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AN EULERIAN METHOD FOR REPRESENTING π^2 BY SERIES

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ABSTRACT. Following a method of Euler, the author presents three apparently new series representations of π^2.

1. Introduction. Leonhard Euler was so impressed by his now famous formula

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{1}$$

that he offered several different proofs of it. Among these methods of proof we wish to highlight a certain one, which according to G. Turnwald [**4**, p. 331] was rediscovered by Boo Rim Choe [**2**, p. 662–663] some 244 years later. This method is briefly described as follows:

(i) For $0 \le x < 1$, first evaluate the integral

$$\int_0^x \frac{dt}{\sqrt{1-t^2}}$$

directly, and then evaluate it by expanding the integrand, followed by termwise integration, to get

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{x^{2n+1}}{2n+1}. \tag{2}$$

(ii) In (2) let $x \to \sin x$, integrate the resulting equation from 0 to $\pi/2$, and simplify by appeal to Wallis's formula in the form

$$\int_0^{\pi/2} \sin^{2k+1} x\, dx = \frac{1}{2k+1} \frac{2^{2k}}{\binom{2k}{k}}, \qquad k \ge 0, \tag{3}$$

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to get

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

It does not seem to have been noticed that one can get different series representations of π^2 by replacing the integral under step (i) by other integrals, like

$$\int_0^x \sqrt{1-t^2}\,dt, \qquad \int_0^x t^2\sqrt{1-t^2}\,dt, \qquad \int_0^x \frac{t^2\,dt}{\sqrt{1-t^2}},$$

and thereafter following the method exactly. The three series representations of π^2 corresponding to these three integrals are

$$\pi^2 = 12 - 16\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)^2}, \tag{4}$$

$$\pi^2 = \frac{128}{9} - 128\sum_{k=1}^{\infty} \frac{k+1}{(4k^2-1)(2k+3)^2}, \tag{5}$$

$$\pi^2 = 4 + 32\sum_{k=1}^{\infty} \frac{k}{(2k-1)(2k+1)^2}. \tag{6}$$

In Section 2 we give a detailed proof of (4) and sketch proofs of (5) and (6). We then compare the rates of convergence of the series (4), (5), (6) with that of (1). Finally, in our concluding remarks, we mention a couple of series representations of $1/\pi$, as they, too, are closely related to the method.

2. Proofs of (4), (5), (6). To establish (4), we choose x such that $0 \le x < 1$ and observe, first of all, that

$$\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}\sin^{-1} x + \frac{x}{2}\sqrt{1-x^2}. \tag{7}$$

On the other hand, by Newton's binomial theorem, we expand the integrand and integrate termwise to get

$$\int_0^x \sqrt{1-t^2}\,dt = x - \sum_{k=1}^{\infty} \frac{\binom{2k-2}{k-1}}{k2^{2k-1}} \cdot \frac{x^{2k+1}}{2k+1}. \tag{8}$$

Now, (7) and (8) imply

$$\frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2} = x - \sum_{k=1}^{\infty}\frac{\binom{2k-2}{k-1}}{k2^{2k-1}}\cdot\frac{x^{2k+1}}{2k+1}. \tag{9}$$

In (9), let $x \to \sin x$, and integrate both sides of the resulting equation from 0 to $\pi/2$ to get

$$\frac{\pi^2}{16} + \frac{1}{4} = 1 - \sum_{k=1}^{\infty}\frac{\binom{2k-2}{k-1}}{k2^{k-1}}\cdot\frac{1}{(2k+1)^2}\cdot\frac{2^{2k}}{\binom{2k}{k}} = 1 - \sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)^2},$$

which upon obvious simplification yields (4). (Wallis's formula (3) here plays an important role in the simplification.)

To prove (5), we begin with the expansion

$$t^2(1-t^2)^{1/2} = t^2 - \sum_{j=2}^{\infty}\frac{\binom{2j-4}{j-2}}{(j-1)2^{2j-3}}\cdot t^{2j},$$

and thereafter follow steps (i) and (ii) of the method. And, to see (6), we begin with

$$t^2(1-t^2)^{-\frac{1}{2}} = \sum_{j=1}^{\infty}\frac{\binom{2j-2}{j-1}}{2^{2j-2}}\cdot t^{2j}.$$

(In both expansions, we assume that $|t| < 1$.)

For the series on the right sides of (4), (5), we note that

$$\frac{1}{(2k-1)(2k+1)^2} = 0(k^{-3}), \qquad \frac{k+1}{(4k^2-1)(2k+3)^2} = 0(k^{-3}),$$
$$\text{as } k \to \infty.$$

Hence, both (4) and (5) are better tools for actually approximating π^2 than (1). However, (6) is no better than (1), since the typical terms of both series have the same order of magnitude. We should add that series converging more rapidly than any of these can be found in [**1**, p. 384–385].

Concluding remarks. Euler's method can be modified to yield series representations of $1/\pi$. Two of these are:

$$\frac{1}{\pi} = \frac{1}{2} - \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{\binom{2n}{n}}{n+1}\right)^2 \frac{2n+1}{2^{4n}},$$

$$\frac{1}{\pi} = \frac{3}{16} + \frac{9}{4}\sum_{n=1}^{\infty}\left\{\frac{\binom{2n-2}{n-1}}{n}\right\}^2 \frac{4n^2-1}{2^{4n}(n+1)^2}.$$

Detailed discussions of these series representations are presented elsewhere. However, we should mention the work of Ramanujan [**3**, 36–38], who presented 17 such series representations, 3 of which he proved within the framework of elliptic function theory. Recently, J.M. Borwein and P.B. Borwein [**1**, 177–187] have shown that all of Ramanujan's series representations of $1/\pi$ can be justified within the theory of hypergeometric functions.

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