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REPRESENTATION OF THE ATTAINABLE SET FOR LIPSCHITZIAN DIFFERENTIAL INCLUSIONS

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1. Introduction. In this paper we consider the Cauchy problem

$$\text{(CP)} \qquad x' \in F(t,x), \qquad x(0) = \xi,$$

where F is Lipschitzian with respect to x, with values that are closed (not necessarily convex nor bounded) subsets of $\mathbf{R}^n$ and ξ ranges in a compact subset Ξ of $\mathbf{R}^n$. We show that the map that assigns to each ξ the set of solutions of (CP), $S(\xi)$, can be continuously represented as

$$S(\xi) = g(\xi, \mathcal{U}).$$

The same result holds for the map from ξ to the attainable set at time T, $\mathcal{A}_T(\xi)$, which in general is not a closed set. Similar representations of set valued maps were known in case the values are compact convex; see [**3, 7, 8**].

In order to obtain our representation, we prove first a continuous selection theorem from the map $S(\xi)$, which is more precise than the result presented in [**2**]. Moreover, we do not assume the boundedness of the values of F, and our proof is considerably simpler than the proof in [**2**]. In particular, we do not need either Liapunov's theorem on the range of a vector measure or any previous existence result.

2. Notation and preliminary results. In what follows we denote by $\mathrm{dl}\,(A,B)$ the Hausdorff distance between the sets $A, B \subset \mathbf{R}^n$ (see [**6**]). The distance of a point x from a set A, $d(x,A)$, is $\inf\{|x-a| : a \in A\}$. I is the interval $[0,T]$; the characteristic function of a subset E of I is χ_E. We consider AC the space of absolutely continuous functions from I to $\mathbf{R}^n$ with norm $||x||_{AC} := |x(0)| + \int_0^T |x'(\tau)|\,d\tau$. We assume

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that Ξ is a compact subset of $\mathbf{R}^n$ with diameter D. F is a set valued map from $I \times \mathbf{R}^n$ into the subsets of $\mathbf{R}^n$ satisfying the following

Condition C. (a) the values of F are closed, nonempty subsets of $\mathbf{R}^n$;

(b) $t \mapsto F(t,x)$ is measurable [**5**];

(c) there exists k in $L^1(I)$ such that

$$\mathrm{dl}\,[F(t,x), F(t,x')] \leq k(t)|x - x'| \text{ a.e. on } I;$$

(d) there exists y in AC such that

$$t \mapsto d[y'(t), F(t, y(t))] \quad \text{is in } L^1(I).$$

It is known, from the results of Filippov [**4**] and Himmelberg-Van Vleck [**6**] that, under the above condition, problem (CP) admits at least one AC solution for each ξ in Ξ. We denote the set of all such solutions, with the topology of AC, by $S(\xi)$. The attainable set at T, $\mathcal{A}_T(\xi)$, is the subset of $\mathbf{R}^n$ defined as $\{x(T) : x \in S(\xi)\}$.

To construct the selection we shall use the following

Proposition. *Let $v_0, \dots, v_m$ be in L^1, and let $(I_j(\xi))$ be a partition of I into a finite number of subintervals with endpoints depending continuously on ξ. Consider the map*

$$\varphi : \xi \to \xi + \int_0^t \sum_{j=0}^{m} \chi_{I_j(\xi)}(\tau) v_j(\tau)\, d\tau.$$

Then there exists α in $L^1(I)$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\xi' - \xi| < \delta \text{ implies } |\varphi(\xi')'(t) - \varphi(\xi)'(t)| \leq \alpha(t)\chi_E(t),$$

for some set E with measure $(E) \leq \varepsilon$.

3. Main results.

Theorem. *Let F satisfy condition* (C)*; let s_0 be in $S(\xi_0)$. Then there exists a continuous $\varphi : \Xi \to AC$, a selection from $S(\xi)$, such that $\varphi(\xi_0) = s_0$.*

Proof. The proof is essentially Filippov's construction of successive approximations. As a function of the initial data, each approximation would not be continuous. We modify it in order to obtain continuity, by interpolating through continuous partitions of the interval I, as in [**1**].

$$\text{(a)} \qquad \text{Set } y : \Xi \to AC \text{ to be } y(\xi)(t) := \xi + \int_0^t s_0'(\tau)\, d\tau,$$

and notice that y is continuous and verifies

$$\begin{aligned} d[y(\xi)'(t), F(t, y(\xi)(t))] &= d[(s_0'(t), F(t, y(\xi)(t))] \\ &\le \mathrm{dl}\,[F(t, y(\xi_0)(t)), F(t, y(\xi)(t))] \le k(t)|\xi_0 - \xi|. \end{aligned}$$

Choose $v^0(\xi)(t)$ to be a measurable selection from $F(t, y(\xi)(t))$ [**5**] such that

$$|y(\xi)'(t) - v^0(\xi)(t)| = d[y(\xi)'(t), F(t, y(\xi)(t))] \le k(t)|\xi_0 - \xi|.$$

Hence $v^0(\xi)$ belongs to L^1. Fix some $\eta > 0$ and define

$$\delta(\xi) := \min\{2^{-3}\eta, |\xi - \xi_0|/2\} \text{ for } \xi \ne \xi_0, \qquad \delta(\xi_0) := 2^{-3}\eta.$$

Cover Ξ with balls $B(\xi, \delta(\xi))$, and let $(B(\xi_j, \delta(\xi_j)))_{j=0,\dots,m}$ be a finite subcovering; in particular, ξ_0 belongs only to $B(\xi_0, \delta(\xi_0))$. Let $(p_j)_{j=0,\dots,m}$ be a continuous partition of unity subordinate to this covering, and define $I_0(\xi) := [0, Tp_0(\xi)]$ and, for $j > 0$,

$$I_j(\xi) := [T(p_0(\xi) + \cdots + p_{j-1}(\xi)), T(p_0(\xi) + \cdots + p_j(\xi))].$$

Set

$$y^1(\xi)(t) := \xi + \int_0^t \sum_{j=0}^m \chi_{I_j(\xi)}(\tau) v^0(\xi_j)(\tau)\, d\tau.$$

From the Proposition, it follows that y^1 is continuous from Ξ to AC. Moreover, $y^1(\xi_0) = s_0$, since $I_0(\xi) = [0, T]$. We have

$$\begin{aligned} (1) \quad \int_0^t |y^1(\xi)' - y(\xi)'|\, d\tau &\le \int_0^t \sum_j |v^0(\xi_j) - y(\xi)'|\, d\tau \\ &\le \int_0^t \sum_j \chi_{I_j(\xi)} k(\tau)|\xi_0 - \xi_j|\, d\tau \le Dm(t), \end{aligned}$$

where $m(t) := \int_0^t k(\tau)\,d\tau$.

Fix t and let j be such that $t \in I_j(\xi)$. Then

(2)
$$\begin{aligned} d[y^1(\xi)'(t), F(t, y(\xi)(t))] &= d[v^0(\xi_j)(t), F(t, y(\xi)(t))] \\ &\le \mathrm{dl}\,[F(t, y(\xi_j)(t)), F(t, y(\xi)(t))] \le k(t)|\xi_j - \xi| \\ &\le 2^{-3}\eta k(t). \end{aligned}$$

This estimate is independent of j, hence it holds on I. By the same reasoning,

$$\begin{aligned} (3) \quad d[y^1(\xi)'(t), F(t, y^1(\xi)(t))] &\le d[y^1(\xi)'(t), F(t, y(\xi)(t))] \\ &+ \mathrm{dl}\,[F(t, y(\xi)(t)), F(t, y^1(\xi)(t))] \le k(t)[2^{-3}\eta + Dm(t)]. \end{aligned}$$

(b) In general we claim that for $n = 1, 2, \ldots$, we can define a continuous map $y^n : \Xi \to AC$ verifying $y^n(\xi_0) = s_0$ and

(i)
$$\int_0^t |y^n(\xi)' - y^{n-1}(\xi)'|\,d\tau \le D\frac{m^n(t)}{n!} + \eta 2^{-n-1}\left[2^{-2} + \sum_{i=1}^{n} \frac{(2m(t))^i}{i!}\right];$$

(ii)
$$d[y^n(\xi)'(t), F(t, y^{n-1}(\xi)(t))] \le \eta 2^{-n-2} k(t);$$

(iii)
$$d[y^n(\xi)'(t), F(t, y^n(\xi)(t))] \le Dk(t)\frac{m^n(t)}{n!} + \eta 2^{-n-1} k(t) \sum_{i=0}^{n} \frac{(2m(t))^i}{i!};$$

(iv) there exists α^n in L^1 such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\xi' - \xi| < \delta \quad \text{implies} \quad |y^n(\xi)'(t) - y^n(\xi')'(t)| \le \alpha^n(t)\chi_E(t),$$

for some $E \subset I$ with measure $(E) \le \varepsilon$.

From the definition of y^1 and the Proposition, this claim holds for $n = 1$. Assume that it holds for $n-1$.

Choose $v^{n-1}(\xi)(t) \in F(t, y^{n-1}(\xi)(t))$ such that

$$\begin{aligned}|y^{n-1}(\xi)'(t) - v^{n-1}(\xi)(t)| &= d[y^{n-1}(\xi)'(t), F(t, y^{n-1}(\xi)(t))] \\ &\leq Dk(t)\frac{m^{n-1}(t)}{(n-1)!} + \eta 2^{-n} k(t) \sum_{i=0}^{n-1} \frac{(2m(t))^i}{i!}.\end{aligned}$$

By (iv) of the recursive hypothesis, there exists $\delta_n > 0$ such that $|\xi' - \xi| < \delta_n$ implies

$$|y^{n-1}(\xi')'(t) - y^{n-1}(\xi)'(t)| \leq \alpha^{n-1}(t)\chi_E(t),$$

for some E such that $\int_E \alpha^{n-1}(t)\,dt \leq \eta 2^{-n-3}$.

Define

$$\begin{aligned}\delta_n(\xi) &:= \min\{\delta_n, 2^{-n-3}\eta, |\xi - \xi_0|/2\} \quad \text{for } \xi \neq \xi_0, \\ \delta_n(\xi_0) &:= \min\{\delta_n, 2^{-n-3}\eta\}.\end{aligned}$$

Cover Ξ with balls $B(\xi, \delta_n(\xi))$ and let

$$B(\xi_j^n, \delta_n(\xi_j^n)), \qquad j = 0, \dots, m_n,\ \xi_0^n = \xi_0,$$

be a finite subcover; in particular, ξ_0 belongs only to $B(\xi_0, \delta_n(\xi_0))$. Let $(p_j^n)_{j=0,\dots,m_n}$, be a continuous partition of unity subordinate to this covering, and define $I_0^n(\xi) := [0, Tp_0^n(\xi)]$ and, for $j > 0$,

$$I_j^n(\xi) := [T(p_0^n(\xi) + \cdots + p_{j-1}^n(\xi)), T(p_0^n(\xi) + \cdots + p_j^n(\xi))].$$

Set

$$y^n(\xi)(t) := \xi + \int_0^t \sum_{j=0}^{m_n} \chi_{I_j^n(\xi)}(\tau) v^{n-1}(\xi_j^n)(\tau)\,d\tau.$$

From the Proposition, it follows that y^n is continuous from Ξ into AC. Moreover, $y^n(\xi_0) = s_0$ since $I_0^n(\xi_0) = [0, T]$. We have

$$\begin{aligned}
&\int_0^t |y^n(\xi)' - y^{n-1}(\xi)'|\, d\tau \\
&\quad\le \int_0^t \sum_j \chi_{I_j^n(\xi)} |v^{n-1}(\xi_j^n) - y^{n-1}(\xi)'|\, d\tau \\
&\quad\le \int_0^t \sum_j \chi_{I_j^n(\xi)} |v^{n-1}(\xi_j^n) - y^{n-1}(\xi_j^n)'|\, d\tau \\
&\qquad + \int_0^t \sum_j \chi_{I_j^n(\xi)} |y^{n-1}(\xi_j^n)' - y^{n-1}(\xi)'|\, d\tau \\
&\quad\le \int_0^t \Big(\sum_j \chi_{I_j^n(\xi)}\Big) \left[Dk(t)\frac{m^{n-1}(t)}{(n-1)!} + \eta 2^{-n} k(t) \cdot \sum_{i=0}^{n-1} \frac{(2m(t))^i}{i!} \right] d\tau \\
&\qquad + \int_0^t \Big(\sum_j \chi_{I_j^n(\xi)}\Big) \alpha^{n-1}(\tau)\chi_E(\tau)\, d\tau \\
&\quad\le D\frac{m^n(t)}{n!} + \eta 2^{-n-1} \sum_{i=1}^{n} \frac{(2m(t))^i}{i!} + \eta 2^{-n-3}.
\end{aligned}$$

Hence, point (i) of the recursive hypothesis holds. Fix t and let j be such that $t \in I_j^n(\xi)$. Then

$$\begin{aligned}
&d[y^n(\xi)'(t), F(t, y^{n-1}(\xi)(t))] \\
&\quad = d[v^{n-1}(\xi_j^n)(t), F(t, y^{n-1}(\xi)(t))] \\
&\quad \le \mathrm{dl}\,[F(t, y^{n-1}(\xi_j^n)(t)), F(t, y^{n-1}(\xi)(t))] \\
&\quad \le k(t)\left[|\xi_j^n - \xi| + \int_0^t |y^{n-1}(\xi_j^n)' - y^{n-1}(\xi)'|\, d\tau \right] \\
&\quad \le k(t)[\eta 2^{-n-3} + \eta 2^{-n-3}] = \eta 2^{-n-2} k(t).
\end{aligned}$$

This estimate is independent of j, so it holds on I. Thus (i) is proved.

By the same reasoning,

$$\begin{aligned} d[y^n(\xi)'(t), F(t, y^n(\xi)(t))] &\le d[y^n(\xi)'(t), F(t, y^{n-1}(\xi)(t))] \\ &\quad + \mathrm{dl}\,[F(t, y^{n-1}(\xi)(t)), F(t, y^n(\xi)(t))] \\ &\le k(t)\Bigg[\eta 2^{-n-2} + D\frac{m^n(t)}{(n)!} \\ &\qquad + \eta 2^{-n-1}\sum_{i=1}^{n}\frac{(2m(t))^i}{i!} + \eta 2^{-n-3}\Bigg] \\ &\le Dk(t)\frac{m^n(t)}{n!} + \eta 2^{-n-1}k(t)\sum_{i=0}^{n}\frac{(2m(t))^i}{i!}. \end{aligned}$$

Applying the Proposition to y^n the recurrence is completed.

(c) From (i) we have that

$$||y^n(\xi) - y^{n-1}(\xi)||_{AC} \le D\frac{m^n(t)}{n!} + \eta 2^{-n-1}e^{2m(t)},$$

so that the sequence of continuous functions $y^n : I \to AC$ converges uniformly to a continuous function φ such that $\varphi(\xi_0) = s_0$. By (iii), $\varphi(\xi)$ belongs to $S(\xi)$. □

The following corollaries show that the solution set map $S(\xi)$ and the attainable set map $\mathcal{A}_T(\xi)$ can be continuously parametrized, and in particular that they are analytic sets.

Corollary 1. *There exists a closed subset $\mathcal{U}$ of a separable Banach space X and a continuous function $g : \Xi \times \mathcal{U} \to AC$ such that $g(\xi, \mathcal{U}) = S(\xi)$ for any ξ in Ξ.*

Proof. Set X to be the separable Banach space of continuous maps φ from the compact Ξ into the separable Banach space AC, with the usual sup norm, and let $\mathcal{U} \subset X$ be the set of continuous selections from the map $\xi \to S(\xi)$. Define g to be the evaluation map $g(\xi, u) := u(\xi)$. Then the continuity of g is obvious, and the above theorem gives $g(\xi, \mathcal{U}) = S(\xi)$. □

Corollary 2. *There exists a closed subset $\mathcal{U}$ of a separable Banach space X and a continuous function $h : \Xi \times \mathcal{U} \to \mathbf{R}^n$ such that $h(\xi, \mathcal{U}) = \mathcal{A}_T(\xi)$ for any ξ in Ξ.*

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