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NONINVERTIBILITY OF INVARIANT DIFFERENTIAL OPERATORS ON LIE GROUPS OF POLYNOMIAL GROWTH

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In recent years weighted L^2 spaces have been useful in proving solvability results for invariant differential operators on Lie groups (e.g., [**2, 3**]). This is done by showing that the operators in question are boundedly invertible on a suitable weighted L^2 space.

In this note we present a result which demonstrates some of the limitations of this approach. We show that left invariant differential operators on a connected Lie group, G, of polynomial growth, are not boundedly invertible on $L^2(G, \omega(x)\,dx)$ where dx is the right Haar measure and $\omega(x)$ is a polynomial weight. This should be considered in the context of Levy-Bruhl's use of exponential weights [**2**].

For a measurable subset A of G, let $|A|$ denote the measure of A.

Definition 1. A connected, locally compact group, G, has polynomial growth if there is a polynomial p such that for each compact neighborhood U of e, there is a constant $C(U)$ so that $|U^n| \le C(U)p(n)$ $(n = 1, 2, \dots)$ $(U^n = \{u_1 \cdot u_2 \cdot \dots \cdot u_n | u_i \in U,\ 1 \le i \le n\}.)$ (J. Jenkins has given a characterization of the locally compact groups with polynomial growth in [**1**].)

Note that since G is connected, its growth will be determined by the behavior of $|U^n|$ for any one compact neighborhood U of e.

Definition 2. A nonnegative measurable function ω on a connected Lie group has polynomial growth if there is a polynomial q such that for each compact neighborhood U of e there is a constant $C(U)$ so that

$$\int_{U^n} \omega(x)\,dx \le C(U)q(n), \qquad n = 1, 2, 3, \dots .$$

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Theorem 3. *Let G be a connected Lie group, ω a function of polynomial growth on G with $\omega(x) > 0$ a.e. dx and D a left invariant differential operator on G with zero constant term. Then D does not have a bounded inverse on $L^2(G, \omega(x)dx)$.*

Proof of Theorem 3. Let $U = U^{-1}$ be a compact neighborhood of e in G. To prove the theorem, it suffices to show that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ with $L^2(G, \omega(x)dx)$ norms uniformly bounded away from 0 such that Df_n goes to 0 in $L^2(G, \omega(x)dx)$ as n goes to infinity.

As D is the sum of monomials of the form $X_m X_{m-1} \ldots X_1$, where the X_is are left invariant vector fields on G, it will suffice to show that $X_m X_{m-1} \ldots X_1 f_n$ goes to 0. Let $\phi > 0$ be a C^∞ function with support in U such that $\int_G \phi(x)\, dx \neq 0$.

Define

$$f_n(x) = \left(\int_{U^{n-1}} \omega(x)\, dx \right)^{-1/2} \chi_{U^n} * \phi(x)$$

where χ_{U^n} is the characteristic function of U^n. A straightforward calculation shows that

$$||f_n(x)||_\omega^2 \geq \left| \int_U \phi(x)\, dx \right|^2$$

where $||\cdot||_\omega$ denotes the norm in $L^2(G, \omega(x)dx)$.

From left invariance, it follows that

$$X_m X_{m-1} \cdots X_m f_n = X_m(\chi_{U^n} * X_{m-1} \cdots X_2 X_1 \phi)$$

where $X_{m-1} \cdots X_2 X_1 \phi$ is C^∞ with support in U. Let $\psi =$

$X_{m-1} \cdots X_2 X_1 \phi$.

For sufficiently small t, $\exp tX_m \in U$. Thus, if $y \in U$, $x \notin U^{n+2}$, then $x \cdot \exp X_m \cdot y^{-1} \notin U^n$. From this, the right invariance of Haar measure, and the fact that $\operatorname{supp}\psi \subseteq U$, it follows that

$$||X_m(\chi_{U^n} * \psi)||_\omega^2 = \int_G \left| \frac{d}{dt}(\chi_{U^n} * \psi)(x \cdot \exp tX_m)|_{t=0} \right|^2 \omega(x)\,dx$$

$$= \int_G \left| \lim_{t\to 0} \frac{1}{t} \left[\int_U (\chi_{U^n}(x \cdot \exp tX_m \cdot y^{-1})\psi(y)\,dy \right.\right.$$

$$\left.\left. - \int_U \chi_{U^n}(x \cdot y^{-1})\psi(y)dy \right] \right|^2 \omega(x)\,dx$$

$$= \int_{U^{n+2}-U^{n-2}} \left| \lim_{t\to 0} \frac{1}{t} \left[\int_U \chi_{U^n}(x \cdot y^{-1})\psi(y \cdot \exp tX_m)\,dy \right.\right.$$

$$\left.\left. - \int_U \chi_{U^n}(x \cdot y^{-1})\psi(y)\,dy \right] \right|^2 \omega(x)\,dx$$

$$= \int_{U^{n+2}-U^{n-2}} \left| \int_U \chi_{U^n}(x \cdot y^{-1}) \cdot X_m\psi(y)\,dy \right|^2 \omega(x)\,dx$$

$$\leq K|U|^2 \int_{U^{n+2}-U^{n-2}} \omega(x)\,dx \qquad \text{where} \qquad K = \left(\int_U |X_m\psi(y)|\,dy \right)^2$$

depends only on U, X_m and ψ. Thus,

$$||X_m f_n||_\omega^2 \leq K|U|^2 \left(\int_{U^{n+2}-U^{n-2}} \omega(x)\,dx \right) \Big/ \left(\int_{U^{n-1}} \omega(x)\,dx \right).$$

Let

$$a_n = \left(\int_{U^{n+2}-U^{n-2}} \omega(x)\,dx \right) \Big/ \left(\int_{U^{n-1}} \omega(x)\,dx \right).$$

It suffices to show that there exists a subsequence of a_ns converging to 0.

Suppose no such sequence exists. Then, given $\varepsilon > 0$, there exists N such that for $n \geq N$

$$\int_{U^{n+2}-U^{n-2}} \omega(x)\,dx > \varepsilon \int_{U^{n-1}} \omega(x)\,dx.$$

If we set $b_n = \int_{U^{n-2}} \omega(x)\,dx$, then this can be rewritten as

$$b_{n+4} - b_n > \varepsilon b_{n+1}.$$

Since b_n is a nondecreasing sequence of positive numbers, we have

$$b_{n+4} > (\varepsilon + 1) b_n$$

for all $n \geq N$. It follows that

$$b_{N+4m\cdot n} > (\varepsilon + 1)^{mn} b_N > \varepsilon^m b_N n^m.$$

For fixed $m > \deg q$, q as in definition 2, and sufficiently large n, this leads to a contradiction to our polynomial growth assumption. In this context, this assumption implies that

$$b_{N+4_{mn}} \leq c n^{\deg q}$$

where c is a positive constant independent of n. □

Corollary 4. *Let G be a Lie group of polynomial growth. Let ω be a measurable function with $\omega(x) > 0$ a.e. dx and satisfying*

$$\omega(x) \leq r(n)$$

for all $x \in U^n$, U a compact neighborhood of e, $r(n)$ a polynomial. Then the left invariant differential operators on G with zero constant term do not have bounded inverses on $L^2(G, \omega(x)\, dx)$.

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