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EXAMPLES OF CYLINDRICAL SHOCK WAVE CONVERSION BY FOCUSING

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1. Introduction. Cylindrical waves are solutions of the wave equation in two space dimensions:

$$U_{tt} - U_{xx} - U_{yy} = 0. \tag{1.1}$$

Cylindrical shock waves are discontinuous solutions of (1.1). The theory of shock waves was initiated by B. Riemann in the nineteenth century and developed by H. Hugoniot and J. Hadamard [**9, 10**]. A simple and general framework in which to develop a theory of shock waves became available in 1950–1951 with the appearance of L. Schwartz's theory of distributions [**16**].

The purpose of this expository article is to present examples of cylindrical shock waves and their conversion into new shock waves by the process of focusing. The construction of the examples is based directly on the concepts and results of the Schwartz theory. All the examples are rotationally symmetric cylindrical waves; that is, waves of the special form

$$U(x,y,t) = u(r,t), \quad \text{where} \quad r = \sqrt{x^2+y^2}. \tag{1.2}$$

Examples are constructed of waves (1.2) that are piecewise analytic for $t < 0$ with finite jump discontinuities on the backward wave cone

$$\Gamma_- = \{(x,y,t) \mid t + r = 0 \text{ and } t \le 0\} \tag{1.3}$$

in three-dimensional space-time. Physically, the sections of Γ_- by the planes $t = t_0 < 0$ correspond to a family of contracting concentric circular wave fronts that converge to a focus at the origin at time $t = 0$. It is shown below that, after passing through the focus at time $t = 0$, the wave function $u(r,t)$ tends to infinity logarithmically at every point of the forward wave cone

$$\Gamma_+ = \{(x,y,t) \mid t - r = 0 \text{ and } t \ge 0\}. \tag{1.4}$$

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Physically, the sections of Γ_+ by the planes $t = t_0 > 0$ correspond to a family of expanding concentric circular wave fronts that bear a logarithmically infinite shock wave. Note that in space-time the singularities on Γ_+ are a barrier to continuing $u(r,t)$ into the interior of Γ_+. Elementary methods provide no way to pass this barrier. The power of Schwartz's theory will be demonstrated by showing that it provides a unique continuation of $u(r,t)$ to all of space-time.

Another effect of focusing is a reduction in the degree of differentiability of $u(r,t)$ at the focus. It is shown below that if $u(r,0) \in C^k[0,\infty)$ and $u_t(r,0) \equiv 0$, then $u(0,t) \in C^{k-1/2}[0,\infty)$. Here use is made of the theory of fractional derivatives [**3**] and k may be any real number such that $k \geq 2.5$.

2. The Cauchy problem for cylindrical waves. Cylindrical shock waves are constructed below by solving the Cauchy problem for the wave equation (1.1) with discontinuous Cauchy data,

$$U(x,y,0) = F(x,y), \qquad U_t(x,y,0) = G(x,y) \qquad \forall (x,y) \in \mathbf{R}^2. \tag{2.1}$$

A solution of (1.1), (2.1) is interpreted as a Schwartz distribution $U \in \mathcal{D}'(\mathbf{R}^3)$ that satisfies

$$U_{tt} - U_{xx} - U_{yy} = F \otimes \delta' + G \otimes \delta \quad \text{in } \mathcal{D}'(\mathbf{R}^3), \tag{2.2}$$

and

$$\operatorname{supp} U \subset \mathbf{R}^2 \times [0,\infty), \tag{2.3}$$

where the derivatives in (2.2) are understood in the distribution-theoretic sense. The fundamental theorem for the problem (2.2), (2.3) is

Theorem 1. *The Cauchy problem* (2.2), (2.3) *has a unique solution* $U \in \mathcal{D}'(\mathbf{R}^3)$ *for every pair* $F, G \in \mathcal{D}'(\mathbf{R}^2)$.

This result has been known since the early days of distribution theory. An elementary proof of the theorem, and references to the literature, are contained in a recent paper by the author [**18**]. Theorem 1 is used below to construct shock waves and to characterize their unique continuations beyond the singularities that are produced by focusing.

If the distributions F, G in (2.2) are generated by locally integrable functions, then equation (2.2) is equivalent to the identity

$$\text{(2.4)}\quad U(\Phi_{tt} - \Phi_{xx} - \Phi_{yy}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{-F(x,y)\Phi_t(x,y,0) + G(x,y)\Phi(x,y,0)\}\,dx\,dy$$

where $\Phi \in \mathcal{D}(\mathbf{R}^3) = C_0^\infty(\mathbf{R}^3)$ is an arbitrary Schwartz testing function. In particular, if $U \in C^2(\mathbf{R}^2 \times [0,\infty))$, then (2.2), (2.3) are equivalent to the classical wave equation (1.1) and the initial conditions (2.1). The proof is by integration by parts.

3. The focusing of shock waves. The purpose of this section is to construct a rotationally symmetric solution of the wave equation that has a finite jump discontinuity on the backward wave cone Γ_-. Note that (1.2) implies that

$$\text{(3.1)}\qquad u(r,t) = \frac{1}{2\pi}\int_0^{2\pi} U(r\cos\theta, r\sin\theta, t)\,d\theta.$$

Moreover, the right hand side of (3.1) is meaningful for all $r \in \mathbf{R}$ and defines an even function of r. Hence, if $U(x,y,t)$ is defined for all $(x,y,t) \in \mathbf{R}^2 \times (t_1,t_2)$, then $u(r,t)$ is defined for all $(r,t) \in \mathbf{R} \times (t_1,t_2)$. In addition, (3.1) implies that

$$\text{(3.2)}\quad U(x,y,t) \in C^k(\mathbf{R}^2 \times (t_1,t_2)) \implies u \in C^k(\mathbf{R} \times (t_1,t_2)) \qquad \forall k = 0,1,2,\dots .$$

Finally, the usual representation of the Laplace operator in polar coordinates implies that if U satisfies the wave equation (1.1), then u satisfies

$$\text{(3.3)}\qquad u_{tt} - u_{rr} - \frac{1}{r}u_r = 0.$$

Equation (3.3) is sometimes called Darboux's equation. It is simply the wave equation for a rotationally symmetric function $u(r,t)$.

The principal example of this article is a solution of (3.3) that has finite jump discontinuities at the points of the backward wave cone Γ_- and vanishes in its interior. The search for such a solution is complicated by the fact that equation (3.3) has no pure progressing wave solutions. Hence, a *generalized progressing wave*, of the type emphasized by F.G. Friedlander [**5**, p. 63], will be sought. The characteristic cone Γ_- has the equation $t + r = 0$. Thus, Friedlander's expansions suggest the *Ansatz*

$$u(r,t) = H(r+t)\sum_{n=0}^{\infty} A_n(r)(t+r)^n, \qquad \forall t < 0, \tag{3.4}$$

where $H(x)$ denotes Heaviside's function:

$$H(x) = 0 \quad \forall x < 0, \quad \text{and} \quad H(x) = 1 \quad \forall x > 0. \tag{3.5}$$

Equation (3.4) implies that the jump in $u(r,t)$ across the wave front Γ_- is

$$[u](r) \equiv u(r, -r+0) - u(r, -r-0) = A_0(r) \qquad \forall r > 0. \tag{3.6}$$

Moreover, Friedlander's transport equation [**5**, p. 51], specialized to circular wave fronts, gives $[u](r) = a_0/r^{1/2}$. Hence, one has

$$A_0(r) = a_0/r^{1/2}. \tag{3.7}$$

Next, substituting (3.4) into (3.3) gives, after some algebra, the recursion relation

$$A_n'' + \frac{1}{r}A_n' + (n+1)\left(2A_{n+1}' + \frac{1}{r}A_{n+1}\right) = 0 \qquad \forall n = 0, 1, 2, \dots, \tag{3.8}$$

where the primes denote r-derivatives. Combining this equation with $n = 0$ and equation (3.7) yields a solution $A_1(r) = a_1/r^{3/2}$. Proceeding in this way, one finds that (3.7), (3.8) have solutions of the form

$$A_n(r) = a_n/r^{n+1/2} \qquad \forall n = 0, 1, 2, \dots, \tag{3.9}$$

where a_0 is an arbitrary constant and the remaining constants are determined by the recursion relation

$$\frac{a_{n+1}}{a_n} = \frac{1}{2}\left(\frac{n+1/2}{n+1}\right)^2, \qquad \forall n = 0, 1, 2, \dots. \tag{3.10}$$

It is easy to verify that the solution of (3.10) is

$$a_n = \frac{[(1/2)(3/2)(5/2)\cdots((2n-1)/2)]^2}{(n!)^2}\left(\frac{1}{2^n}\right)a_0. \tag{3.11}$$

On taking $a_0 = 1/2^{1/2}$ in (3.11) and substituting (3.11) and (3.9) in (3.4), one finds that

$$u(r,t) = H(r+t)\frac{1}{(2r)^{1/2}}\sum_{n=0}^{\infty}\frac{[(1/2)(3/2)(5/2)\cdots((2n-1)/2)]^2}{(n!)^2}\left(\frac{t+r}{2r}\right)^n. \tag{3.12}$$

Up to this point, the calculation of $u(r,t)$ has been purely formal. However, on comparing the series in (3.12) with the hypergeometric series

$$F(a,b,c,z) = \sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)n!}z^n, \tag{3.13}$$

one finds that

$$u(r,t) = \frac{H(r+t)}{(2r)^{1/2}}F(1/2,1/2,1,(t+r)/(2r)). \tag{3.14}$$

Solutions of the wave equation of this type were first mentioned to the author by Dr. R.N. Buchal. Recall that the hypergeometric series, when it does not terminate, converges in the unit disk $|z| < 1$ in the complex plane. It follows that the function $F(1/2,1/2,1,(t+r)/(2r))$ is defined and analytic for $-3r < t < r$, $r > 0$. Hence, equation (3.14) defines a function $u(r,t)$ in the space-time domain $t < r$ with boundary Γ_+. Moreover, $u(r,t)$ is piecewise analytic there, with finite jump discontinuity $[u](r) = 1/(2r)^{1/2}$ on the backward wave cone Γ_- and satisfies the wave equation except on Γ_-. The last statement follows from the derivation and can also be verified by direct differentiation. An independent verification that (3.14) defines a distribution solution of the wave equation in the domain $t < r$, $r \geq 0$, is given in Section 6 below.

The function $u(r,t)$ defined by (3.14) is the principal example of a converging cylindrical wave that will be studied here. The converging wave fronts $r + t = 0$, $t < 0$, have a focus at the origin at $t = 0$ and a singularity may be expected there. In fact, if (r,t) approaches the origin along the straight line $(t+r)/2r = k$, $0 < k < 1$, then $t = (2k-1)r$, and hence one has

$$u(r,(2k-1)r) = \frac{1}{(2r)^{1/2}} F(1/2,1/2,1,k) \to \infty \quad \text{when } r \to 0. \tag{3.15}$$

Thus, $u(r,t)$ does indeed have an unbounded singularity at the focus. It is surprising that it also has an unbounded singularity at every point of the forward wave cone Γ_+! It is clear that if $(r,t) \to (r_0,r_0)$, $r_0 > 0$, then in (3.14) one has $(t+r)/2r \to 1$, a point on the circle of convergence of the hypergeometric series, and hence one may expect that $u(r,t) \to \infty$. To make a more precise statement, the identity [**4**, p. 318]

$$F(1/2,1/2,1,k^2) = \frac{2}{\pi}\mathbf{K}(k), \qquad \forall |k| < 1, \tag{3.16}$$

may be used. Here $\mathbf{K}(k)$ is the complete elliptic integral of the first kind, defined by

$$\mathbf{K}(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad \forall |k| < 1. \tag{3.17}$$

It is known that [**4**]

$$\log 4 < \mathbf{K}(k) + \log\sqrt{1-k^2} < \frac{\pi}{2}. \tag{3.18}$$

Combining this with (3.14) and (3.16) gives, after some simplification,

$$u(r,t) = \frac{-\log(r-t)}{\pi(2r)^{1/2}} + O(1), \quad \text{for } r-t \to 0+,\ t \geq t_0 > 0. \tag{3.19}$$

In particular, $u(r,t) \to +\infty$ at every point of Γ_+.

The singularities of $u(r,t)$ on Γ_+ are a barrier to its continuation, by elementary methods, into the interior of Γ^+. Nevertheless, a unique continuation of $u(r,t)$ to all of space-time, as a Schwartz distribution, is guaranteed by Theorem 1 above. To see this, one need only note that the Cauchy data $u(r,t_0)$, $u_t(r,t_0)$ are piecewise analytic, and hence in $\mathcal{D}'(\mathbf{R}^2)$, for any $t_0 < 0$. The Cauchy problem with rotational symmetry is solved in the next section. The results are used in Section 6 to derive an explicit representation of $u(r,t)$ in the interior of Γ_+.

4. The Cauchy problem with rotationally symmetric smooth data. An explicit solution of the Cauchy problem for the wave equation (1.1) is constructed in this section for the case of rotationally symmetric Cauchy data (2.1):

$$F(x,y) = f(r), \quad G(x,y) = g(r) \quad \text{where } r = \sqrt{x^2+y^2}. \tag{4.1}$$

Of course, the classical Parseval formula provides a representation of the solution as a double integral [**18**]. However, instead of evaluating this integral it will be more expedient to use another method that makes use of the symmetry of the data.

To begin, the case of smooth data $F, G \in C^\infty(\mathbf{R}^2)$ and corresponding classical solution $U(x,y,t) \in C^\infty(\mathbf{R}^2 \times [0,\infty))$ is considered. Note that (4.1) implies

$$U(x,y,t) = u(r,t). \tag{4.2}$$

To see this, note that (4.1) implies that $U(r\cos(\theta+\alpha), r\sin(\theta+\alpha), t)$ is a solution of the Cauchy problem with data (4.1) for every fixed value of α. It follows from the uniqueness of classical solutions [**18**] that $U(x,y,t) = U(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha, t)$ for every α. Choosing $\alpha = -\theta$ gives $U(x,y,t) = U(r,0,t)$ which is equivalent to (4.2).

The preceding remarks imply that the solution $u(r,t)$ corresponding to smooth symmetric data (4.1) is characterized by the properties

$$u(r,t) \in C^\infty(\mathbf{R} \times [0,\infty)) \quad \text{and} \quad u(-r,t) = u(r,t), \tag{4.3}$$

$$u_{tt} - u_{rr} - \frac{1}{r}u_r = 0 \qquad \forall (r,t) \in \mathbf{R} \times [0,\infty), \tag{4.4}$$

$$u(r,0) = f(r), \qquad u_t(r,0) = g(r) \qquad \forall r \in \mathbf{R}. \tag{4.5}$$

Of course, the limiting form of the equation (4.4) holds at $r = 0$.

The solution method makes use of the linear integral operator L defined by

$$\mathcal{F}(s) \equiv L\{f\}(s) = \int_0^s \frac{rf(r)}{\sqrt{s^2 - r^2}}\, dr. \tag{4.6}$$

A second representation of $\mathcal{F}$ which is useful below is obtained by making the change of variable $r = sx$ in (4.6). The result is

$$\mathcal{F}(s) = s\int_0^1 \frac{xf(sx)}{\sqrt{1 - x^2}}\, dx. \tag{4.7}$$

The operator L was used extensively in the classical literature to integrate Darboux's equation (4.4); see, e.g., [**2**, p. 700]. Its utility is based on

Theorem 2. *The operator L has the following properties*:

$$f \in C^k(\mathbf{R}) \Longrightarrow \mathcal{F} \in C^k(\mathbf{R}) \qquad \forall 0 \le k \le \infty, \tag{4.8}$$

$$f \text{ even } \Longrightarrow \mathcal{F} \text{ odd}, \tag{4.9}$$

$$L\left\{f''(r) + \frac{1}{r}f'(r)\right\} = \mathcal{F}'' \qquad \forall f \in C^2(\mathbf{R}), \tag{4.10}$$

$$\begin{aligned} f(r) = L^{-1}\{\mathcal{F}\}(r) &= \left(\frac{2}{\pi}\right)\frac{1}{r}\frac{\partial}{\partial r}\int_0^r \frac{s\mathcal{F}(s)}{\sqrt{r^2 - s^2}}\, ds \\ &= \left(\frac{2}{\pi}\right)\frac{1}{r}\frac{\partial}{\partial r}L\{\mathcal{F}\}(r) \qquad \forall r > 0, \end{aligned} \tag{4.11}$$

while $f(0) = \mathcal{F}'(0)$.

Proof. Properties (4.8), (4.9) are clear from the representation (4.7). To verify (4.10) note that by (4.7) one has
(4.12)

$$\begin{aligned} \mathcal{F}'(s) &= \int_0^1 \frac{xf(sx)}{\sqrt{1-x^2}}\, dx + s\int_0^1 \frac{x^2 f'(sx)}{\sqrt{1-x^2}}\, dx \\ &= \int_0^1 \frac{xf(sx)}{\sqrt{1-x^2}}\, dx + s\int_0^1 \frac{f'(sx)}{\sqrt{1-x^2}}\, dx - s\int_0^1 f'(sx)\sqrt{1-x^2}\, dx. \end{aligned}$$

On integrating by parts in the last integral, one finds, after simplification,

$$\mathcal{F}'(s) = f(0) + s\int_0^1 \frac{f'(sx)}{\sqrt{1-x^2}}\,dx. \tag{4.13}$$

A second differentiation then gives

$$\begin{aligned}\mathcal{F}''(s) &= \int_0^1 \frac{f'(sx)}{\sqrt{1-x^2}}\,dx + s\int_0^1 \frac{xf''(sx)}{\sqrt{1-x^2}}\,dx \\ &= s\int_0^1 \frac{xf''(sx)}{\sqrt{1-x^2}}\,dx + s\int_0^1 \frac{x(f'(sx)/sx)}{\sqrt{1-x^2}}\,dx,\end{aligned} \tag{4.14}$$

which is equivalent to (4.10).

Finally, the verification of (4.11) is based on the theory of fractional integration. The Riemann–Liouville fractional integral of order α is defined by

$$f_\alpha^+(x) = \frac{1}{\Gamma(\alpha)}\int_0^x f(x)(x-t)^{\alpha-1}\,dt \qquad \forall\alpha > 0. \tag{4.15}$$

The operator has the property [**3**]

$$(f_\alpha^+)_\beta^+ = f_{\alpha+\beta}^+ \qquad \forall\alpha,\beta > 0. \tag{4.16}$$

This may be applied to the operator L by making the changes of variable $r^2 = \rho$, $s^2 = \sigma$ in (4.6). On has

$$\mathcal{F}(\sqrt{\sigma}) = \frac{1}{2}\int_0^\sigma \frac{f(\sqrt{\rho})}{\sqrt{\sigma-\rho}}\,d\rho = \frac{\sqrt{\pi}}{2}(f(\sqrt{\rho}))_{1/2}^+(\sigma), \tag{4.17}$$

since $\Gamma(1/2) = \sqrt{\pi}$. Applying the operator $(\ )_{1/2}^+$ to (4.17) and using property (4.16) gives

$$(\mathcal{F}(\sqrt{\sigma}))_{1/2}^+(\rho) = \frac{\sqrt{\pi}}{2}(f(\sqrt{x}))_1^+(\rho) = \frac{\sqrt{\pi}}{2}\int_0^\rho f(\sqrt{x})\,dx. \tag{4.18}$$

Hence, one has

$$f(\sqrt{\rho}) = \frac{2}{\sqrt{\pi}}\left(\frac{\partial}{\partial\rho}\right)(\mathcal{F}(\sqrt{\sigma}))_{1/2}^+(\rho), \tag{4.19}$$

and, therefore,

$$f(r) = \left(\frac{2}{\pi}\right) \frac{1}{r} \frac{\partial}{\partial r} \int_0^r \frac{s\mathcal{F}(s)}{\sqrt{r^2 - s^2}}\, ds \qquad \forall r > 0. \tag{4.20}$$

Note that the limiting value of $f(r)$ for $r \to 0$ may be inferred from equation (4.13): It is $f(0) = \mathcal{F}'(0)$. This completes the proof of (4.11). $\square$

Theorem 2 will be used to construct the solution $u(r,t) \in C^\infty(\mathbf{R}^2 \times [0,\infty))$ of the Cauchy problem (4.3)–(4.5) with data that satisfy

$$f(r), g(r) \in C^\infty(\mathbf{R}) \quad \text{and} \quad f(-r) = f(r),\ g(-r) = g(r) \quad \forall r \in \mathbf{R}. \tag{4.21}$$

The notation

$$v(s,t) = L\{u(r,t)\}(s) = \int_0^s \frac{ru(r,t)}{\sqrt{s^2 - r^2}}\, dr = s \int_0^1 \frac{xu(sx,t)}{\sqrt{1 - x^2}}\, dx \tag{4.22}$$

will be used. It follows from Theorem 2 that $v(s,t)$ has the properties

$$v(s,t) \in C^\infty(\mathbf{R} \times [0,\infty)) \quad \text{and} \quad v(-s,t) \equiv -v(s,t), \tag{4.23}$$

$$v_{tt} - v_{ss} = 0 \qquad \forall (s,t) \in \mathbf{R} \times [0,\infty), \tag{4.24}$$

$$v(s,0) = \mathcal{F}(s), \qquad v_t(s,0) = G(s) \qquad \forall s \in \mathbf{R}, \tag{4.25}$$

where $\mathcal{F} = L\{f\}$ and $G = L\{g\}$ are in $C^\infty(\mathbf{R})$ and are odd functions. Properties (4.23)–(4.25) imply that $v(s,t)$ is given by d'Alembert's formula

$$v(s,t) = \frac{1}{2}\left(\mathcal{F}(s+t) + \mathcal{F}(s-t) + \int_{s-t}^{s+t} G(x)\, dx\right). \tag{4.26}$$

Inverting the L-transform by means of Theorem 2, equation (4.11) gives

$$u(r,t) = \left(\frac{2}{\pi}\right) \frac{1}{r} \frac{\partial}{\partial r} \int_0^r \frac{sv(s,t)}{\sqrt{r^2 - s^2}}\, ds \qquad \forall r \neq 0, \tag{4.27}$$

and

$$u(0,t) = v_s(0,t). \tag{4.28}$$

Note that for data $f(r), g(r)$ that satisfy (4.21), the function $v(s,t)$ is defined by (4.26) for *all* real s and t and $v(s,t) \in C^\infty(\mathbf{R}^2)$. This implies

Theorem 3. *For all* $f(\sqrt{x^2+y^2}), g(\sqrt{x^2+y^2}) \in C^\infty(\mathbf{R}^2)$*, equations* (4.27), (4.28) *define a function* $u(r,t) \in C^\infty(\mathbf{R}^2)$ *which satisfies the wave equation everywhere in three-dimensional space-time and has the desired initial values* (4.5)*.*

The proof is a direct verification, based on Theorem 2. The wave equation (4.4) follows on calculating u_{tt} from (4.27), by differentiating under the integral sign and then using (4.24) and (4.10). The initial values (4.5) follow from (4.11) applied to $u(r,0)$ and $u_t(r,0)$.

5. The Cauchy problem with rotationally symmetric data in $L_p^{\rm loc}, p > 1$. The solution formulas (4.26), (4.27) have been established for smooth data only. The purpose of this section is to extend their validity to a class of data that includes the focusing shock wave (3.14) of Section 3. If Ω is a domain in $\mathbf{R}^n$ it will be convenient to use the notation $L_p^{\rm loc}(\Omega)$ to denote the set of all measurable functions $f : \Omega \to \mathbf{R}$ that are in $L_p(K \cap \Omega)$ for all compact sets $K \subset \mathbf{R}^n$. Note that the Cauchy data for the soluton (3.14) at $t = 0$ are

$$u(r,0) \equiv f(r) = \frac{1}{(2r)^{1/2}} F(1/2, 1/2, 1, 1/2) = \frac{c}{r^{1/2}}, \tag{5.1}$$

$$u_t(r,0) \equiv g(r) = \frac{1}{(2r)^{3/2}} F'(1/2, 1/2, 1, 1/2) = \frac{c'}{r^{3/2}}, \tag{5.2}$$

where c and c' are finite constants. It follows that $f(\sqrt{x^2+y^2}) \in L_p^{\rm loc}(\mathbf{R}^2)$ for $1 \le p < 4$ and $g(\sqrt{x^2+y^2}) \in L_p^{\rm loc}(\mathbf{R}^2)$ for $1 \le p < 4/3$. For the study of shock waves it might seem simplest to work with $p = 1$. However, for a technical reason, which will appear below, it will be expedient to choose a value $p > 1$.

The first goal of this section is to show that if

$$f(\sqrt{x^2+y^2}), g(\sqrt{x^2+y^2}) \in L_p^{\rm loc}(\mathbf{R}^2, dx\,dy) \quad \text{for some } p>1, \tag{5.3}$$

then the equation

$$w(r,t) = \begin{cases} \int_0^r \frac{sv(s,t)}{\sqrt{r^2-s^2}}\,ds, & \text{for } t \ge 0, \\ 0, & \text{for } t<0, \end{cases} \tag{5.4}$$

defines a function in $L_p^{\rm loc}(\mathbf{R}^2, dr\,dt)$. Note that to verify this it will suffice to show that

$$\int_0^{\rm R}\int_0^{\rm R} |w(r,t)|^p\,dr\,dt < \infty \qquad \forall {\rm R} > 1. \tag{5.5}$$

The general case then follows because $w(r,t)$ is even in r and vanishes for $t<0$. After (5.5) has been verified it will be shown that

$$u(r,t) = \left(\frac{2}{\pi}\right)\frac{1}{r}\frac{\partial w(r,t)}{\partial r}, \tag{5.6}$$

where the differentiation is interpreted in a distribution-theoretic sense, defines the distribution solution of the Cauchy problem for all data that satisfy (5.3).

The starting point for the analysis of $\mathcal{F}, G, v(s,t)$ and $w(r,t)$ is the observation that, for any $p \ge 1$, one has

$$\begin{aligned} f(\sqrt{x^2+y^2}) \in L_p^{\rm loc}(\mathbf{R}^2) &\Longleftrightarrow f(r) \in L_p^{\rm loc}(\mathbf{R}_+, r\,dr) \\ &\Longleftrightarrow f(\sqrt{\rho}) \in L_p^{\rm loc}(\mathbf{R}_+), \end{aligned} \tag{5.7}$$

where $\mathbf{R}_+ = [0,\infty)$. Recall that $G(\sqrt{\sigma})$ is a fractional integral of $(\sqrt{\pi}/2)g(\sqrt{\rho})$, by (4.17). Moreover, in the definition (4.15) of the fractional integral, $x^{\alpha-1}/\Gamma(\alpha) \in L_1^{\rm loc}(\mathbf{R}_+)$. Hence, $g \in L_1^{\rm loc}(\mathbf{R}_+)$ implies $g_\alpha^+ \in L_1^{\rm loc}(\mathbf{R}_+)$ by Fubini's theorem. (This fact is the familiar property that $L_1(\mathbf{R})$ is closed under convolution.) Thus, if $g(\sqrt{\rho}) \in L_1^{\rm loc}(\mathbf{R}_+)$, then one has $G(\sqrt{\sigma}) \in L_1^{\rm loc}(\mathbf{R}_+)$, and hence $G(s) \in L_1^{\rm loc}(\mathbf{R}_+, s\,ds)$, or, equivalently, $sG \in L_1^{\rm loc}(\mathbf{R}_+)$. But in the definition (4.26) of $v(s,t)$ one must have $G(s) \in L_1^{\rm loc}(\mathbf{R}_+)$ and not just $sG \in L_1^{\rm loc}(\mathbf{R}_+)$. The required property will be obtained from the following classical

Theorem of Hardy and Littlewood [11]. *For all* $p > 1$ *and all* $\alpha > 0$ *there exists a constant* $K = K(p, \alpha) > 0$ *such that*

$$\int_0^\infty |x^{-\alpha} f_\alpha^+(x)|^p \, dx \leq K \int_0^\infty |f(x)|^p \, dx. \tag{5.8}$$

Moreover, for all $\mathrm{R} > 0$ *one has*

$$\int_0^{\mathrm{R}} |x^{-\alpha} f_\alpha^+(x)|^p \, dx \leq K \int_0^{\mathrm{R}} |f(x)|^p \, dx. \tag{5.9}$$

In particular, $f(x) \in L_p^{\mathrm{loc}}(\mathbf{R}_+) \Rightarrow x^{-\alpha} f_\alpha^+(x) \in L_p^{\mathrm{loc}}(\mathbf{R}_+)$.

Note that (5.9) may be obtained from (5.8) by setting $f(x) \equiv 0$ for $x \geq \mathrm{R}$, since the values $f_\alpha^+(x)$ for $0 \leq x \leq \mathrm{R}$ depend on the values $f(x)$ for $0 \leq x \leq \mathrm{R}$ only. Hardy and Littlewood showed by an example that their result is false for $p = 1$. It is this fact that necessitates the choice of $p > 1$ below.

The theorem of Hardy and Littlewood will be used to prove the following theorem which is the key to the extension of the solution formulas (4.26), (4.27) to the case of Cauchy data in L_p^{loc}.

Theorem 4. *If the data* f, g *satisfy* (5.3) *so that* $f(\sqrt{\rho})$, $g(\sqrt{\rho}) \in L_p^{\mathrm{loc}}(\mathbf{R}_+)$, *then*

$$\mathcal{F}(s), G(s) \in L_p^{\mathrm{loc}}(\mathbf{R}_+, ds) \cap L_p^{\mathrm{loc}}(\mathbf{R}_+, s\, ds), \tag{5.10}$$

$$v(s,t) \in L_p^{\mathrm{loc}}(\mathbf{R}^2, s\, ds\, dt), \tag{5.11}$$

and

$$w(r,t) \in L_p^{\mathrm{loc}}(\mathbf{R}^2, dr\, dt). \tag{5.12}$$

Moreover, for all $\mathrm{R} > 1$, *there exists a constant* $K = K(\mathrm{R}, p)$ *such that*

$$||w||_{p,\mathrm{R}} \leq K(||f||_{p,2\mathrm{R}} + ||g||_{p,2\mathrm{R}}), \tag{5.13}$$

where

$$||w||_{p,\mathrm{R}} = \left(\int_0^{\mathrm{R}} \int_0^{\mathrm{R}} |w(r,t)|^p \, dr\, dt \right)^{\frac{1}{p}}, \tag{5.14}$$

and

$$\|f\|_{p,\mathrm{R}} = \left(\int_0^{\mathrm{R}} |f(r)|^p r\, dr \right)^{\frac{1}{p}}. \tag{5.15}$$

Proof. Throughout the proof the symbol $K = K(p, \mathrm{R})$ denotes a generic constant, i.e., one whose numerical value may vary from one inequality to the next. The notation $\sigma = s^2$, $\rho = r^2$ will also be used. To verify (5.10) note that (4.17) and the Hardy–Littlewood theorem imply that for all $\mathrm{R} > 1$ one has

$$\int_0^{\mathrm{R}} |\sigma^{-1/2}\mathcal{F}(\sqrt{\sigma})|^p \, d\sigma \le K \int_0^{\mathrm{R}} |f(\sqrt{\rho})|^p \, d\rho, \tag{5.16}$$

or, equivalently,

$$\int_0^{\mathrm{R}} |\mathcal{F}(\sigma)|^p s^{1-p} \, ds \le K \int_0^{\mathrm{R}} |f(r)|^p r\, dr. \tag{5.17}$$

Moreover, since $p > 1$, one has $s^{1-p} \le \mathrm{R}^{1-p}$ for all $0 \le s \le \mathrm{R}$, and hence (5.17) implies

$$\int_0^{\mathrm{R}} |\mathcal{F}(s)|^p \, ds \le K \int_0^{\mathrm{R}} |f(r)|^p r\, dr. \tag{5.18}$$

In particular, $\mathcal{F} \in L_p^{\mathrm{loc}}(R_+, ds)$. To complete the proof of (5.10) note that the inequality $x^{-\alpha p} \le \mathrm{R}^{-\alpha p}$ for $0 \le x \le \mathrm{R}$ and (5.9) imply

$$\int_0^{\mathrm{R}} |f_\alpha^+(x)|^p \, dx \le K \int_0^{\mathrm{R}} |f(x)|^p \, dx, \tag{5.19}$$

whence by (4.17) one has

$$\int_0^{\mathrm{R}} |\mathcal{F}(\sqrt{\sigma})|^p \, d\sigma \le K \int_0^{\mathrm{R}} |f(\sqrt{\rho})|^p \, d\rho \tag{5.20}$$

and, therefore,

$$\int_0^{\mathrm{R}} |\mathcal{F}(s)|^p s\, ds \le K \int_0^{\mathrm{R}} |f(r)|^p r\, dr. \tag{5.21}$$

In particular, $\mathcal{F} \in L_p^{\mathrm{loc}}(\mathbf{R}_+, s\,ds)$. This completes the proof of (5.10).

For the proof of (5.11) it will be convenient to write

$$v = v_{(f,g)} = v_{(f,0)} + v_{(0,g)} = v^+_{(f,0)} + v^-_{(f,0)} + v^+_{(0,g)} + v^-_{(0,g)} \tag{5.22}$$

where

$$v^{\pm}_{(f,0)} = \left(\frac{1}{2}\right) \mathcal{F}(s \pm t), \tag{5.23}$$

$$v^{\pm}_{(0,g)} = \pm \left(\frac{1}{2}\right) \mathcal{H}(s \pm t), \tag{5.24}$$

and

$$\mathcal{H}(s) = \int_0^s G(x)\,dx. \tag{5.25}$$

The definition of $v^+_{(f,0)}$ implies that

(5.26)

$$\begin{aligned} ||v^+_{(f,0)}||^p_{p,\mathrm{R}} &\equiv \int_0^{\mathrm{R}} \int_0^{\mathrm{R}} |v^+_{(f,0)}(s,t)|^p s\,ds\,dt \\ &= K \int_0^{\mathrm{R}} \int_0^{\mathrm{R}} |\mathcal{F}(s+t)|^p s\,ds\,dt \le K \int_0^{\mathrm{R}} \int_0^{\mathrm{R}} |\mathcal{F}(s+t)|^p\,ds\,dt. \end{aligned}$$

Next, the change of variables $s' = s + t$, $t' = s - t$ will be made in the last integral. On noting that the square $0 \le s \le \mathrm{R}$, $0 \le t \le \mathrm{R}$ is contained in the square $0 \le s' \le 2\mathrm{R}$, $-\mathrm{R} \le t' \le \mathrm{R}$, one has

$$\begin{aligned} ||v^+_{(f,0)}||^p_{p,\mathrm{R}} &\le K \int_{-\mathrm{R}}^{\mathrm{R}} \int_0^{2\mathrm{R}} |\mathcal{F}(s')|^p\,ds'\,dt' \\ &= K \int_0^{2\mathrm{R}} |\mathcal{F}(s)|^p\,ds \le K \int_0^{2\mathrm{R}} |f(r)|^p r\,dr, \end{aligned} \tag{5.27}$$

by (5.18). In particular, $v^+_{(f,0)} \in L_p^{\mathrm{loc}}(\mathbf{R}^2, s\,ds)$. A similar calculation gives

$$||v^-_{(f,0)}||^p_{p,\mathrm{R}} \le K \int_0^{2\mathrm{R}} |f(r)|^p r\,dr, \tag{5.28}$$

and $v^-_{(f,0)} \in L_p^{\rm loc}(\mathbf{R}^2, s\,ds)$. Finally, $v(s,t)_{(0,g)} \in C(\mathbf{R}^2)$ and, hence, $v_{(0,g)} \in L_p^{\log}(\mathbf{R}^2, s\,ds)$. Property (5.11) now follows from (5.27), (5.28) and the triangle inequality for the L_p-norm (Minkowski's inequality).

To prove (5.12) it will be convenient to write

$$w = w_{(f,g)} = w_{(f,0)} + w_{(0,g)}, \tag{5.29}$$

in analogy with (5.22). Then

$$w_{(f,0)} = \frac{\sqrt{\pi}}{2}(v_{(f,0)}(\sqrt{\sigma},t))^+_{1/2}(\rho), \tag{5.30}$$

and, hence,

$$\int_0^{\rm R} |w_{(f,0)}(r,t)|^p\,dr \le K \int_0^{\rm R} |v_{(f,0)}(s,t)|^p s\,ds; \tag{5.31}$$

see the proof of (5.18). Integrating over $0 \le t \le {\rm R}$ then gives

$$||w_{(f,0)}||_{p,{\rm R}} \le K||v_{(f,0)}||_{p,{\rm R}} \le K||f||_{p,2{\rm R}} \tag{5.32}$$

by (5.27), (5.28) and the triangle inequality. To estimate the term $w_{(0,g)}$ one can apply the Hölder inequality to

$$v_{(0,g)}(s,t) = \frac{1}{2}\int_{s-t}^{s+t} G(x)\,dx \tag{5.33}$$

to obtain

$$\begin{aligned} |v_{(0,g)}(s,t)| &\le \frac{1}{2}\int_{s-t}^{s+t} |G(x)|\,dx \le K\bigg(\int_{s-t}^{s+t} |G(x)|^p\,dx\bigg)^{1/p} \\ &\le K\bigg(\int_{-{\rm R}}^{2{\rm R}} |G(x)|^p\,dx\bigg)^{1/p} \le K\bigg(\int_0^{2{\rm R}} |G(x)|^p\,dx\bigg)^{1/p}, \end{aligned} \tag{5.34}$$

where at the last step the fact that G is an odd function has been used. Integrating over $0 \le s \le {\rm R}$, $0 \le t \le {\rm R}$ then gives

$$||v_{(0,g)}||^p_{p,{\rm R}} \le K\int_0^{2{\rm R}} |G(s)|^p\,ds \le K\int_0^{2{\rm R}} |g(r)|^p r\,dr, \tag{5.35}$$

by (5.21). This last result implies

(5.36) $$||w_{(0,g)}||_{p,\mathrm{R}} \le K||v_{(0,g)}||_{p,\mathrm{R}} \le K||g||_{p,2\mathrm{R}},$$

in analogy with (5.32). Finally, (5.29), (5.32) and (5.36) imply (5.13) by the triangle inequality. This completes the proof of Theorem 4. □

The construction of the solution $u(r,t)$ of the Cauchy Problem with rotationally symmetric data in L_p^{loc}, $p>1$, will be completed by interpreting the r-derivative in (5.6) in a distribution-theoretic sense. Note that $w(\sqrt{x^2+y^2},t) \in L_1^{\mathrm{loc}}(\mathbf{R}^3)$. This follows easily from Theorem 4 and Hölder's inequality. It follows that, for all testing functions $\phi(x,y,t) \in \mathcal{D}(\mathbf{R}^3)$, one has

$$w(\phi) = \int_0^\infty \int_0^\infty w(r,t)\left(\int_0^{2\pi} \phi(r\cos\theta, r\sin\theta, t)\,d\theta\right) r\,dr\,dt = w(\tilde{\phi}),$$

where

$$\tilde{\phi}(r,t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(r\cos\theta, r\sin\theta, t)\,d\theta.$$

The alternative formula

$$\tilde{\phi}(r,t) = \frac{1}{2\pi}\int_0^{2\pi} \phi(x\cos\psi - y\sin\psi, x\sin\psi + y\cos\psi, t)\,d\psi$$

shows that $\phi \in \mathcal{D}(\mathbf{R}^3) \Rightarrow \tilde{\phi} \in \mathcal{D}(\mathbf{R}^3)$ and $\phi_n \to 0$ in $\mathcal{D}(\mathbf{R}^3) \Rightarrow \tilde{\phi}_n \to 0$ in $\mathcal{D}(\mathbf{R}^3)$. Thus, if $u \in \mathcal{D}(\mathbf{R}^3)$ is *any* distribution, then $\tilde{u}(\phi) = u(\tilde{\phi})$ defines another distribution on $\mathbf{R}^3$. A distribution $u \in \mathcal{D}(\mathbf{R}^3)$ will be said to have circular symmetry $\Leftrightarrow$ $\tilde{u} = u$. Such a distribution satisfies $u(\phi) = \tilde{u}(\phi) = u(\tilde{\phi})$ and, hence, is determined by its values on the circularly symmetric testing functions. It was shown that $\phi(\sqrt{x^2+y^2},t) \in \mathcal{D}(\mathbf{R}^3) \Leftrightarrow \phi(r,t)$ has a unique extension to a function in

$$\mathcal{D}_e(\mathbf{R}^2) = \mathcal{D}(\mathbf{R}^2) \cap \{\phi \mid \phi(-r,t) = \phi(r,t)\}.$$

The following lemma will be needed.

Lemma. *If $\phi(r,t) \in C^\infty(\mathbf{R}^2)$ and $\phi(0,t) \equiv 0$, then $(\phi(r,t))/r \in C^\infty(\mathbf{R}^2)$. In particular, if $\phi \in \mathcal{D}_e(\mathbf{R}^2)$, then $(1/r)\partial\phi(r,t)/\partial r \in \mathcal{D}_e(\mathbf{R}^2)$.*

The Lemma is a consequence of Taylor's theorem. The simple proof will be omitted. Note that if $U \in \mathcal{D}'(\mathbf{R}^3)$ is any distribution then the lemma implies that

$$V(\phi) = -U\left(\frac{1}{r}\frac{\partial\tilde{\phi}(r,t)}{\partial r}\right)$$

defines a circularly symmetric distribution $V \in \mathcal{D}'(\mathbf{R}^3)$. The notation

$$V = \frac{1}{r}\frac{\partial U}{\partial r}$$

will be used. This is motivated by the observation that if $U = f(\sqrt{x^2+y^2}, t) \in C^1(\mathbf{R}^2)$ and $f(0,t) \equiv 0$, then $V = (1/r)\partial f/\partial r \in C^1(\mathbf{R}^2)$.

Theorem 4 and the Lemma above imply that (5.6) defines a distribution $u \in \mathcal{D}'(\mathbf{R}^3)$. The principal result of this section may now be formulated as follows.

Theorem 5. *Let $p > 1$. Then for all $f(\sqrt{x^2+y^2}), g(\sqrt{x^2+y^2}) \in L_p^{\mathrm{loc}}(R^2, dx\,dy)$, the distribution*

$$u(r,t) = \left(\frac{2}{\pi}\right)\frac{1}{r}\frac{\partial w(r,t)}{\partial r}, \tag{5.6}$$

where $w(r,t)$ is defined as in Theorem 4, is the unique solution of the Cauchy problem (2.2), (2.3) with data $F(x,y) = f(\sqrt{x^2+y^2}), G(x,y) = g(\sqrt{x^2+y^2})$.

Proof. It is clear from (5.6) and the definition of $w(r,t)$ that $\operatorname{supp} u \subset \mathbf{R}^2 \times [0,\infty)$ and u has circular symmetry. Hence, to prove Theorem 5 it will suffice to verify the differential equation (2.2), in the form (2.4), with $\Phi(x,y,t) = \phi(\sqrt{x^2+y^2}, t)$ and $\phi \in \mathcal{D}_e(\mathbf{R}^2)$. Equation (2.4), with U replaced by the distribution u of (5.6) and Φ replaced by $\phi \in \mathcal{D}_e(\mathbf{R}^2)$, can be written

$$\begin{aligned} &\int_0^\infty \int_0^\infty w(r,t)\frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\partial^2\phi}{\partial t^2}\right) dr\,dt \\ &\qquad = \frac{\pi}{2}\int_0^\infty \left\{-f(r)\frac{\partial\phi(r,0)}{\partial t} + g(r)\phi(r,0)\right\} r\,dr. \end{aligned} \tag{5.37}$$

In deriving (5.37) from (2.4), the distributional definition of the r-derivative in (5.6) was used to move the derivative to the testing function. Notice that, for any locally integrable functions w, f, g, the two sides of equation (5.37) are finite and define circularly symmetric distributions. The proof of Theorem 5 will be completed by showing that these distributions coincide when w is related to f, g as in Theorem 4. To show this, note that (5.37) holds and is equivalent to the classical wave equation (1.1) and Cauchy data (2.1), when $f(\sqrt{x^2+y^2}), g(\sqrt{x^2+y^2}) \in C^\infty(\mathbf{R}^2)$ and $u(r,t)$ and $w(r,t)$ are the corresponding functions in $C^\infty(\mathbf{R}^2)$ of Theorem 3. Moreover, it is well known that, for any fixed $p \geq 1$, the set $\mathcal{D}(\mathbf{R}_+)$ is dense in $L_p(\mathbf{R}_+, r\,dr)$: see, for example, [**12**, p. 3]. To apply this result to the proof of (5.37), let $\phi \in \mathcal{D}_e(\mathbf{R}^2)$ be given and choose $\mathrm{R} > 0$ such that $\operatorname{supp}\phi(r,t) \subset \{(r,t) \mid r^2+t^2 \leq \mathrm{R}^2\}$. Let f_n, g_n be sequences in $\mathcal{D}(\mathbf{R}_+)$ such that $f_n \to f$ and $g_n \to g$ in $L_p(0, 2\mathrm{R})$. If $w_n(r,t)$ is the classical solution of the Cauchy problem with data f_n, g_n, then

$$\text{(5.38)}\quad \int_0^\infty \int_0^\infty w_n(r,t)\frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\partial^2\phi}{\partial t^2}\right) dr\,dt = \frac{\pi}{2}\int_0^\infty \left\{-f_n(r)\frac{\partial\phi(r,0)}{\partial t} + g_n(r)\phi(r,0)\right\} r\,dr.$$

Moreover, inequality (5.13) of Theorem 4, written for $w-w_n$, $f-f_n$ and $g-g_n$, implies that $w_n \to w$ in $L_p(\{(r,t) \mid r^2+t^2 \leq \mathrm{R}^2\})$ when $n \to \infty$. Passage to the limit $n \to \infty$ in (5.38) now gives (5.37), by Hölder's inequality, becasue the coefficient of w_n in the left-hand integral of (5.38) is continuous and, hence, is in $L_q(\{(r,t) \mid r^2+t^2 \leq \mathrm{R}^2\})$ where $1/p+1/q=1$. Similar remarks apply to the integrals containing f_n and g_n. This completes the proof of Theorem 5. □

6. Applications to the focusing of shock waves. Two applications of Theorem 5 to the analysis of shock wave conversion by focusing are given in this section. The first is a calculation of the unique continuation to the interior of the forward wave cone Γ_+ of the converging finite jump shock wave of Section 3. The second application is an analysis of the reduction in differentiability of the solution, due to focusing, along the focal line $r=0$, $t \geq 0$ in space-time.

The converging shock wave $u(r,t)$ of Section 3 has the representation (3.14) in the space-time domain $t < r$, $r \geq 0$. At each point of

Γ_+, which is the boundary of this domain, $u(r,t)$ has the logarithmic singularity described by (3.19). The unique continuation of $u(r,t)$ to the interior of Γ_+ will now be calculated by applying the solution formula (5.6) of Theorem 5 to the Cauchy data for $u(r,t)$ which are given by (5.1), (5.2). Recall that both $f(r)$ and $g(r)$ are in $L_p^{\text{loc}}(\mathbf{R}_+, r\,dr)$ for $1 \le p < 4/3$, and hence Theorem 5 is applicable to the data (5.1), (5.2).

The constants c, c' in (5.1), (5.2) are known to have the values [**15**, pp. 40-41]

$$c = F(1/2, 1/2, 1, 1/2) = \frac{\Gamma(1/2)}{(\Gamma(3/4))^2},$$

$$c' = F'(1/2, 1/2, 1, 1/2) = \frac{\Gamma(1/2)}{4(\Gamma(5/4))^2}. \tag{6.1}$$

To construct the wave function $u(r,t)$ for these data, one needs the L-transforms $\mathcal{F}(s), G(s)$. These may be found by means of their relation to fractional integrals, equation (4.17), and a table of fractional integrals [**3**]. The surprisingly simple result is

$$\mathcal{F}(s) = s^{1/2}, \qquad G(s) = \frac{1}{2}s^{-1/2} \qquad \forall s \ge 0. \tag{6.2}$$

Moreover, $\mathcal{F}(s)$ and $G(s)$ are odd functions, by Theorem 2. It follows that the function $v(s,t)$, defined by (4.26), can be written

$$v(s,t) = \frac{1}{2}\{\mathcal{F}(s+t) + \mathcal{F}(s-t) + \mathcal{H}(s+t) - \mathcal{H}(s-t)\}, \tag{6.3}$$

where

$$\mathcal{F}(s) = \mathcal{H}(s) = \int_0^s G(x)\,dx = s^{\frac{1}{2}} \qquad \forall s \ge 0, \tag{6.4}$$

while

$$\mathcal{F}(s) \text{ is } \textit{odd} \quad \text{and} \quad \mathcal{H}(s) \text{ is } \textit{even.} \tag{6.5}$$

Note that these properties imply that

$$v(s,t) = \begin{cases} \sqrt{t+s} - \sqrt{t-s}, & \text{for } s \le t \\ \sqrt{t+s}, & \text{for } s \ge t. \end{cases} \tag{6.6}$$

Recall that the wave function $w(r,t)$ is defined by (5.4). In particular, for $t \geq 0$, one has

$$w(r,t) = \int_0^r \frac{sv(s,t)}{\sqrt{r^2 - s^2}}\, ds = r \int_0^1 \frac{\sigma v(r\sigma, t)}{\sqrt{1-\sigma^2}}\, d\sigma. \tag{6.7}$$

$w(r,t)$ will be calculated first for points of the set $\{(r,t) \mid 0 < r < t\}$ interior to Γ_+. Note that in this case only the top line of the definition of $v(s,t)$ enters into the integration in (6.7) and one has

$$w(r,t) = \int_0^r \frac{s(\sqrt{t+s} - \sqrt{t-s})}{\sqrt{r^2-s^2}}\, ds. \tag{6.8}$$

Simple changes of variable give the alternative representations

$$w(r,t) = -\int_{-r}^r \frac{s\sqrt{t-s}}{\sqrt{r^2-s^2}}\, ds = -r\int_{-1}^1 \frac{x\sqrt{t-rx}}{\sqrt{1-x^2}}\, dx. \tag{6.9}$$

It is clear from the last integral that, in the interior of Γ_+, $w(r,t)$ is an analytic function of r and t. Moreover, the derivative $w_r(r,t)$ can be calculated by differentiating under the integral sign. The result is

$$w_r(r,t) = -\int_{-1}^1 \frac{x\sqrt{t-rx}}{\sqrt{1-x^2}}\, dx + \frac{r}{2}\int_{-1}^1 \frac{x^2}{\sqrt{t-rx}\sqrt{1-x^2}}\, dx, \tag{6.10}$$

or, after simplification,

$$w_r(r,t) = -\frac{1}{2}\int_{-1}^1 \frac{x(2t-3rx)}{\sqrt{(t-rx)(1-x^2)}}\, dx. \tag{6.11}$$

This last integral is an *elliptic integral*. It can be reduced by simple, but lengthy, algebraic procedures to standard elliptic integrals; see [**1**,**4**]. The result of this reduction can be written

$$w_r(r,t) = \frac{r}{\sqrt{r+t}} \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \text{with } k^2 = \frac{2r}{t+r}. \tag{6.12}$$

The integral in (6.12) is the complete elliptic integral $\mathbf{K}(k)$ of (3.17). Note that $0 < k^2 < 1$ when (r,t) is in the interior of Γ_+, and hence $\mathbf{K}(k)$ is finite. Finally, the relation (3.16) implies the alternative description

$$w_r(r,t) = \frac{\pi}{2}\frac{r}{\sqrt{r+t}} F(1/2, 1/2, 1, k^2).$$

Combining this with equation (5.6) gives the result

(6.13)

$$u(r,t)=\left(\frac{2}{\pi}\right)\frac{1}{r}w_r(r,t)=\frac{1}{\sqrt{r+t}}F(1/2,1/2,1,2r/(t+r)) \qquad \forall\, 0<r<t.$$

This representation implies that the shock wave $u(r,t)$ is analytic in the interior of Γ_+. In particular, one has

$$u(0,t)=t^{-1/2} \quad \text{for } t>0.$$

Moreover, the estimates (3.18) imply that

$$u(r,t)=\frac{-\log(t-r)}{\pi(2r)^{1/2}}+O(1), \quad \text{for } t-r\to 0_+,\ t\geq t_0>0.$$

Thus, $u(r,t)$ has the same logarithmic singularity on both sides of Γ_+.

It is interesting to evaluate the integral in (6.7) for $t<r$, $r\geq 0$. For $-r<t<r$ one gets an elliptic integral which reduces to the representation (3.14). Of course, it is clear from the uniqueness statement of Theorem 1 that this must happen. For $t<-r$ the integral (6.7) reduces to 0. This calculation verifies that the function $u(r,t)$ defined by (3.14) is indeed a solution of the wave equation (1.1), in the distribution-theoretic sense, in the domain $t<r$, $r\geq 0$.

The calculation of $u(r,t)$ was carried out in the open sets $r<t$ and $r>t$ only. Hence, there is the possibility that the distribution $u(r,t)$ might contain singular terms, such as delta functions, localized on Γ_+. However, it is easy to verify from (6.6), (6.7) that $w(r,t)$ is continuous at the points of Γ_+. It follows that $u(r,t)$, defined by (5.6), is the piecewise analytic function constructed abaove. A proof may be based on integration by parts in the definition of the distributional derivative in (5.6).

To exhibit the reduction in differentiability along the focal line $r=0$, $t>0$, Cauchy data $f(r)\in C^2[0,\infty)$, $g(r)\equiv 0$ will be considered. For such data the representation (5.6) of Theorem 5 implies that $u(r,t)\in C^2$ for $r>0$ so that no differentiability is lost at points other than the focus $r=0$. At the focus, one has the special relation $u(0,t)=v_s(0,t)$; see (4.28). Thus

$$u_t(0,t)=\mathcal{F}''(t)=\frac{\sqrt{\pi}}{2}(h\sqrt{\rho}))^+_{1/2}(\tau), \qquad \tau=t^2, \tag{6.14}$$

where, by (4.10),

$$h(r) = f''(r) + \frac{1}{r} f'(r), \qquad \forall r > 0. \tag{6.15}$$

For a given function $h(r) \in L_1^{\mathrm{loc}}(\mathbf{R}_+)$, this equation has a unique solution $f(r)$ such that $f(r) \in C^1[0, \infty)$ and $f(0) = 0$, $f'(0) = 0$. It is given by

$$f(r) = \int_0^r \rho \left(\ln \frac{r}{\rho} \right) h(\rho)\, d\rho. \tag{6.16}$$

If a function $h(r) \in C[0, \infty)$ is selected that has no (fractional) derivatives on $[0, \infty)$, then (6.16) defines Cauchy data $(f(r), 0)$ such that $f(r) \in C^2[0, \infty)$ and $f(r) \notin C^m[0, \infty)$ for $m > 2$. It then follows from (6.14) that $u(0, t) \in C^{1.5}[0, \infty)$ but $u(0, t) \notin C^m[0, \infty)$ for $m > 1.5$. This shows clearly that focusing reduces the degree of differentiability of the solution by exactly 0.5. Alternatively, (6.14) implies that, for all $k \geq 2$, Cauchy data $f(r), 0)$ generate a solution $U(x, y, t) = u(r, t) \in C^k(\mathbf{R}^3)$ if and only if $f(r) \in C^{k+0.5}[0, \infty)$.

7. Related literature. In recent years the theory of the propagation of singularities of distribution solutions of hyperbolic partial differential equations has been an active field of research, beginning with the work of Hörmander on wave front sets in 1971 [**13, 14**]. This research has led to a new branch of analysis called *microlocal analysis*; see [**6,7**] for recent surveys of the field and many references to the journal literature. The results presented in this article can undoubtedly be derived by the techniques of microlocal analysis. However, in order to follow such a treatment, a reader would have to make a large excursion into a complex and difficult area of modern analysis. The goal of this article is to present a simple and rigorous discussion, within the context of classical analysis, of some typical examples of the conversion by focusing of cylindrical shock waves.

The converging shock wave (3.14) was derived above by means of F.G. Friedlander's *generalized progressive wave* formalism. An alternative method to obtain it may be based on the Lie theory of families of self-similar solutions of the Darboux equation (3.3) [**17**]. One such family is defined by

$$u = r^k F(\xi), \qquad \xi = \frac{t + r}{2r}. \tag{7.1}$$

The function u defined by (7.1) is a solution of (3.3) if and only if $F(\xi)$ satisfies the hypergeometric equation

$$\xi(1-\xi)F''(\xi) + \left(\left(\frac{1}{2}-k\right) - (1-2k)\xi\right)F' - k^2F(\xi) = 0, \tag{7.2}$$

which has the hypergeometric function

$$F(\xi) = F(-k, -k, 1/2-k, \xi) \tag{7.3}$$

as a solution. The parameter k must have the value $k = -1/2$ to match to the zero solution inside the cone Γ_-; see (3.7). This gives

$$u = \frac{1}{(2r)^{1/2}} F(1/2, 1/2, 1, \xi), \qquad \xi = \frac{t+r}{2r} < 1. \tag{7.4}$$

which is precisely the solution (3.14) in the space-time domain $t < r$ with boundary Γ_+.

The continuation of the shock wave (3.14) to the interior of Γ_+ is defined by (6.14) which may be written

$$u = \frac{1}{(2r)^{1/2}}(\xi)^{-\frac{1}{2}} F(1/2, 1/2, 1, \xi^{-1}), \qquad \xi = \frac{t+r}{2r} > 1. \tag{7.5}$$

An alternative approach to shock wave continuation that is found in the literature is analytic continuation [**8**]. The hypergeometric function in (7.4) has a logarithmic singularity at $\xi = 1$ and is analytic in the cut ξ-plane with branch cut along the real axis from $\xi = 1$ to $\xi = +\infty$ [**1**, Eq. 15.3.1]. The analytic continuation of $F(1/2, 1/2, 1, \xi)$ to $\xi > 1$ may be obtained from [**1**, Eq. 15.3.13] which implies that, for real $z > 1$, one has

$$\begin{aligned} &F(1/2, 1/2, 1, z, \pm i0) \\ &\quad = \left\{\frac{\pm i}{\pi} z^{-1/2} F(1/2, 1/2, 1, z^{-1})\right\}(\ln z \pm i\pi) \mp \frac{i}{\pi} z^{-\frac{1}{2}} \sum_{n=0}^{\infty} \gamma_n z^{-n}, \end{aligned} \tag{7.6}$$

where the γ_n are real. In particular, one has

$$\begin{aligned} &\frac{1}{2}\{F(1/2, 1/2, 1, z+i0) + F(1/2, 1/2, 1, z-i0)\} \\ &\qquad\qquad = z^{-1/2} F(1/2, 1/2, 1, z^{-1}). \end{aligned} \tag{7.7}$$

Thus, if the solution (7.4) is continued analytically in the parameter ξ from the interval $-1 < \xi < 1$ to the two sides of the branch cut $\xi > 1$, then the mean value of the two continuations is real-valued and coincides with the physical continuation (7.5) provided by distribution theory.

8. Concluding remarks. Schwartz's existence and uniqueness theorem for the Cauchy problem for the wave equation (Theorem 1 above) implies that any distributional solution of the wave equation in a half-space $\{(x, y, t) \mid t < t_0\}$ has a unique continuation to the whole three-dimensional space-time. This result played a fundamental role above in continuing the shock wave across the shock front Γ_+. Indeed, if the distributional wave equation is relaxed at even one space-time point, then the uniqueness of the continuation is lost. It is for this reason that so much care was taken to verify that the wave function (5.6) of Theorem 5 satisfies the distributional wave equation (2.2) of Theorem 1.

To verify the assertion of the preceding paragraph, recall that the fundamental solution of the wave equaton is the distribution defined by the locally integrable function

$$G(r,t) = \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}; \tag{8.1}$$

cf. [**18**]. The distribution derivatives

$$G^{(n)}(r,t) = \frac{\partial^n}{\partial t^n}G(r,t) \tag{8.2}$$

are uniquely characterized by the properties

$$G^{(n)}_{tt} - G^{(n)}_{rr} - \frac{1}{r}G^{(n)}_{r} = \frac{\delta(r)\delta^{(n)}(t)}{2\pi r}, \tag{8.3}$$

and $\operatorname{supp} G = \Gamma_+$. The support of the right hand side of equation (8.3) is the space-time origin. Thus, if the wave equation (2.2) is relaxed at this one point then $G^{(n)}$ may be added to any solution of the shock continuation problem to obtain another different solution, and hence, uniqueness is lost.

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