ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 2, Spring 1991

NODAL PROPERTIES OF SOLUTIONS OF PARABOLIC EQUATIONS

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1. Introduction. In this note we review the known facts about the zero set of a solution of a scalar parabolic equation

$$(1)\quad u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u, \qquad x_0 < x < x_1, 0 < t < T.$$

In particular, we discuss some applications to spectral theory, the dynamics of nonlinear diffusion equations, and the geometric heat equation for plane curves.

2. The zero number. Let u be a classical solution of (1) and assume u is continuous on the rectangle $[x_0, x_1] \times [0, T]$. Moreover, assume that

$$u(x_i, t) \neq 0 \quad \text{for } i = 0, 1 \quad \text{and} \quad 0 \leq t \leq T.$$

Then, for each $t \in [0, T]$ we define the set $Z(t) = \{x \in [x_0, x_1] \mid u(t, x) = 0\}$, and we let $z(t)$ denote the number of elements of $Z(t)$. The set $Z(t)$ is a compact subset of the open interval (x_0, x_1).

Finally, we always assume the following about the coefficients a, b and c:

$$(2)\quad \begin{array}{l} a, a_x, a_{xx}, a_t, b_x, b_t \text{ and } c \text{ are continuous on } [x_0, x_1] \times [0, T]. \\ \text{Moreover, } a(x,t) \text{ is strictly positive.} \end{array}$$

In this situation we have the following:

Theorem A. *For any $0 < t \leq T$, $z(t)$ is finite. If, for some $0 < t_0 < T$, the function $u(t_0)$ has a double zero, then for all*

* Supported by a NATO Science Fellowship and by the Netherlands Organization for the advancement of pure research (Z.W.O.)
This paper was written in 1987 for the Proceedings of the Conference on Nonlinear Partial Differential Equations held in Provo, Utah, in March, 1987.

$t_1 < t_0 \le t_2$ *we have* $z(t_1) > z(t_2)$. Here a double zero is a point where both u and u_x vanish.

This theorem shows that the number of zeros, $z(t)$, does not increase with time. The theorem is a refinement of a result of Nickel, Matano and Henry (see [**7,6,5**]). If the coefficients and the solution are real analytic, then Theorem A was proven in [**3**]. The general case was proven in [**2**].

The idea of the proof in the analytic case is to study the Taylor series of a solution $u(t,x)$ near its multiple zeros. If $(\hat{t},\hat{x})$ is such a zero, then repeated differentiation of the equation (1) shows that, up to rescaling and higher order terms, one has

$$u(\hat{t}+\tau,\hat{x}+\delta) = \frac{\delta^m}{m!} + \frac{\tau}{1!}\frac{\delta^{m-2}}{(m-2)!} + \frac{\tau^2}{2!}\frac{\delta^{m-4}}{(m-4)!} + \dots \tag{3}$$

with $m \ge 2$. Using the Newton polygon method, one then finds that the zero set of $u(x,t)$ near $(\hat{t},\hat{x})$ consists of a finite number of curves. Furthermore, if m is even, all these curves lie in the region $t \le \hat{t}$. If m is odd, there is one additional curve that intersects the line $t = \hat{t}$ transversally (see Figure 1). In either case, the number of zeros of $u(t,\cdot)$ drops as t increases beyond $\hat{t}$.

It should be noted that the polynomials given in [**3**] are special solutions of the heat equation $u_\tau = u_{\delta\delta}$ and that it can be instructive to study their graphs (see Figure 2).

The boundary conditions $u(x_i,t) \ne 0$ are not the only ones under which Theorem A holds. More general conditions were discussed in [**2**], and one we would like to mention here is the periodic case.

If the functions u, a, b and c are periodic in x with period 1 (so that they are defined on $\mathbf{R} \times [0,T]$) and satisfy (2) on $\mathbf{R} \times [0,T]$ instead of $[x_0,x_1] \times [0,T]$, then Theorem A remains valid if one defines $z(t)$ to be the number of zeros of $u(t,\cdot)$ in the interval $[0,1)$.

3. Time-dependent Sturm Liouville theory. Let $c(x,t)$ be a continuous function on $\mathbf{R} \times [0,T]$ satisfying $c(x+1,t) \equiv c(x,t)$. Then we define a linear operator L on $C(\mathbf{R}/Z)$ by the following recipe. Given

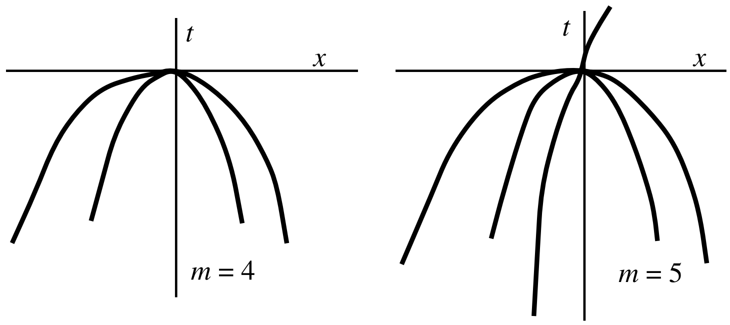

FIGURE 1. The zeroset near a multiple zero.

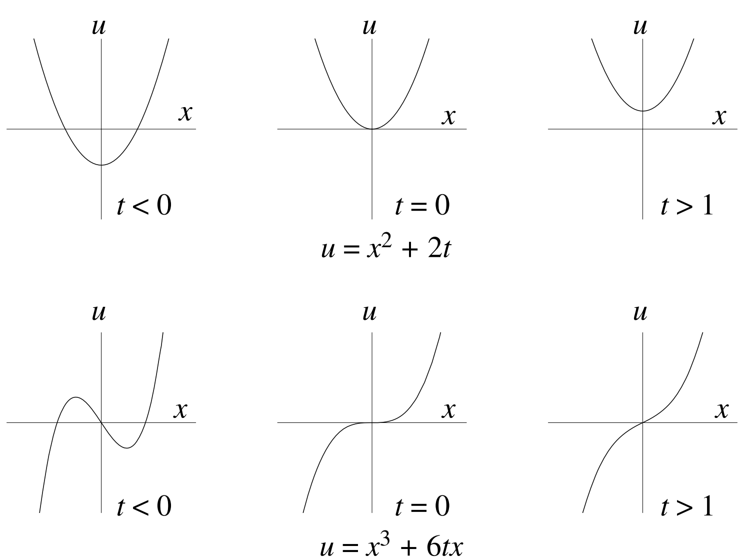

FIGURE 2. Some special solutions of the heat equation.

$f \in C(\mathbf{R}/Z)$ one computes Lf by solving the initial value problem

$$u_t = u_{xx} + c(x,t)u, \qquad x \in \mathbf{R}/Z, \quad t \in [0,T]$$
$$u(x,0) = f(x) \tag{4}$$

and defining $Lf(x) = u(x,T)$.

Standard results on the smoothing property of parabolic equations imply that L is a bounded compact operator on $E = C(\mathbf{R}/Z)$. Thus, its spectrum consists of, at most, a countable number of eigenvalues, clustering at $\lambda = 0$. We denote these eigenvalues by $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots$ and order them so that $|\lambda_j| \geq |\lambda_{j+1}|$. Each eigenvalue is assumed to occur as often as its algebraic multiplicity.

If $c(x,t) = c(x)$ does not depend on t, then we may write $L = \exp(A)$ where $-A$ is the Hill's operator $-A = -(d/dx)^2 - c(x)$. In this case it is known that the eigenvalues λ_j come in pairs, i.e., $\lambda_{2n} > \lambda_{2n+1}$ for all $n \geq 0$. Also, the eigenfunctions belonging to λ_{2n-1} and λ_{2n} have exactly $2n$ zeros in one period interval $0 \leq x < 1$.

Using Theorem A one can show that this also holds in the general case where c does depend on time. More precisely, if L is defined as above, then we have

$$\lambda_0 > |\lambda_1| \geq |\lambda_2| > |\lambda_3| \geq |\lambda_4| > \cdots . \tag{5}$$

In particular, for any $n \geq 1$, $\{\lambda_{2n-1}, \lambda_{2n}\}$ is a spectral set for the operator L, so that its corresponding spectral subspace $F_n \subset E$ is well defined. This space is two dimensional and any real function $f \in F_n$ has exactly $2n$ zeros, all of which are simple.

The proof of these statements is contained in [**1,3**]. The key ingredient is the following observation: for any $f \in E$, Lf has only a finite number of zeros, and $z(Lf) \leq z(f)$. If Lf has a multiple zero, then $z(Lf) < z(f)$. This follows from Theorem A and the definition of L. It immediately implies that, if f is a real eigenfunction, its zeros are all simple (since $z(f) = z(Lf)$). A lengthier argument along the same lines leads to the statements we just made.

4. Rotating waves. We consider the initial value problem

$$u_t = f(u, u_x, u_{xx}), \qquad x \in \mathbf{R}/Z, t > 0$$
$$u(x,0) = u_0(x) \tag{6}$$

in which $f(u,p,s)$ is a C^∞ function of its arguments and $f_s(u,p,s) > 0$.

In [**3**] the dynamics of the semiflow generated by such an equation was studied in the semilinear case (i.e., $f(u,p,s) = s + g(u,p)$ for some other function g).

Using Theorem A we can prove the following: Let $u(t,x)$ be a periodic solution of (6), i.e., $u(t+T,x) \equiv u(t,x+1) \equiv u(t,x)$ and suppose u is so smooth that u_t and u_{xx} are Hölder continuous. By parabolic regularity theory the solution u is then actually C^∞. We now have:

u is either constant, or a rotating wave, i.e.,
of the form $U(x-ct)$ for some $c \in \mathbf{R}$.

To prove this we observe that any linear combination w of u_t and u_x is a solution of

$$\begin{aligned} w_t &= a \cdot w_{xx} + b \cdot w_x + c \cdot w \\ w(x+1,t) &\equiv w(x,t+T) \equiv w(x,t) \end{aligned} \tag{7}$$

where $a = f_s(u, u_x, u_{xx})$, $b = f_p(u, u_x, u_{xx})$ and $c = f_u(u, u_x, u_{xx})$.

So if $w \neq 0$, then for any time t, $w(\cdot, t)$ has only a finite number of zeros, $z(t)$.

Furthermore, $z(t)$ is nonincreasing, and by periodicity $z(t+T) = z(t)$. Hence, $z(t)$ must be constant, and Theorem A implies that $w(\cdot,t)$ never has a multiple zero.

Now choose a point where $u(x,t)$ attains its maximal value, say (x_0, t_0). Then both u_x and u_t vanish at (x_0, t_0) and there must be a linear combination $w = \alpha u_x + \beta u_t$ such that w_x also vanishes at this point. The foregoing considerations show that $w \equiv 0$, and we are left with two cases. If $\beta = 0$, then $w = \alpha u_x \equiv 0$, so that u is constant. Otherwise, we have $u_t + c u_x = 0$ with $c = \alpha/\beta$ so that u can be written as $u(x-ct)$.

In [**3**] many other results were derived; in particular, the existence of connecting orbits between different rotating waves was studied.

5. The geometric heat equation. Let X be a regular curve in the plane, i.e., a C^1 mapping from $\mathbf{R}/Z$ into $\mathbf{R}^2$ whose derivative never vanishes. The curve may have self-intersections.

We shall use the letter u to denote the parameter in $\mathbf{R}/Z$ on the curve (i.e., $X = X(u), u \in \mathbf{R}/Z$).

If the curve is C^2, then its curvature k is well defined. The geometric heat equation is the following

$$\frac{\partial X}{\partial t} = kN \quad \text{or} \quad \frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial s^2} \tag{8}$$

where N is the unit normal to the curve, and s denotes arclength along the curve. The second form of the equation is slightly misleading since $\partial/\partial t$ stands for a derivative w.r.t. t with constant $u \in \mathbf{R}/Z$, and not constant s. A more precise version is

$$\begin{gathered} X_t = |X_u|^{-1}(|X_u|^{-1}X_u)_u, \qquad X(u+1,t) \equiv X(u,t) \\ u \in \mathbf{R}/Z,\ t > 0. \end{gathered} \tag{9}$$

This is a degenerate system of parabolic PDEs. Local solvability in time was shown in [**4**] for C^∞ initial data.

It is known that, if $X(u,t)$ $(0 \le t < T)$ is a solution of (9) whose initial value has no self-intersections, then for all $0 < t < T$, the curve $X(\cdot, t)$ also has no self-intersections (see [**4**]).

Using Theorem A we can say a little more.

Let $X(u,t)$ be a solution of (9). Choosing rectangular coordinates x, y in the plane any small enough portion of the family of curves $X(u,t)$ can be represented as the graph of a function $y = w(x,t)$. A lengthy computation shows that (9) is, locally at least, equivalent to the following equation for w.

$$w_t = \frac{w_{xx}}{1 + (w_x)^2} \stackrel{\text{def}}{=} F(w_x, w_{xx}). \tag{10}$$

Since this is a quasilinear parabolic equation, the curves $X(\cdot, t)$ are, for each t, real analytic.

If we have two solutions of (9), say X_1 and X_2, then for any $t > 0$, they either coincide or they have only a finite number of intersections, say $i(t)$.

Near a point of intersection both curves can be represented by two solutions w^1 and w^2 of (10) (if one chooses the y-axis in the right direction).

The difference $v = w^1 - w^2$ satisfies a linear equation of the form

$$v_t = a(x,t)v_{xx} + b(x,t)v_x$$

(just subtract equation (10) for w^1 and w^2, and apply the mean value theorem to F).

By Theorem A, the number of zeros of v cannot increase and in fact decreases if $v(t,\cdot)$ has a multiple zero. Since zeros of $v(t,\cdot)$ correspond to intersections of X_1 and X_2, we arrive at the following conclusion.

At any time $t > 0$ for which the curves X_1 and X_2 are defined, their number of intersections, $i(t)$, is finite.

If for some $t_0 > 0$, X_1 and X_2 have a nontransversal intersection, then $i(t)$ drops as t increases t_0.

A similar argument shows that the number of self-intersections of a solution $X(t,u)$ of (9) cannot increase with time.

To conclude this discussion we note that the curvature k as a function of normalized arclength s satisfies

$$k_t = k_{ss} + (\beta k)_s = k_{ss} + \beta k_s + \beta_s k \tag{11}$$

where $\beta(s,t) = \int_0^s k(s',t)^2\,ds' - s\int_0^1 k(s',t)^2\,ds'$ [**1**].

The normalized arclength is defined to be ordinary arclength divided by the total length of the curve. Thus k and β are periodic functions of s, with period 1.

If we apply Theorem A to (11), then we find:

for any $t > 0$ the curve $X(\cdot,t)$ has a finite number of flexpoints.

This number does not increase with time.

(Recall that a flexpoint is a point on the curve where k vanishes.)

Differentiating (11) with respect to s, and using $\beta_{ss} = 2k\cdot k_s$, we see that k_s also satisfies an equation of the form (1) so that Theorem A can again be applied.

For any $t > 0$ the curve $X(\cdot,t)$ has a finite number of vertices.

This number does not increase with time.

(A vertex of a plane curve is a point where the curvature reaches a local maximum or minimum [**8**, Vol. 2].)

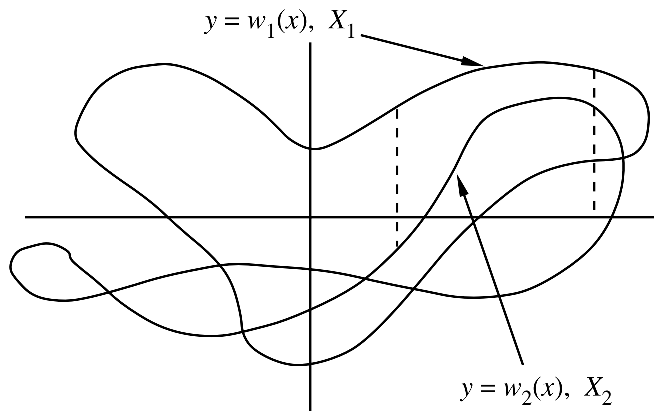

FIGURE 3. Two plane curves.

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