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SOME REMARKS ON REPRODUCING KERNEL KREIN SPACES

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ABSTRACT. The one-to-one correspondence between positive functions and reproducing kernel Hilbert spaces was extended by L. Schwartz to a (onto, but not one-to-one) correspondence between difference of positive functions and reproducing kernel Krein spaces. After discussing this result, we prove that a matrix valued function $K(z,\omega)$ symmetric and jointly analytic in z and $\bar{\omega}$ in a neighborhood of the origin is the reproducing kernel of a reproducing kernel Krein space. We conclude with an example showing that such a function can be the reproducing kernel of two different Krein spaces.

1. Introduction. In this paper we study some points in the theory of reproducing kernel Krein spaces, namely existence theorems and a nonuniqueness counterexample. We begin with a brief review of reproducing kernel spaces, which helps setting the framework and gives some motivation.

Let $K(z,\omega)$ be a $\mathbf{C}_{n\times n}$-valued function for z and ω in some set Ω, and let V be a vector space of $\mathbf{C}_n$-valued functions defined on Ω, endowed with some hermitian form $[\ ,\ ]_V$. The function K is a reproducing kernel for V if the following two conditions hold:

(a) for any ω in Ω and c in $\mathbf{C}_n$, the function $K_\omega c : z \to K(z,\omega)c$ belongs to V.

(b) for any f in V, ω in Ω and c in $\mathbf{C}_n$,

$$[f, K_\omega c]_V = c^* f(\omega) \tag{1.1}$$

($\mathbf{C}_{n\times l}$ denotes the vector space n-rows l-columns matrices with complex entries, $\mathbf{C}_{n\times 1}$ is written $\mathbf{C}_n$ and A^* denotes the adjoint of the matrix A.)

It is easy to check that $(V, [\ ,\ ]_V)$ has at most one reproducing kernel, which is moreover a symmetric function (also called hermitian)

$$K(z,\omega) = (K(\omega,z))^*. \tag{1.2}$$

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Conversely, given a $\mathbf{C}_{n\times n}$-valued symmetric function $K(z,\omega)$ on Ω, let $\overset{\circ}{\mathcal{K}}$ denote the vector space spanned by the functions $K_\omega c$, $\omega \in \Omega$, $c \in \mathbf{C}_n$, and let $[\ ,\ ]_{\overset{\circ}{\mathcal{K}}}$ denote the hermitian form defined by

$$[K_\omega c, K_\nu d]_{\overset{\circ}{\mathcal{K}}} = d^* K(\nu,\omega)c. \tag{1.3}$$

It is also easy to check that the form $[\ ,\ ]_{\overset{\circ}{\mathcal{K}}}$ is well defined and that any other reproducing kernel space $(V, [\ ,\ ]_V)$ with reproducing kernel K contains isometrically $\overset{\circ}{\mathcal{K}}$, that is, $V \supset \overset{\circ}{\mathcal{K}}$ and

$$[f,g]_{\overset{\circ}{\mathcal{K}}} = [f,g]_V$$

for f and g in $\overset{\circ}{\mathcal{K}}$.

In general, the space $\overset{\circ}{\mathcal{K}}$ has no nice topological structure, and the case where one can find a reproducing kernel Krein space V $(\supset \overset{\circ}{\mathcal{K}})$ with reproducing kernel K is of interest.

If the function K is positive: for any integer r, points $\omega_1, \dots, \omega_r$ in Ω and vectors $c_1, \dots, c_r$ in $\mathbf{C}_n$, the hermitian forms

$$\sum_{i,j=1}^{r} \alpha_i \alpha_j^* c_j^* K(\omega_i, \omega_j) c_i \qquad (\alpha_i \in \mathbf{C}; i = 1, \dots, r) \tag{1.4}$$

are positive, the $\overset{\circ}{\mathcal{K}}$ is a prehilbert space and can be completed in a unique way into a reproducing kernel Hilbert space with reproducing kernel K. Conversely, a reproducing kernel Hilbert space has a reproducing kernel which is a positive function (see [**3**]).

This one-to-one correspondence between positive functions and reproducing kernel Hilbert spaces has been extended by L. Schwartz [**10**, Proposition 40] and P. Sorjonen [**11**] to the case of functions which have a finite number ν of negative squares, that is, such that all the hermitian forms defined in (1.4) have at most ν negative squares and at least one of them has exactly ν negative squares. The corresponding reproducing kernel spaces are then Pontryagin spaces.

When K is the reproducing kernel of a reproducing kernel Krein space, it is not difficult to show that K can be written as a difference of two positive functions. This condition is also sufficient, as proved in [**10**]. A proof of this result will be recalled in Section 2, since L. Schwartz's study does not seem to be widely known among people working in reproducing kernel spaces.

It may happen that a given function is the reproducing kernel of two different reproducing kernel Krein spaces [**10**].

It may be difficult to check in general that a given symmetric function can be written as a difference of two positive functions, and sufficient conditions which insure the existence of an associated reproducing kernel Krein space are called for. In the third section we prove that a $\mathbf{C}_{n\times n}$-valued function $K(z,\omega)$ jointly analytic for z and $\bar{\omega}$ in a neighborhood of the origin is the reproducing kernel of an associated reproducing kernel Krein space.

Analyticity is a strong requirement and one could think that the associated Krein space is then unique. The fourth section presents an example of a function K, jointly analytic in z and $\bar{\omega}$ in a neighborhood of the origin and which is the reproducing kernel of two different reproducing kernel Krein spaces.

To conclude this introduction, we recall a few facts on positive functions and on Krein spaces. Let $K(z,\omega)$ be a $\mathbf{C}_{n\times n}$-valued function positive for z and ω in Ω. The associated reproducing kernel Hilbert space will be denoted by $H(K)$, the inner product by $\langle\ ,\ \rangle_K$ and the associated norm by $||\ ||_K$.

One introduces a partial ordering on positive functions on Ω by

$$K \le K_1 \text{ if and only if } K_1 - K \text{ is a positive function.}$$

It is an easy exercise to check that $K \le K_1$ if and only if $H(K)$ is contractively included in $H(K_1)$, that is, if and only if $H(K) \subset H(K_1)$ and the inclusion map is a contraction from $H(K)$ into $H(K_1)$ [**3**].

A space $\mathcal{K}$ endowed with an hermitian form $[\ ,\ ]$ is a Krein space if it admits a decomposition $\mathcal{K} = \mathcal{K}_+ + \mathcal{K}_-$, where

(1) $\mathcal{K}_+$ endowed with $[\ ,\ ]$ is a Hilbert space;

(2) $\mathcal{K}_-$ endowed with $-[\ ,\ ]$ is a Hilbert space;

(3) $\mathcal{K}_+ \cap \mathcal{K}_- = \{0\}$ and $[k_+, k_-] = 0$ for $k_\pm \in \mathcal{K}_\pm$.

Let $f = f_+ + f_-$ be the decomposition of $f \in \mathcal{K}$ along $\mathcal{K}_+ + \mathcal{K}_-$. The hermitian form

$$\langle f, f\rangle = [f_+, f_+] - [f_-, f_-]$$

is positive and $(\mathcal{K}\langle\ ,\ \rangle)$ is a Hilbert space.

The map

$$\sigma f = f_+ - f_-$$

is both selfadjoint and unitary in $(\mathcal{K}\langle\ ,\ \rangle)$, and

$$[f, g] = \langle f, \sigma g\rangle. \tag{1.5}$$

Conversely, every Krein space can be obtained in such a way: starting from a Hilbert space $(\mathcal{K}, \langle\ ,\ \rangle)$ and an operator σ both selfadjoint and unitary in $\mathcal{K}$, the space $\mathcal{K}$ endowed with the form defined in (1.5) is a Krein space.

Krein spaces were introduced in the fifties by Ginzburg and appear in an implicit way in the work of Nevanlinna and Pesonen (see [**5**, p. 118] for references). They were rediscovered independently by Schwartz [**10**], who called them hermitian spaces.

We refer to Bognar's book [**5**] for further information on Krein spaces.

2. Characterization of kernel functions. This section is of a survey nature, and results are mostly due to L. Schwartz [**10**]. It is included because, to the best of our knowledge, the results are not widely known. Moreover, since [**10**] is written in the language of Hilbert subspaces of topological vector spaces, we have chosen to provide proofs as well.

Lemma 2.1 appears in Ando's lecture notes [**2**].

The main result is

Theorem 2.1 (Schwartz). *Let $K(z, \omega)$ be a $\mathbf{C}_{n\times n}$-valued symmetric function defined for z and ω in some set Ω. Then there is an associated reproducing kernel Krein space if and only if $K = K_1 - K_2$ where K_1 and K_2 are positive functions on Ω. When K admits such a decomposition, one can choose K_1 and K_2 such that $H(K_1) \cap H(K_2) = \{0\}$. If K_1 and K_2 satisfy this condition, the set of functions of the* *form* $f = f_1 + f_2$, $f_i \in H(K_i)$ $i = 1, 2$, *with indefinite inner product*

$$[f, f] = \langle f_1, f_1\rangle_{K_1} - \langle f_2, f_2\rangle_{K_2},$$

is a reproducing kernel Krein space with reproducing kernel K.

In order to prove this theorem, we need the following two lemmas.

Lemma 2.1 (Ando). *Let* K_1 *and* K_2 *be two* $\mathbf{C}_{n\times n}$*-valued functions, positive on* Ω. *Then the intersection* $H(K_1)\cap H(K_2)$ *endowed with the inner product*

$$\langle f, f\rangle = \langle f, f\rangle_{K_1} + \langle f, f\rangle_{K_2}$$

is a reproducing kernel Hilbert space contractively included in $H(K_1)$ *and* $H(K_2)$.

Lemma 2.2 (Schwartz). *Let* K_1 *and* K_2 *be two* $\mathbf{C}_{n\times n}$*-valued functions, positive on* Ω, *and let* $I(K_1, K_2)$ *denote the set of all functions* K *positive on* Ω *and such that* $K \le K_1$ *and* $K \le K_2$. *Then* $I(K_1, K_2)$ *is inductive.*

Proof of Lemma 2.1. The intersection $H = H(K_1)\cap H(K_2)$ endowed with the inner product $\langle\ ,\ \rangle$ is clearly a pre-Hilbert space. Let (f_p) be a Cauchy sequence in H. Then it is also a Cauchy sequence in $H(K_1)$ and $H(K_2)$, and thus there exists f in $H(K_1)$ and g in $H(K_2)$ such that

$$\lim_{p\to\infty} f_p = f$$

in the $H(K_1)$ norm, and

$$\lim_{p\to\infty} f_p = g$$

in the $H(K_2)$ norm.

Let ω be in Ω and c be in $\mathbf{C}_n$,

$$c^*f(\omega) = \langle f, K_1(\cdot,\omega)c\rangle_{K_1} = \lim_{p\to+\infty}\langle f_p, K_1(\cdot,\omega)c\rangle_{K_1} = \lim_{p\to+\infty} c^*f_p(\omega)$$

and, similarly,

$$c^*g(\omega) = \lim_{p\to+\infty} c^*f_p(\omega).$$

Hence, $f = g$, and f belongs to H. It is easy to check that $\lim f_p = f$ in H, and so H is a Hilbert space. Moreover, by the definition of its norm, H is contractively included in $H(K_1)$ and $H(K_2)$.

Finally, the inequalities

$$|c^* f(\omega)|^2 \le ||f||_{K_1}^2 \cdot c^* K_1(\omega,\omega)c \le ||f||_K^2 c^* K_1(\omega,\omega)c$$

express that H is a reproducing kernel Hilbert space. $\square$

Let K be the reproducing kernel of H. By the contractive inclusions, we have

$$K \le K_1 \tag{2.1}$$

$$K \le K_2. \tag{2.2}$$

In some special cases, the space $H(K_1) \cap H(K_2)$ is closely related to the overlapping subspaces defined by de Branges (see [**6**], [**7**]).

Proof of Lemma 2.2. Let $(K_j)_{j\in J}$ be an ordered subset of $I(K_1, K_2)$. We first remark that, for each z in Ω and c in $\mathbf{C}_n$, $(c^* K_j(z,z)c)_{j\in J}$ is an increasing bounded sequence of positive numbers.

Let ω be in Ω, d in $\mathbf{C}_n$, and $i \le j$ $(i, j \in J)$. Then, $H(K_i)$ is contractively included in $H(K_j)$ and, in particular,

$$||K_i(\cdot,\omega)d||_{K_j}^2 \le ||K_i(\cdot,\omega)d||_{K_i}^2 = d^* K_i(\omega,\omega)d. \tag{2.3}$$

Moreover,

$$c^* K_i(z,\omega)d - c^* K_j(z,\omega)d = \langle K_i(\cdot,\omega)d - K_j(\cdot,\omega)d, K_j(\cdot,z)c\rangle_{K_j}$$

and the Cauchy Schwartz inequality and (2.3) lead easily to

$$|c^* K_i(z,\omega)d - c^* K_j(z,\omega)d|^2 \le (c^* K_i(z,z)c)(d^* K_i(\omega,\omega)d - d^* K_j(\omega,\omega)d).$$

Hence, $K(z,\omega) = \lim_i K_i(z,\omega)$ exists for any z and ω in Ω. It is clearly positive and in $I(K_1, K_2)$, and hence this latter is inductive. $\square$

By Zorn's lemma, the set $I(K_1, K_2)$ admits a maximum element K_{Max}. Suppose that $H(K_1 - K_{\text{Max}}) \cap H(K_2 - K_{\text{Max}}) \ne \{0\}$. By

Lemma 2.1, this intersection is then a reproducing kernel Hilbert space with a nonzero reproducing kernel K satisfying

$$K \leq K_1 - K_{\mathrm{Max}}$$

and

$$K \leq K_2 - K_{\mathrm{Max}}\,,$$

contradicting the maximality of K_{Max}.

As a corollary, we have

Corollary 2.1. *Let K be a difference of two positive functions: $K = K_1 - K_2$. Then, without loss of generality, one can choose K_1 and K_2 such that $H(K_1) \cap H(K_2) = \{0\}$.*

The kernels K_1 and K_2 are then called "étrangers" in [**10**].

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. One direction is clear. Suppose that $K(z, \omega)$ is the reproducing kernel of some reproducing kernel Krein space $(\mathcal{K}, [\ ,\ ])$ and let $\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ be a decomposition of $\mathcal{K}$ into a direct and orthogonal sum of a Hilbert space $\mathcal{K}_+$ and of an antiHilbert space $\mathcal{K}_-$ (i.e., $\mathcal{K}_-$ endowed with $-[\ ,\ ]$ is a Hilbert space); let P_+ (respectively, P_-) denote the orthogonal projection from $\mathcal{K}$ onto $\mathcal{K}_+$ (respectively, $\mathcal{K}_-$).

$$\xi^* K(z, \omega)\eta = [K_\omega \eta, K_z \xi] = [P_+ K_\omega \eta, P_+ K_z \xi] - (-[P_- K_\omega \eta, P_- K_z \xi])$$

exhibits K as a difference of two positive functions.

To prove the converse statement, let $K = K_1 - K_2$ be a decomposition of the symmetric function K into a difference of two positive functions. By Corollary 2.1, we can suppose that $H(K_1) \cap H(K_2) = \{0\}$. Then the space

$$\mathcal{K} = \{f = f_1 + f_2; f_i \in H(K_i), i = 1, 2\},$$

endowed with the inner product

$$\langle f, f\rangle = \langle f_1, f_1\rangle K_1 + \langle f_2, f_2\rangle K_2,$$

is a Hilbert space. Moreover, the map σ defined by

$$\sigma f = f_1 - f_2$$

is easily seen to be selfadjoint and unitary from $\mathcal{K}$ to $\mathcal{K}$, $K_\omega c$ belongs to $\mathcal{K}$ for any ω in Ω and c in $\mathbf{C}_n$; with

$$[f, f]_K = \langle f, \sigma f\rangle,$$

we have

$$c^* f(\omega) = [f, K_\omega c],$$

and so $\mathcal{K}$ is a reproducing kernel Krein space with reproducing kernel K. □

We now give an example, due to L. Schwartz, of a symmetric function which cannot be written as a difference of two positive functions.

Theorem 2.2. *Let B be a reflexive real Banach space with a norm $|| \; ||_B$ not equivalent to a quadratic norm and let B' be its dual. Let $E = B \times B'$, endowed with the norm $|| \; ||_E$,*

$$||(x, \varphi)||_E^2 = ||x||_B^2 + ||\varphi||_{B'}^2,$$

and the hermitian form $[\ ,\ ]_E$,

$$[(x, \varphi), (y, \psi)]_E = \varphi(y) + \psi(x).$$

Then $[\ ,\ ]_E$ cannot be represented as a difference of two positive functions on E.

Proof of Theorem 2.2. We proceed in a number of steps and denote by capital letters $X, Y, \ldots$ the elements of E.

Step 1. The space E endowed with $[\ ,\ ]_E$ is not decomposable and the norm $|| \; ||_E$ is an admissible majorant for $[\ ,\ ]_E$ (i.e., the form $(X, Y) \to [X, Y]_E$ is jointly continuous in X and Y in the topology induced by $|| \; ||_E$ and any continuous linear functional on E has the form $X \to [X, Y]_E$ for some Y in E).

Proof of Step 1. See [**5**; Example 5.6, p. 90].

We can thus identify E with its dual E' and with $\overset{\circ}{\mathcal{K}} = \text{l.s.}\{K_Y, Y \in E\}$ where l.s. stands for linear span and K_Y denotes the function $X \to [X,Y]_E$.

Suppose that the function $K(X,Y) = [X,Y]_E$ can be written as a difference of two functions positive on E. Then, by Theorem 2.1, there exists a reproducing kernel Krein space $\mathcal{K}$ with reproducing kernel K, and thus,

$$[X,Y]_E = [K_X, K_Y]_{\overset{\circ}{\mathcal{K}}} = \langle \sigma K_X, K_Y \rangle_{\mathcal{K}}$$

where $\langle\ ,\ \rangle_{\mathcal{K}}$ makes $\mathcal{K}$ into a Hilbert space and σ is both selfadjoint and unitary in the Hilbert space $(\mathcal{K}, \langle\ ,\ \rangle_{\mathcal{K}})$.

Step 2. There exists $M < \infty$ such that

$$||K_X||_{\mathcal{K}} \leq M||X||_E, \qquad X \in E \tag{2.4}$$

Proof of Step 2. We show that the map $i : i(X) = K_X$ is closed. The inequality (2.4) will follow by the closed graph theorem.

Let X_n be a sequence in E which converges in E (to X_∞) and such that K_{X_n} converges in $\mathcal{K}$ (to an element Y in $\mathcal{K}$). Then each K_{X_n} defines a linear continuous functional in $\mathcal{K}$, namely, $X \to [X, X_n]_{\mathcal{K}}$, and these functionals converge to some linear continuous function which, by the Riesz representation theorem, is of the form $X \to [X,Y]_{\mathcal{K}}$ for some Y in $\mathcal{K}$. We show that $Y = X_\infty$. Let U be in E,

$$\begin{aligned}[Y - X_\infty, U]_{\mathcal{K}} &= [Y,U]_{\mathcal{K}} - [X,U]_{\mathcal{K}} \\ &= [Y,U]_{\mathcal{K}} - [X,U]_E \\ &= \lim [X_n, U]_{\mathcal{K}} - [X_n, U]_E;\end{aligned}$$

hence, $Y = X_\infty$ since E is dense in $\mathcal{K}$, which ends the proof of Step 2.

By inequality (2.4), each element Y in $\mathcal{K}$ defines a linear continuous functional on E by $X \to [X,Y]_E$, and, thus, after identification of the functional with Y,

$$E \subset \mathcal{K} \subset E'.$$

Since $E = E'$, we deduce $E = \mathcal{K}$, a contradiction, since E is not decomposable. □

About spaces with two norms, we refer to the paper [**9**].

3. An existence theorem. In this section we consider the special case where the function $K(z,\omega)$ is jointly analytic in z and $\bar{\omega}$ for z and ω in a neighborhood of the origin. The following theorem improves a result of [**1**] where functions $K(z,\omega)$ of a certain type (displacement kernels) were considered. As in [**1**], the associated reproducing kernel Krein space is described as an operator range. The characterization of reproducing kernel Krein spaces as operator range goes back to Schwartz [**10**]. About operator ranges and reproducing kernel Hilbert spaces, we also mention [**4**] and [**8**].

Theorem 3.1. *Let $K(z,\omega)$ be a $\mathbf{C}_{n\times n}$-valued function symmetric and jointly analytic in z and $\bar{\omega}$ for $|z|<r$, $|\omega|<r$. Then, there exists a reproducing kernel Krein space of $\mathbf{C}_n$-valued functions analytic for $|z|<r'$ (where $r'\le r$), with reproducing kernel K.*

Proof. We proceed in a number of steps and first suppose $r>1$. We will denote by H_n^2 the space of $n\times 1$ vectors with entries in H^2, the classical Hardy space of the circle, with inner product

$$\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} g^*(e^{it})f(e^{it})\,dt.$$

Step 1. If $r>1$, the operator P defined by

$$(Pf)(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z,e^{it})f(e^{it})\,dt$$

is bounded and selfadjoint from H_n^2 into itself; moreover, for any ω in the unit disk $\mathbf{D}$ and any c in $\mathbf{C}_n$, the function $K_\omega c$ belongs to $\operatorname{Ran} P$, the range of P.

Proof of Step 1. The function Pf is analytic in $\mathbf{D}$. Let $M = \sup ||K(z,\omega)||$, where the supremum is on z and ω of modulus less than r. By the Cauchy-Schwartz inequality,

$$||Pf||(z) \le M||f||_{H_n^2},$$

and, hence, Pf belongs to H_n^2 and $||P|| \leq M$.

Moreover, for f and g in H_n^2,

$$\langle Pf, g\rangle_{H_n^2} = \langle f, Pg\rangle_{H_n^2} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^*(e^{it})K(e^{it}, e^{it'})f(e^{it'})\, dt\, dt',$$

and, hence, P is selfadjoint.

Finally, for $f(z) = c/(1 - z\omega^*)$, $|\omega| < 1$, the Cauchy's formula leads to

$$(Pf)(z) = K(z, \omega)c$$

which ends the proof of Step 1.

The Steps 2 and 3 are taken from [**1**]. Proofs are provided for completeness.

Before proceeding to Step 2 we define on $\operatorname{Ran} P$ two hermitian forms $\langle\ ,\ \rangle_P$ and $[\ ,\ ]_P$ by

$$\langle Pu, Pv\rangle_P = \langle |P|u, v\rangle_{H_n^2}$$

and

$$[Pu, Pv]_P = \langle Pu, v\rangle_{H_n^2},$$

where $|P|$ is the absolute value of the selfadjoint operator P.

These two forms are easily seen to be well defined, and $\operatorname{Ran} P$ endowed with $\langle\ ,\ \rangle_P$ is clearly a prehilbert space.

Step 2. Let $\mathcal{K}$ denote the closure of $\operatorname{Ran} P$ in the $\langle\ ,\ \rangle_P$ inner product. Then, $\mathcal{K}$ endowed with $[\ ,\ ]_P$ is a Krein space.

Proof of Step 2. Let σ denote the signum of P and note that $\sigma(\operatorname{Ran} P) \subset \operatorname{Ran} P$. For u and v in H_n^2, we have

$$\langle \sigma Pu, \sigma Pv\rangle_P = \langle Pu, Pv\rangle_P$$
$$\langle \sigma Pu, Pv\rangle_P = \langle Pu, \sigma Pv\rangle_P$$

so that σ extends to a continuous operator both self adjoint and unitary from $\mathcal{K}$ into $\mathcal{K}$. Moreover, the equality

$$[Pu, Pv]_P = \langle \sigma Pu, Pv\rangle_P$$

expresses that $\mathcal{K}$ endowed with $[\ ,\ ]_P$ is a Krein space.

Step 3. The space $\mathcal{K}$ defined in Step 2 is a reproducing kernel Krein space included (in general nonisometrically) in H_n^2 with reproducing kernel $K(z,\omega)$.

Proof of Step 3. Let $\rho_\omega(z) = 1 - z\omega^*$. Then, as seen in Step 1, $Pc/\rho_\omega = K_\omega c$,

$$[Pu, K_\omega c]_P = \langle Pu, c/\rho_\omega\rangle_{H_n^2} = c^*(Pu)(\omega),$$

which permits one to conclude that $\mathcal{K}$ is a reproducing kernel Krein space of functions analytic in $\mathbf{D}$ with reproducing kernel $K(z,\omega)$.

To show that $\mathcal{K} \subset H_n^2$, we use the easily checked inequality

$$\langle Pu, Pu\rangle_{H_n^2} \le \|\,|P|^{\frac{1}{2}}\,\|^2 \langle Pu, Pu\rangle_{\mathcal{K}},$$

where $\|\,|P|^{1/2}\,\|$ is the norm of $|P|^{1/2}$ as an operator from H_n^2 into H_n^2. Hence, a Cauchy sequence in $\mathcal{K}$ is a Cauchy sequence in H_n^2. Let $(Pu_l)_{l\ge 0}$ be a Cauchy sequence of elements in $\operatorname{Ran} P$ in the $\mathcal{K}$-norm, and let f and g be the limit of this sequence in the $\mathcal{K}$ norm and in the H_n^2 norm, respectively. Then, for ω in $\mathbf{D}$,

$$\begin{aligned} c^* f(\omega) &= \lim [Pu_l, K_\omega c]_P \\ &= \lim [Pu_l, Pc/\rho_\omega]_P \\ &= \lim \langle Pu_l, c/\rho_\omega\rangle_{H_n^2} \\ &= \langle g, c/\rho_\omega\rangle_{H_n^2} \\ &= c^* g(\omega). \end{aligned}$$

Thus, $f = g$, which concludes the proof of Step 3.

To conclude the proof of the theorem, it remains to consider the case where $r \le 1$. If the function $K(z,\omega)$ is jointly analytic in z and ω in the disk of radius r, the function $K_\rho(z,\omega) = K(\rho z, \rho\omega)$ is jointly analytic for z, ω of modulus strictly bigger than 1, if ρ is small enough.

Let $\mathcal{K}_\rho$ be the reproducing kernel Krein space with reproducing kernel K_ρ built as in Steps 1–3, and let $\mathcal{K}$ be defined by

$$\mathcal{K} = \{f; f(z) = F(z/\rho); F \in \mathcal{K}_\rho\}$$

with inner product

$$[f,g]_{\mathcal{K}} = [F,G]_{\mathcal{K}_\rho}$$

if $g(z) = G(z/\rho)$.

The function $z \to K(z,\rho\omega)$ belongs to $\mathcal{K}$, and, for

$$[f, K(\cdot,\rho\omega)c]_{\mathcal{K}} = [F, K(\rho\cdot,\rho\omega)c]_{\mathcal{K}_\rho} = c^*F(\rho\omega) = c^*f(\omega)$$

which concludes the proof. □

4. A nonuniqueness counterexample. The fact that different Krein spaces may be associated to the same reproducing kernel function was first noted by L. Schwartz, who called such kernels multiplicity kernels and gave criteria for multiplicity and uniqueness, and also presented an example of a multiplicity kernel. We here present another example of a multiplicity kernel.

Let $r < 1$ and let $\mathcal{K}_r$ be the space of functions which are restrictions to $|z| < r$ of an element of H_2^2 with norm

$$||f||^2_{\mathcal{K}_r} = ||F||_{H_2^2} \tag{4.1}$$

if

$$f = F_{|D_r}.$$

The norm (4.1) is well defined since the restriction of an element of H_2^2 *to* D_r uniquely characterizes this element.

The space $\mathcal{K}_r$ is in fact easily seen to be the reproducing kernel Hilbert space with reproducing kernel

$$\frac{I_2}{1 - z\omega^*},$$

where z and ω are restricted to D_r, and where $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Let us define on $\mathcal{K}_r$ the indefinite inner product

$$[f,g]_{\mathcal{K}_r} = \langle F, JG\rangle_{H_2^2},$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The space $\mathcal{K}_r$ with the hermitian form $[\ ,\ ]_{\mathcal{K}_r}$ is also easily seen to be a reproducing kernel Krein space with reproducing kernel

$$\frac{J}{1 - z\omega^*}, \tag{4.2}$$

with z and ω in D_r.

We now construct another Krein space with the same reproducing kernel. Let us first define

$$T = \begin{pmatrix} \operatorname{ch}\theta & \operatorname{sh}\theta \\ \operatorname{sh}\theta & \operatorname{ch}\theta \end{pmatrix},$$

where $\theta \in \mathbf{R}$ is such that $e^{-\theta} = r$.

The matrix T satisfies

$$e^{-\theta} I_2 \leq T \leq e^{\theta} I_2 \tag{4.3}$$

and

$$TJT^* = J \tag{4.4}$$

(this latter property being called J-unitarity).

Theorem 4.1. *Let $\mathcal{K}$ denote the set of $\mathbf{C}_2$-valued power series $f(z) = \sum_0^\infty f_n z^n$ such that*

$$||f||_{\mathcal{K}}^2 = \sum_0^\infty f_n^* T^{2n} f_n < \infty, \tag{4.5}$$

and define on $\mathcal{K}$ the indefinite inner product

$$[f, f]_{\mathcal{K}} = \sum_0^\infty f_n^* J f_n. \tag{4.6}$$

Then $\mathcal{K}$ is a reproducing kernel Krein space of $\mathbf{C}_2$-valued functions analytic in D_r with reproducing kernel (4.2).

Proof of Theorem 4.1. To ease the presentation, the proof is divided in steps.

Step 1. $\mathcal{K}$ is a Hilbert space of functions analytic in D_r.

Proof of Step 1. From (4.3) one sees that elements of $\mathcal{K}$ are analytic in D_r; $(\mathcal{K}, \langle\ ,\ \rangle_{\mathcal{K}})$ is clearly a prehilbert space. Let us show that it is complete. Let $(f^{(p)})$ be a Cauchy sequence of elements of $\mathcal{K}$; the sequences

$$n \to T^n f_n^{(p)}$$

then form a Cauchy sequence in the Hilbert space l_2^2 of $\mathbf{C}_2$-valued sequences $(\mu_0, \mu_1, \dots)$ such that $\sum_0^\infty \mu_n^* \mu_n < \infty$. There thus exists an element (g_n) of l_2^2 such that

$$g_n = \lim_{p\to\infty} T^n f_n^{(p)} \quad \text{(limit in the } l_2^2 \text{ norm).}$$

Let $f(z) = \sum_0^\infty T^{-n} g_n z^n$. One shows without difficulty that f is in $\mathcal{K}$ and that

$$\lim ||f - f^{(p)}||_{\mathcal{K}} = 0,$$

hence, the completeness of $\mathcal{K}$.

Step 2. Let σ be defined by

$$(\sigma f)(z) = \sum_0^\infty T^{-2n} J f_n z^n.$$

Then σ is both unitary and selfadjoint from $\mathcal{K}$ into $\mathcal{K}$.

Proof of Step 2. To compute $\langle \sigma f, \sigma f\rangle_{\mathcal{K}}$, one makes use of the J-unitarity of the matrix T:

$$\begin{aligned}\langle \sigma f, \sigma f\rangle_{\mathcal{K}} &= \sum_0^\infty f_n^* J T^{-2n} T^{2n} T^{-2n} J f_n \\ &= \sum_0^\infty f_n^* J T^{-2n} J f_n \\ &= \sum_0^\infty f_n^* T^{2n} f_n = \langle f, f\rangle_{\mathcal{K}}\end{aligned}$$

since $JT^{-2n}J = T^{2n}$.

From Steps 1 and 2 it follows that $\mathcal{K}$ endowed with $[\ ,\ ]_{\mathcal{K}}$ (defined in (4.3)) is a Krein space.

Step 3. Let $\omega \in D_r$ and c be in $\mathbf{C}_2$. The function $K_\omega c : z \to Jc/(1 - z\omega^*)$ belongs to $\mathcal{K}$ and, for f in $\mathcal{K}$,

$$[f, K_\omega c] = c^* f(\omega).$$

Proof of Step 3. The power expansion

$$\frac{Jc}{1 - z\omega^*} = \sum_0^\infty Jc\, \omega^{*n} z^n$$

leads to

$$\begin{aligned}\langle K_\omega c, K_\omega c\rangle_{\mathcal{K}} &= \sum_0^\infty |\omega|^{2n} c^* J T^{2n} J c \\ &\le \sum_0^\infty |\omega|^{2n} c^* c e^{2\theta n} \\ &\le \frac{c^* c}{1 - (|\omega| e^{\theta})^2} < \infty,\end{aligned}$$

and, thus, $K_\omega c$ belongs to $\mathcal{K}$.

Let $f : f(z) = \sum f_n z^n$ be in $\mathcal{K}$. Then

$$[f, K_\omega c] = \sum \omega^n c^* J J f_n = c^* f(\omega),$$

which concludes the proof of Step 3.

To conclude the proof, it remains to show that indeed $\mathcal{K}$ and $\mathcal{K}_r$ are different Krein spaces.

Take $f(z) = \sum_0^\infty n^\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} z^n$ where $\alpha \ge 0$. Then f belongs to $\mathcal{K}$ since

$$\langle f, f\rangle_{\mathcal{K}} = \sum_0^\infty (1, -1) T^{2n} \begin{pmatrix} 1 \\ -1 \end{pmatrix} n^{2\alpha} = 2 \sum_0^\infty n^{2\alpha} e^{-2n\theta} < \infty.$$

On the other hand, f does not belong to $\mathcal{K}_r$, since

$$\sum_0^\infty f_n^* f_n = \sum_0^\infty 2n^{2\alpha} = +\infty.$$

Similarly, the function $f(z) = \sum_1^\infty \frac{1}{n}\begin{pmatrix}1\\1\end{pmatrix} z^n$ is in H_2^2, and, thus, its restriction to D_r is in $\mathcal{K}_r$. It is not in $\mathcal{K}$ since

$$\sum_1^\infty f_n^* T^{2n} f_n = 2\sum_1^\infty \frac{1}{n^2} e^{2n\theta} = +\infty.$$

So, neither $\mathcal{K} \subset \mathcal{K}_r$ nor $\mathcal{K}_r \subset \mathcal{K}$. □

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