4(\Omega)$ we deduce

$$\int_t^{t+1}\int_\Omega |D_t\gamma(\tilde{u})|^2\,dx\,ds \le C \tag{3.2}$$

where $C$ is a constant dependent on $||\beta(u_0)||_1$ and on the $S^4$ bound of $f$ in $L^4(\Omega)$ but independent of $t$ in $R_+$.

By approximation of our problem by nondegenerate parabolic problems and taking into account (2.9) and the $S^2$ boundedness of $f$ in $L^2(\Omega)$ we obtain

$$\int_t^{t+1} ||\gamma(\tilde{u})||_1^2\,ds \le C \tag{3.3}$$

where $C$ is a constant dependent on $\int_\Omega \Phi(u_0)\,dx$ and on the $S^2$ bound of $f$ in $L^2(\Omega)$ but independent of $t$ in $R_+$ (formally is a multiplication of (1.1) by $u$).

From (3.3) we can deduce, by standard methods,

$$\int_t^{t+1}\int_\Omega |D_t\gamma(u)|^2\,dx\,ds \le C, \qquad \int_t^{t+1} ||\gamma(u)||_1^2\,ds \le C. \tag{3.4}$$

From (3.4) the relative compactness in $L^2(\Omega)$ of the trajectory of $\gamma(u)$ can be easily proved.

**4. Proof of Corollaries 1, 2.** (a) *Corollary 1.* We prove at first the almost periodicity of $\beta(u)$. We observe that in our case $\beta(s) = y(s)$ and $\beta'(s)$ is bounded on $R$ (then the results of section 3 hold again for $f$ $S^2$ bounded in $L^2(\Omega)$). From equation (1.1) we can easily prove that

$$\int_0^1 \int_\Omega \beta(u(t+\eta, x)v(\eta, x)\, d\eta\, dx$$

is almost periodic for every $v \in C_0^1(0,1; H_0^1(\Omega))$.

Taking into account the boundedness of $\gamma(u)$ in $L^2(\Omega)$, we obtain that $\gamma(u)$ is $S^2$ weakly almost periodic in $L^2(\Omega)$ and from (3.2) and the relative compactness of the trajectory of $u$ in $L^2(\Omega)$ we deduce that $\gamma(u)$ is $S^2$ almost periodic in $L^2(\Omega)$.

Fix a sequence $\{s_n\}$ in $R$; we can suppose $(\forall \varepsilon > 0)$

$$\int_0^1 \int_\Omega |\beta(u(t+\eta+s_n, x)) - \chi(t+\eta, x)|^2\, d\eta\, dx \le \varepsilon, \qquad \forall t \in R \tag{4.1}$$

for a suitable function $\chi(t,x)$ and $n \ge n_\varepsilon$.

From (1) there exists, $\forall t \in R$, $t^* \in (t, t+\varepsilon^{1/2})$ such that

$$\int_\Omega |\beta(u(t^*+s_n, x)) - \chi(t^*, x)|^2\, dx \le \varepsilon^{1/2} \tag{4.2}$$

($t^*$ can depend on $n$ and the set of the points $t^*$ is not negligible).

From (3.2) we have

$$\int_\Omega |\beta(u(t^*+s_n, x)) - \beta(u(t+s_n, x))|\, dx \le C\varepsilon^{1/4} \tag{4.3}$$

and also

$$\int_\Omega |\chi(t^*, x) - \chi(t, x)|\, dx \le C\varepsilon^{1/4}. \tag{4.4}$$

From (4.3), (4.4) we can deduce the almost periodicity of $\beta(u)$ in $L^1(\Omega)$; then, taking into account the relative compactness of the trajectory of $\beta(u) = \gamma(u)$ in $L^2(\Omega)$, the almost periodicity of $\beta(u)$ in $L^2(\Omega)$ follows.

We observe now that in our case $\Phi(u) = |\beta(u)|^2/2$; then, taking into account the almost periodicity of $\beta(u)$ in $L^2(\Omega)$ and the relation

$$D_t\left(\int_\Omega \Phi(u)\,dx\right) + ||\beta(u)||_1^2 = \int_\Omega f\beta(u)\,dx,$$

the $S^2$ almost periodicity of $\beta(u)$ in $H_0^1(\Omega)$ easily follows.

Now let $\tilde{u}$ be the solution of the Cauchy problem with initial data $\tilde{u}(0) = u_0$, where $u_0$ is in $H^{-1}(\Omega)$ and $\beta(u_0)$ is in $H_0^1(\Omega)$; suppose also $\beta(u(0)) \in H_0^1(\Omega)$.

From (3.2), (3.3) we have that the trajectory of $\beta(\tilde{u}) = \gamma(\tilde{u})$ is relatively compact in $L^2(\Omega)$. We multiply (1.1) by $-\Delta^{-1}(\tilde{u} - u)$, where $-\Delta^{-1}$ denotes the inverse of the operator of $-\Delta$ with Dirichlet homogeneous boundary conditions; we obtain

$$\frac{1}{2}D_t(||\tilde{u} - u||_{-1}^2) + \int_\Omega (\beta(\tilde{u}) - \beta(u))(\tilde{u} - u)\,dx = 0. \tag{4.5}$$

From (4.5), taking into account the boundedness of $\tilde{u}$, $u$ in $H^{-1}(\Omega)$, we obtain that $(\beta(\tilde{u}) - \beta(u))$ $(\tilde{u} - u)$ is integrable on $R_+ \times \Omega$. Then

$$\lim_{t\to+\infty} \int_t^{t+1} \int_\Omega (\beta(\tilde{u}) - \beta(u))(\tilde{u} - u)\,dx\,ds = 0. \tag{4.6}$$

We observe that $(\beta(\tilde{u}) - \beta(u))$ $(\tilde{u} - u) \geq |\beta(\tilde{u}) - \beta(u)|^2$; then

$$\lim_{t\to+\infty} \int_t^{t+1} \int_\Omega |\beta(\tilde{u}) - \beta(u)|^2\,dx\,ds = 0. \tag{4.7}$$

From (3.2), (3.3), recalling that $\beta = \gamma$, we obtain

$$\lim_{t\to+\infty} (\beta(\tilde{u}) - \beta(u)) = 0 \text{ in } L^2(\Omega). \tag{4.8}$$

We observe that if $u_0$ $(u(0))$ is in $H^{-1}(\Omega)$, (4.8) holds again while $\Phi(\tilde{u})$ $(\Phi(u))$ is integrable on $(0,1) \times \Omega$ and $\beta(\tilde{u})$ $(\beta(u))$ is in $L^2(\delta, 1; H_0^1(\Omega))$, $\forall\delta > 0$; then we can repeat the preceding proof only changing the initial instant.

Let now $u_1$ and $u_2$ be two solutions of (1.1) almost periodic in $H^{-1}(\Omega)$; (4.5) holds again with $\tilde{u}$ and $u$ replaced by $u_1$ and $u_2$ for $t \in R$. Then

$$D_t(||u_1 - u_2||^2_{-1}) \leq 0 \Longrightarrow D_t(||u_1 - u_2||^2_{-1}) = 0. \tag{4.9}$$

Then

$$\int_\Omega |\beta(u_1) - \beta(u_2)|^2\, dx \leq \int_\Omega (\beta(u_1) - \beta(u_2)(u_1 - u_2)\, dx = 0. \tag{4.10}$$

From (4.10) we have $\beta(u_1) = \beta(u_2)$ a.e. From (1.1) we obtain $D_t u_1 = D_t u_2$ in $H^{-1}(\Omega)$; then $w = u_1 - u_2$ is a vector in $L^2(\Omega)$ independent of $t$.

(b) *Corollary 2.* Firstly we observe that in our case $\gamma(u) = |u|^{(m/2)-1}u$ and that the following algebraic inequality

$$||a|^{(n-1)}a - |b|^{(n-1)}b| \geq c|a - b|^n \tag{4.11}$$

(for every integer $n \geq 1$ and $c > 0$ suitable) can be easily proved (for example by contradiction).

The relative compactness of the trajectory of $\gamma(u)$ in $L^2(\Omega)$ and (4.11) imply the relative compactness of the trajectory of $u$ in $L^m(\Omega)$; then $u$ is almost periodic in $L^m(\Omega)$.

The uniqueness of the $H^{-1}(\Omega)$ almost periodic solution and the asymptotic behavior of the solutions of the Cauchy problem can be proved by the same methods of part (a) taking into account (4.11) and the particular form of $\gamma, \beta$.

During the writing of this paper I was informed through personal communication that A. Damlamian and N. Kenmochi have proved the same result of Corollary 1 independently.

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