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QUASI-MULTIPLIERS OF PEDERSEN'S IDEAL

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ABSTRACT. For the Pedersen ideal K of a C^*-algebra A, the space $QM(K)$ of quasi-multipliers of K is investigated. It is shown that $QM(K)$ is a complete involutive locally convex space under a naturally defined topology $\omega\kappa$. A theory is presented whereby $QM(K)$ is represented by a space of densely defined sesqui-linear forms on a Hilbert space.

1. Introduction. A general C^*-algebra A is often considered as a noncommutative analog of the algebra $C_0(X)$ of all continuous functions, on a locally compact space X, vanishing at infinity. Along this line of thought, Pedersen [**4**] showed the existence of a minimal dense ideal $K(A)$, or K, in the C^*-algebra A, corresponding to the ideal $C_{00}(X)$ of $C_0(X)$ consisting of all continuous functions on X having compact support. Using Pedersen's ideal, Lazar and Taylor [**1,2**] have studied the (double) multiplier algebra $\Gamma(K)$ of the ideal K as an analog of the algebra $C(X)$ of all continuous functions on X. This note is an investigation of another analog of $C(X)$—that being the space $QM(K)$ of quasi-multipliers of Pedersen's ideal.

We recall that for an algebra D, a *quasi-multiplier* on D (or quasi-centralizer) is defined to be a bilinear map M from $D \times D$ into D such that for all a, b, c, d in D we have $M(ab, cd) = aM(b, c)d$. Let us denote the space of all quasi-multipliers of D by $QM(D)$. Let D be an algebra with an involution denoted by $*$. For any M in $QM(D)$ we define $M^*(a, b) = M(b^*, a^*)^*$, where $a, b \in D$. Clearly, this $*$ operation satisfies $M^{**} = M$ and $(\lambda M)^* = \bar{\lambda} M^*$ for $M \in QM(D)$, $\lambda \in C$. Hence, we say $QM(D)$ is an involutive vector space. Finally, if D is a normed *-algebra, we may consider the space of *bounded quasi-multipliers* $QM_b(D)$ consisting of all $M \in QM(D)$ for which $\sup\{||M(a, b)|| : a, b \in D, ||a|| \leq 1, ||b|| \leq 1\}$ is finite. Defining $||M||$ to be this finite supremum, we easily see that $QM_b(D)$ is a normed linear

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space. Furthermore, if D is a Banach algebra, then $QM_b(D)$ is complete [**3**, Theorem 2]. The Cohen-Hewitt factorization theorem shows that for a Banach algebra D with a bounded approximate identity, every quasi-multiplier on D is bounded and jointly continuous [**3**, Theorem 1].

2. The structure of QM(K). Restricting the norm $||\cdot||$ of A to K we may consider K as a normed $*$-algebra. The following theorem identifies the space $QM_b(K)$ of bounded quasi-multipliers on K.

Theorem 2.1. *Every bounded quasi-multiplier on K may be extended uniquely to a quasi-multiplier on A.*

Proof. Let $M \in QM_b(K)$. Let $\{u_\alpha\}$ be an approximate identity for A contained in K. For $a, b \in A$ we have that

$$\begin{aligned} &||M(au_\alpha, u_\alpha b) - M(au_\beta, u_\beta b)|| \\ &\quad \leq ||M(au_\alpha, u_\alpha b - u_\beta b)|| + ||M(au_\alpha - au_\beta, u_\beta b)|| \\ &\quad \leq ||M||\,||au_\alpha||\,||u_\alpha b - u_\beta b|| + ||M||\,||au_\alpha - au_\beta||\,||u_\beta b|| \\ &\quad \leq ||M||\,||a||\,||u_\alpha b - u_\beta b|| + ||M||\,||au_\alpha - au_\beta||\,||b||. \end{aligned}$$

Since au_α converges to a and $u_\alpha b$ converges to b, this implies that $M(au_\alpha, u_\alpha b)$ is a Cauchy net in A. Hence, it converges to an element $\overline{M}(a,b)$ in A.

Now it easily follows that $\overline{M}(\cdot,\cdot)$ is in $QM(A)$. A calculation similar to that above shows that $\overline{M}$ is a well-defined extension of M to $A \times A$. Uniqueness follows from the fact that any quasi-multiplier on A is jointly continuous [**3**]. □

From Theorem 2.1 it follows that the map restricting any M in $QM(A)$ to $K \times K$ provides an imbedding of $QM(A)$ into $QM(K)$ with image $QM_b(K)$. Furthermore, the density of the unit ball of K in that of A implies that this imbedding is an isometry of $QM(K)$ onto $QM_b(K)$. Thus, we may consider $QM(A)$ as a subset of $QM(K)$.

Lemma 2.2. *Let B be a hereditary C^*-subalgebra of $K(A)$. Then the restriction of each M in $QM(K)$ to $B \times B$ is an element of $QM(B)$.*

Proof. Let $M \in QM(K)$ and $a, b \in B$. Applying a variant of the Cohen-Hewitt factorization theorem found in [**1**, Lemma 3.1], we may find $c \in B^+$ and $x, y \in B$ such that $a = cx$ and $b = yc$. Then we have $M(a,b) = M(cx, yc) = cM(x,y)c$. Since B is a hereditary C^*-subalgebra, it follows that $M(a,b)$ is in B. Therefore, $M|_{B\times B}$ is in $QM(B)$. □

A similar argument proves the following lemma.

Lemma 2.3. *Let B and B' be hereditary C^*-subalgebras in K such that $B \subseteq B'$. Then the restriction of each M in $QM(B')$ to $B \times B$ is an element of $QM(B)$.*

For B and B' as in the above lemma we define maps p_B from $QM(K)$ into $QM(B)$ and $P_{BB'}$ from $QM(B')$ into $QM(B)$ by the restriction of an appropriate quasi-multiplier to $B \times B$. It follows easily that the maps p_B and $p_{BB'}$ are linear and preserve the $*$ operation.

We define a topology on $QM(K)$ that is an analog of the compact-open topology on $C(X)$. For each $a, b \in K$, define the function ρ_{ab} from $QM(K)$ into $\mathbf{R}^+$ by $\rho_{ab}(M) = ||M(a,b)||$ for all $M \in QM(K)$. It is clear that ρ_{ab} is a seminorm for all a and b. We denote the topology on $QM(K)$ generated by the family $\{\rho_{ab} : a, b \in K\}$ of seminorms by $\omega\kappa$.

Theorem 2.4. *Under the $\omega\kappa$-topology, $QM(K)$ is complete.*

Proof. Let $\{M_\alpha\}$ be an $\omega\kappa$-Cauchy net in $QM(K)$. Then for all $a, b \in K$, the net $\{M_\alpha(a,b)\}$ is a Cauchy net in A under the norm topology. Hence, there exists an element $M(a,b)$ in A to which $\{M_\alpha(a,b)\}$ converges. Since each M_α is a quasi-multiplier, it follows from the uniqueness of the limit that M is also a quasi-multiplier. Thus, $\{M_\alpha\}$ is $\omega\kappa$-convergent to M. Hence, $(QM(K), \omega\kappa)$ is complete. □

A similar argument shows that the $*$-operation on $QM(K)$ is $\omega\kappa$-continuous. Thus, we may say that under $\omega\kappa$, $QM(K)$ is a complete involutive locally convex space.

Theorem 2.5. *Pedersen's ideal K is $\omega\kappa$-dense in $QM(K)$.*

Proof. Let $M \in QM(K)$ and $c, d \in K$. Let B be the hereditary C^*-subalgebra generated by c and d. Proposition 5.6.2 in Pedersen [**5**] shows that $B \subseteq K$. Let $n \in \mathbf{N}$. Since $p_B[M]$ is contained in $QM(B)$, Theorem 5 in McKennon [**3**] shows that there exists an element $a_n \in B^+$ with $||a_n|| = 1$ such that $||cM(a_n, a_n)d - M(c, d)|| < 1/n$. That is, $\{M(a_n, a_n)\}$ converges in the $\omega\kappa$-topology to M. □

Corollary. *The linear space $QM(K)$ is the $\omega\kappa$-completion of K.*

Let B be a hereditary C^*-subalgebra of K. We give $QM(B)$ the quasi-norm topology generated by $\{\rho_{ab} : a, b \in K\}$, where each such seminorm is restricted to $QM(B)$. It is clear that p_B and p_{BB} are continuous in the quasi-norm topologies, where B and B' are hereditary C^*-subalgebras. By the standard construction of the projective limit of locally convex spaces [**6**, Chapter II, sec. 5], we have the following.

Theorem 2.6. *Under the $\omega\kappa$ topology, $QM(K)$ is the projective limit of the family $\{QM(B) : B$ hereditary C^*-subalgebra of $K\}$ with respect to the mappings $\{p_{BB}, : B \subseteq B'$ hereditary C^*-subalgebras$\}$.*

3. A representation theorem for $QM(K)$. Consider A as a nondegenerate C^*-algebra of operators on a Hilbert space H. As usual, let K denote Pedersen's ideal of A. For a set D of operators on H, the linear span of the set $\{d\xi : d \in D, \xi \in H\}$ will be denoted by $[DH]$. The nondegeneracy of A is equivalent to $[AH] = H$. The inner product on H is denoted by $\langle \cdot, \cdot \rangle$.

Lemma 3.1. *The span $[KH]$ is precisely the set $\{a\xi : a \in K, \xi \in H\}$.*

This follows easily from the factorization theorem found in Lazar and Taylor [**1**, Lemma 3.1]. We use H' to represent the span $[KH]$.

Lemma 3.2. *For each $M \in QM(K)$ define the mapping $\hat{M}$ on $H' \times H'$ into* $\mathbf{C}$ *by*

$$\hat{M}(b\xi, a^*\eta) = \langle M(a,b)\xi, \eta\rangle$$

for $b\xi$, $a^*\eta \in H'$. Then $\hat{M}$ is a well-defined sesqui-linear form on H'.

Proof. Let $M \in QM(K)$. Suppose that $a^*\eta$, $b\xi$, $b'\xi' \in H'$ where $b\xi = b'\xi'$. Let B be the hereditary C^*-subalgebra generated by a, b and b'. We can assume B acts on the closed subspace $[BH]$ of H. Since $p_b(M)$ is in $QM(B)$, Proposition 3.12.3 of Pedersen [**5**] shows that there exists a unique operator m_B in the weak closure of B in the bounded operators on $[BH]$ such that $M(c,d) = cm_Bd$ for all $c, d \in B$. Then we have

$$\begin{aligned}\langle M(a,b)\xi, \eta\rangle &= \langle am_Bb\xi, \eta\rangle = \langle am_Bb'\xi', \eta\rangle \\ &= \langle M(a,b')\xi', \eta\rangle.\end{aligned}$$

Similarly, $\hat{M}(b\xi, a^*\eta)$ is independent of the particular representation of $a^*\eta$. Hence, it follows that $\hat{M}$ is well-defined.

Sesqui-linearity of $\hat{M}$ is a result of an application of the now familiar factorization lemma [**1**, Lemma 3.1]. □

Let μ be a sesqui-linear form on H' and $a, b \in K$. Define ${}_a\mu_b$ to be the sesqui-linear form on H given by ${}_a\mu_b\,(\xi, \eta) = \mu(b\xi, a^*\eta)$ for every $\xi, \eta \in H$. When ${}_a\mu_b$ is continuous, we write $\check{\mu}(a,b)$ for that operator guaranteed by the Riesz representation theorem, such that

$${}_a\mu_b\,(\xi, \eta) = \langle \check{\mu}(a,b)\xi, \eta\rangle$$

for all $\xi, \eta \in H$. We denote by $SQ(H')$ the set of all sesqui-linear forms on H' such that ${}_a\mu_b$ is continuous for every $a, b \in K$.

For $\mu \in SQ(H')$ define $\mu^*(b\xi, a^*\eta) = \overline{\mu(a^*\xi, b\eta)}$ for all $a^*\eta, b\xi \in H'$. Then this definition of $*$ makes $SQ(H')$ into an involutive vector space. For $a, b \in K$, define the seminorm $\hat{\rho}_{ab}$ on $SQ(H')$ by

$$\hat{\rho}_{ab}(\mu) = \sup\{|\mu(b\xi, a^*\eta)| : \xi, \eta \in H, ||\xi|| \le 1, ||\eta|| \le 1\}.$$

This family of seminorms defines a topology, denoted $\omega\hat{\kappa}$, on $SQ(H')$.

Theorem 3.3. (Representation Theorem) *Let A be a C^*-algebra of operators on a Hilbert space H such that $[AH] = H$. Let $H' = [KH]$. Then the map $\wedge$ provides a $*$-isomorphism of involutive vector spaces $QM(K)$ onto $SQ(H')$, under the $\omega\kappa$ and $\omega\hat{\kappa}$ topologies, respectively, whose inverse is given by the map v.*

Proof. Let $M \in QM(K)$. For $a, b \in K$, we have

$$_a(\hat{M})_b(\xi, \eta) = \langle M(a,b)\xi, \eta\rangle$$

for all $\xi, \eta \in H$. By the Riesz representation theorem, $M(a,b)$ is the unique bounded operator on H that makes the above equality true. Hence, we obtain that $\hat{M} \in SQ(H')$ and $(\hat{M})^\vee = M$.

On the other hand, take $\mu \in SQ(H')$. For $a, b, c, d \in K$, we have

$$\langle \overset{\vee}{\mu}(ab, cd)\xi, \eta\rangle = \mu(cd\xi, b^*a^*\eta) = \langle \overset{\vee}{\mu}(b,c)d\xi, a^*\eta\rangle = \langle a\overset{\vee}{\mu}(b,c)\, d\xi, \eta\rangle$$

for all $\xi, \eta \in H$. Thus, by uniqueness, we obtain $\overset{\vee}{\mu}(ab, cd) = a\overset{\vee}{\mu}(b,c)d$. The bilinearity of $\overset{\vee}{\mu}$ follows easily from the definition. Thus, $\overset{\vee}{\mu} \in QM(K)$ and $(\overset{\vee}{\mu})^\wedge = \mu$. Hence $\wedge$ is onto and is obviously linear. A calculation similar to that above shows that $\wedge$ preserves the $*$-operation.

Continuity of $\wedge$ and $\vee$ is clear from the definitions of the $\omega\kappa$ and $\omega\hat{\kappa}$-topologies. □

4. Examples. It is clear that the double multiplier algebra $\Gamma(K)$ of Lazar and Taylor is contained in $QM(K)$. The last example will show how different these two spaces can be.

Example 4.1. If A is an abelian C^*-algebra, then $QM(K)$ can be identified with the set $C(\hat{A})$ of all continuous functions on the spectrum $\hat{A}$. The demonstration of this is similar to the argument in Lazar and Taylor [**1**] identifying $\Gamma(K)$ with $C(\hat{A})$.

Example 4.2. Let A be the C^*-algebra $B_0(H)$ of compact operators on a Hilbert space H, with Pedersen's ideal $B_{00}(H)$, the operators of finite rank. Then $QM(K)$ is represented by the set of all sesqui-linear forms on H. This easily follows from two facts. First, that $[KH]$ equals H. Second, if, for any sesqui-linear form μ on H, we define

$$M(t_{\xi\eta}, t_{\xi'\eta'}) = \mu(\xi', \xi) t_{\eta'\eta}$$

for $\xi, \xi', \eta, \eta' \in H$, (where $t_{\xi\mu}(\zeta) = \langle \zeta, \mu \rangle \xi$ for all $\xi, \mu, \zeta \in H$), then linearly extending M to all of K, we obtain a quasi-multiplier M such that $\tilde{M} = \mu$.

Contrast the above result with the fact that the multiplier algebra $\Gamma(K)$ is the set of all bounded operators on H; as Lazar and Taylor show.

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