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DENSITY AND THE CIRCULAR PROJECTION

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0. Introduction. One aspect of complex analysis deals with classifying the points $a \in \partial D$, D a simply connected domain, by determining whether a certain geometric condition exists or fails to exist in a neighborhood of a. See [**1, 3, 4**]. These geometric conditions sometimes indicate how some part of the boundary of D, say $E \subset \partial D$, geometrically behaves near a. See [**2**]. When some sort of an inner normal at $a \in \partial D$ exists, one can define a set S on the normal that is the image of E under a circular projection. If the set S happens to have certain density properties, does E have them also? We answer this question in a certain setting.

In Section I we discuss the basic definitions and properties of density on the real number line and on a rectifiable Jordan arc. We then consider the curve $\Gamma : y = f(x)$, $0 \le x \le m$, where $f(x)$ satisfies a Lipschitz condition and show that $(x_0, f(x_0))$ is a point of density of a measurable set B of Γ if and only if x_0 is a point of density of $P(B)$, where P is the projection map $P : \Gamma \to [0, m]$. Section 2 considers a point $a_0 \in \Gamma$ where the inner normal exists, defines the circular projections C_R, C_L from Γ into the inner normal at a_0, and shows that with a Lipschitz constant less than one a similar result holds for C_R and C_L. Finally, Section 3 shows that if the Lipschitz constant is greater than one, then the theorem is true in one direction for C_R, C_L but not in the other.

1. Density and projections. We begin our discussion of density on the real number line which we denote by $\mathbf{R}$. Let m denote Lebesgue measure and m^* the outer measure with respect to m. We shall say that *a sequence* $\{I_k\}$ *of intervals in* $\mathbf{R}$ *converges to* $x \in \mathbf{R}$ and write $I_k \to x$, $x \in \mathbf{R}$, if $x \in I_k$ for each k and $\lim_{k\to\infty} \operatorname{diam} I_k = 0$.

Let A be any subset of $\mathbf{R}$. For any measurable set E in $\mathbf{R}$ we define

$$\sigma_A(E) = m^*(A \cap E).$$

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We shall say that *the derivative of σ_A at x exists* if there exists a number $\sigma'_A(x)$ such that for any sequence $\{I_k\}$ of intervals converging to x,

$$\lim_{k\to\infty} \frac{\sigma_A(I_k)}{m(I_k)} = \sigma'_A(x).$$

Since σ_A is a completely additive set function for measurable sets, we are assured [**5**, 168–179] that $\sigma'_A(x)$ exists almost everywhere. Since $\sigma'_A(x)$ is in some way a "measure" of the denseness of A in a neighborhood of x, $\sigma'_A(x)$ is called *the outer density of A at x*. If A is measurable, $\sigma'_A(x)$ is called *the density of A at x*. If $\sigma'_A(x) = 1$, x is called *a point of density of A* and if $\sigma'_A(x) = 0$, x is called *a point of dispersion of A*.

If A is any measurable set and if CA denotes its complement with respect to $\mathbf{R}$, then it immediately follows from the additive properties of m that for measurable A, x is a point of density of A if and only if x is a point of dispersion of CA; that is,

$$\sigma'_A(x) = 1 \quad \text{if and only if} \quad \sigma'_{CA}(x) = 0. \tag{1}$$

Making use of the subadditive properties of m^* we have for any set A that if x is a point of dispersion of CA, then x is a point of density of A; that is,

$$\sigma'_{CA}(x) = 0 \quad \text{implies} \quad \sigma'_A(x) = 1. \tag{2}$$

A direct consequence of the Density Theorem [**5**] shows that the converse is not true. The theorem also describes how dense A is at any of its points.

We move from the setting of $\mathbf{R}$ to that of a rectifiable Jordan arc J. The measure we use on J is naturally arc length which we denote by L. Let L^* denote the outer measure with respect to L. Since the measure induced by L is a Lebesgue measure on J, similar definitions for outer density, density, point of density, point of dispersion exist and results similar to (1) and (2) and the Density Theorem hold. We denote these respectively by (1′), (2′), Density Theorem on J.

Throughout this paper $y = f(x)$ will be a function defined on the closed interval $[0, m]$, with $f(0) = f(m) = 0$ and satisfying the Lipschitz condition

$$|f(x') - f(x')| \le M|x' - x''| \quad \text{for all } x', x'' \in [0, m]. \tag{3}$$

Set $\Gamma = \{(x, f(x)) : x \in [0, m]\}$. From (3) we have that Γ is a rectifiable Jordan arc. Let L denote the arc length measure on Γ and let L^* denote the outer measure corresponding to L. Let $P : \Gamma \to \mathbf{R}$ be defined by $P(x, y) = x$. Denote the closed subarc of Γ from $a = (x, f(x))$ to $b = (y, f(y))$ where $x < y$, by $[a, b]$. The open subarc of Γ from a to b will be denoted by (a, b). Using (3) we have

$$|x - y| \leq L((a, b)) \leq \sqrt{1 + M^2}|x - y| \tag{4}$$

and this can easily be extended to include arbitrary sets. Hence, for any $B \subset \Gamma$,

$$m^*(P(B)) \leq L^*(B) \leq \sqrt{1 + M^2}m^*(P(B)). \tag{5}$$

As a consequence of (5), we have

Lemma 1. *$a_0 = (x_0, f(x_0))$ is a point of dispersion of $B \subset \Gamma$ if and only if x_0 is a point of dispersion of $P(B)$.*

Using this lemma, (1) and (1′), we obtain

Theorem 1. *If $B \subset \Gamma$ is L-measurable, then $a_0 = (x_0, f(x_0))$ is a point of density of B if and only if x_0 is a point of density of $P(B)$.*

We close this section by stating a weaker result that immediately follows from the Density Theorems above and from (5): For almost every $a_0 = (x_0, f(x_0)) \in B$, a_0 is a point of density of B and x_0 is a point of density of $P(B)$.

2. Density and circular projection. We shall be working with the setting described in Section 1 with the additional assumption that $M < 1$ and we identify (x, y) and $x + iy$. From the Lipschitz condition, it follows [**6**, p. 244-246] that $f'(x)$ exists almost everywhere on $[0, m]$ and is a measurable function. Hence, at almost every point of Γ, the tangent exists. Letting $a_0 = (x_0, f(x_0))$ $0 < x_0 < m$ be a point at which the tangent exists and $a = (x, f(x))$, we set

$$\nu^+(a_0) = \lim_{\substack{x \to x_0 \\ x > x_0}} \arg(a - a_0) = \operatorname{Arctan}(f'(x_0))$$

and

$$\nu^-(a_0) = \lim_{\substack{x\to x_0\\ x<x_0}} \arg(a-a_0) = \operatorname{Arctan}(f'(x_0)) + \pi.$$

Let $N(a_0) = \nu^-(a_0) + \pi/2$ and note that $\pi < N(a_0) < 2\pi$. We set

$$L(a_0) = \{a_0 + \rho e^{iN(a_0)} : \rho > 0\},$$
$$L_r(a_0) = \{a_0 + \rho e^{iN(a_0)} : 0 < \rho < r\},$$
$$\Gamma_R = P^{-1}([x_0, m]), \qquad \Gamma_L = P^{-1}([0, x_0]) \quad \text{and}$$
$$\Gamma_R(h) = P^{-1}([x_0, x_0 + h]), \qquad \Gamma_L(h) = P^{-1}([x_0 - h, x_0])$$
$$\text{where } h \le (\frac{1}{2})\min(x_0, m - x_0).$$

Define the right circular projection $C_R : \Gamma_R \to L(a_0)$ by $C_R(a) = a_0 + |a-a_0|e^{iN(a_0)}$ and the left circular projection $C_L : \Gamma_L \to L(a_0)$ by $C_L(a) = a_0 + |a - a_0|e^{iN(a_0)}$. Using arc length measure on Γ_R and Γ_L, and Lebesgue measure on $L(a_0)$, we shall derive measure and density results similar to those in Section 1, only this time under circular projection.

We first establish some mapping properties of C_R. Our first property follows from the fact that the Lipschitz constant M is less than one.

Property 1. C_R is one-to-one. The property is equivalent to: If $0 < x_1 < x_2$, $|y_1| < x_1$, $|y_2| < x_2$, $|y_2 - y_1| < x_2 - x_1$, then $x_1^2 + y_1^2 \ne x_2^2 + y_2^2$. (See Figure 1.) The conditions above imply

$$x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1) > |y_2 - y_1|(|y_2| + |y_1|) \ge |y_2^2 - y_1^2|,$$

from which the result follows.

As an immediate consequence, we have

Property 2. Let a_1, $a_2 \in \Gamma_R$, $a_1 = (x_1, f(x_1))$, $a_2 = (x_2, f(x_2))$. If $x_1 < x_2$, then $|a_1 - a_0| < |a_2 - a_0|$.

Property 3. The circular projection of an arc in Γ_R is an interval on $L(a_0)$ and the circular projection of disjoint arcs in Γ_R are disjoint intervals on $L(a_0)$. If (a_1, a_2) is an open subarc of Γ_R with $a_1 = (x_1, f(x_1))$ and $a_2 = (x_2, f(x_2))$, $x_1 < x_2$, then $m(C_R((a_1, a_2))) = |a_2 - a_0| - |a_1 - a_0|$.

The first two comments follow by using Property 1 and that C_R is continuous. If we let $c = \max_{a\in(a_1,a_2)} |a - a_0|$ and $d = \min_{a\in(a_1,a_2)} |a -$

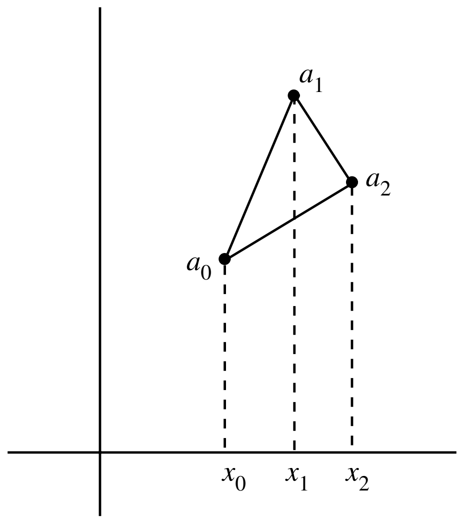

FIGURE 1.

$a_0|$, then $m(C_R((a_1, a_2))) = c - d$ and from Property 2, $c = |a_2 - a_0|$ and $d = |a_1 - a_0|$.

Although the corresponding properties for P were obvious and we used them readily, they needed to be verified for C_R. From Property 3, we have

$$m(C_R((a_1, a_2))) \leq L((a_1, a_2)). \tag{1}$$

We shall establish a result similar to (5) in Section 1. We must first show that there exists a number $K > 0$ such that for all a_1, $a_2 \in \Gamma_R$ where $a_1 = (x_1, f(x_1))$ and $a_2 = (x_2, f(x_2))$, $x_1 < x_2$, the following holds

$$|a_2 - a_1| < K(|a_2 - a_0| - |a_1 - a_0|) \quad \text{for all } a_1, a_2, \varepsilon\Gamma_R. \tag{2}$$

Let C_1 and C_2 be concentric circles with center a_0 whose radii are $|a_1 - a_0|$ and $|a_2 - a_0|$, respectively. Let l be the line passing through a_1 and a_0 and let α be the angle of intersection of l and the line segment

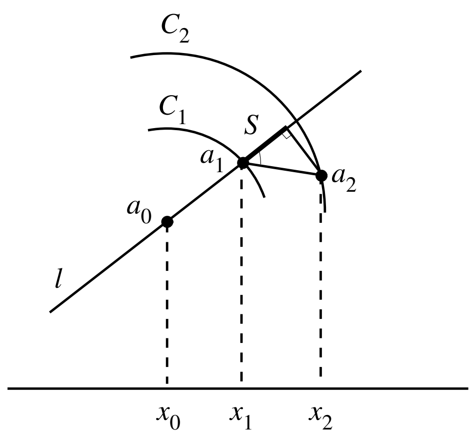

FIGURE 2.

connecting a_1 and a_2. Because $M < 1$, it follows that $\alpha < \pi/2$. The orthogonal projection of this line segment onto l will be denoted by S. (See Figure 2.)

It follows that

$$m(S) \leq |a_2 - a_0| - |a_1 - a_0| \quad \text{and} \quad \alpha = |\arg(a_1 - a_0) - \arg(a_2 - a_1)|.$$

Since both $|\arg(a_2 - a_0)|$ and $|\arg(a_1 - a_0)|$ are strictly less than $\operatorname{Arctan} M$, where $0 < M < 1$ is the Lipschitz constant for $f(x)$, we have that $\alpha < 2\operatorname{Arctan} M < \pi/2$. Setting $K = (\cos(2\operatorname{Arctan} M))^{-1}$, it follows that $\cos\alpha > K^{-1}$. We conclude that

$$K^{-1} < \cos\alpha = \frac{m(S)}{|a_2 - a_1|} < \frac{|a_2 - a_0| - |a_1 - a_0|}{|a_2 - a_1|},$$

from which (2) follows. As a consequence, $L((a_1, a_2)) < Km(C_R(a_1, a_2))$ and combining this with (1) will show that for any $B \subset \Gamma_R$,

$$m^*(C_R(B)) \leq L^*(B) \leq Km^*(C_R(B)). \tag{3}$$

By using the one-to-one and continuous properties of C_R and P, (1) and (1′) of Section 1, and (3) above, the following theorem concerning density under circular projections is readily established.

Theorem 2. *Let $B \subset \Gamma_R$ be L-measurable, then $C_{\mathbf{R}}(B)$ is Lebesgue measurable and*

$$\lim_{h\to 0} \frac{L(B\cap \Gamma_R(h))}{L(\Gamma_R(h))} = 1 \quad \textit{iff} \quad \lim_{r\to 0} \frac{m(C_R(B)\cap L_r(a_0))}{r} = 1.$$

We close this section by noting that similar results hold for C_L on Γ_L.

3. Concluding remarks. We now examine the results of Section 2 under the assumption that $M > 1$. Clearly, Properties 1, 2 and 3 no longer are true. However, (1) of Section 2 can still be shown to hold using a different argument. First note that if C_R is constant on $[a_1, a_2]$, then (1) of Section 2 holds trivially. We therefore assume that C_R is nonconstant on $[a_1, a_2]$. Since it is continuous on the closed subarc $[a_1, a_2]$ it assumes maximum and minimum values at the points a_M and a_m, respectively. We have that

$$\begin{aligned} m(C_R((a_1,a_2)) &= C_R(a_M) - C_R(a_m) = |a_M - a_0| - |a_m - a_0| \\ &\le L((a_1,a_2)), \end{aligned}$$

from which one obtains, for any $B \subset \Gamma_R$,

$$(1) \qquad m^*(C_R(B)) \le L^*(B).$$

In Section 2 we used the fact that M was less than one to argue that α was less than $\pi/2$ and that $\cos\alpha > K^{-1}$, where $K = (\cos(2\operatorname{Arctan} M))^{-1}$. Such results no longer hold if M is greater than one and thus the argument for (2) of Section 2 fails. However, we prove a slightly weaker result

(2)

For any $a_1 \in \Gamma_R$, $\quad a_1 = (x_1, f(x_1)), L((a_0, a_1)) \le \sqrt{1+M^2}|a_1 - a_0|$.

From (4) of Section 1 we have

$$L((a_0,a_1)) \le \sqrt{1+M^2}|x_1 - x_0|.$$

Letting l be the horizontal line through a_0 and $a_2 = (x_1, f(x_0))$, and β the measure of the angle, $\measuredangle a_1 a_0 a_2$, one has that

$$|x_1 - x_0| = \cos\beta |a_1 - a_0|$$

and since $0 \leq \beta < \pi/2$, (2) follows.

We now show that for any L-measurable $B \subset \Gamma_R$,

$$(3) \quad \lim_{h\to 0} \frac{L(B \cap \Gamma_R(h))}{L(\Gamma_R(h))} = 1 \quad \text{implies} \quad \lim_{r\to 0} \frac{m(C_R(B) \cap L_r(a_0))}{r} = 1.$$

Assuming the first limit is one and using the measurability of B and $(1')$ of Section 1, we have

$$(4) \qquad \lim_{h\to 0} \frac{L(CB \cap \Gamma_R(h))}{L(\Gamma_R(h))} = 0.$$

For each r sufficiently small we consider $\Gamma_R \cap C^r$ where C^r is the circle of radius r centered at a_0. There is a unique point of the intersection, say a_r, having the property that the arc (a_0, a_r) is contained in the interior of C^r. Letting $a_r = (x_r, f(x_r))$ and $h_r = x_r - x_0$, we have

$$(5) \quad (a_0, a_r) = \Gamma_R(h_r), \qquad C_R(\Gamma_R(h_r)) = L_r(a_0), \qquad |a_r - a_0| = r.$$

Using (1), (2) and (5) above, we have

$$(6) \qquad \frac{m(C(C_R(B)) \cap L_R(a_0))}{r} \leq \sqrt{1+M^2}\, \frac{L(CB \cap \Gamma_R(h_r))}{L(\Gamma_R(h_r))}.$$

Now, as $r \to 0$, $h_r \to 0$, it follows from (4), (6) and (1) of Section 1 that

$$\lim_{r\to 0} \frac{m(C_R(B) \cap L_r(a_0))}{r} = 1.$$

Again, similar results hold for C_L and Γ_L.

We now show that the converse of (3) is false.

Start with the segment $[0,1]$ of $\mathbf{R}$ and consider the sequence $\{1/n\}_{n=1}^{\infty}$ of $[0,1]$. For each $n, n = 1, 2, \ldots$, let S_n be the arc of a circle of radius $1/n$, centered at the origin such that the angle at the origin subtended by S_n has radian measure $1/n$ and such that S_n lies in the upper half plane as shown in Figure 3. The initial and terminal points of S_n are $(1/n, 0)$ and $((1/n)\cos(1/n), (1/n)\sin(1/n))$, respectively.

Let L_n be the segment connecting $((1/n)\cos(1/n), (1/n)\sin(1/n))$ to $(1/(n+1), 0)$. The curve formed by considering $U(S_n \cup L_n)$ has the following properties

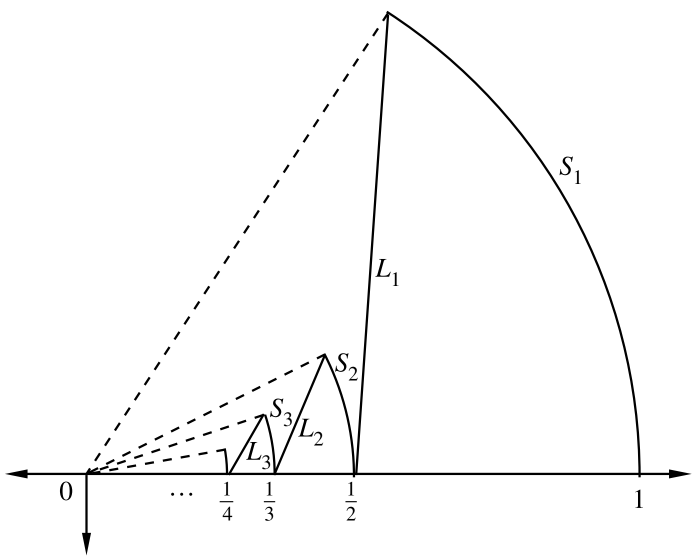

FIGURE 3.

(i) $\lim_{h\to 0}(L(UL_n \cap \Gamma_R(h))/L(\Gamma_R(h))) \neq 1$ and $\lim_{r\to 0}(m(C_R(UL_n) \cap L_r(a_0))/r) = 1$.

(ii) The slope of L_n, denoted m_n, is such that $1 < m_n < 30$ and m_n decreases to 1 as $n \to \infty$.

To get a Lipschitz constant for this curve, we rotate it counterclockwise by η radians, η sufficiently small so that if m'_n denotes the slope of L'_n, the rotated L_n, S'_n the rotated S_n, then

$$m'_n = \frac{m_n + \tan\eta}{1 - m_n \tan\eta} \quad \text{and} \quad \frac{1+\tan\eta}{1-\tan\eta} < m'_n < \frac{30+\tan\eta}{1-30\tan\eta} < 100$$

and the slope at any point of S'_n, $n = 1, 2, 3, \ldots$, in absolute value is less than $(1/\tan\eta)$. We thus have a curve with a Lipschitz constant $M > 1$ such that (i) holds.

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