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AN EXTENSION OF ASKEY-WILSON'S q-BETA INTEGRAL AND ITS APPLICATIONS

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1. One of the remarkable q-extensions of the classical beta integral evaluated by Askey and Wilson [**3**] is:

If $\max(|a|,|b|,|c|,|d|)<1$, then

$$\begin{aligned}
&\int_{-1}^{1} w(z;a,b,c,d)\,dz = K, \text{ where}\\
&w(z;a,b,c,d) = \frac{h(z;1)h(z;-1)h(z;\sqrt{q})h(z;-\sqrt{q})}{h(z;a)h(z;b)h(z;c)h(z;d)\sqrt{1-z^2}},\\
&h(z;a) = \prod_{n=0}^{\infty}(1-2azq^n+q^{2n}) \qquad (1.1)\\
&\qquad = (ae^{i\theta}, ae^{-i\theta};q)_\infty, \qquad z=\cos\theta,\\
&(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j), \text{ whenever it converges,}\\
&(a_1,a_2,\dots,a_n;q)_\infty = (a_1;q)_\infty\dots(a_n;q)_\infty
\end{aligned}$$

and

$$K = \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}.$$

Nassrallah and Rahman [**6**] used (1.1) to obtain q-analogues of Euler's integral representation of Gauss's hypergeometric series ${}_2F_1$:

$$\begin{aligned}
(1.2)\quad &\int_{-1}^{1} w(z;a,b,c,d)\frac{h(z;\lambda)}{h(z;f)}\,dz\\
&= K\frac{(\lambda a,\lambda b,\lambda c,abcf;q)_\infty}{(af,bf,cf,\lambda abc;q)_\infty}\cdot {}_8W_7\left[\frac{\lambda abc}{q};bc,ac,ab,\lambda/d,\lambda/f;q,df\right],
\end{aligned}$$

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where

$$ {}_r\phi_{r+k-1}\left[\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_{r+k-1}\end{matrix};q,z\right] = \sum_{n=0}^{\infty}\frac{(a_1,\dots,a_r;q)_n z^n(-1)^{kn}}{(q,b_1,\dots,b_{r+k-1};q)_n}q^{kn(n-1)/2} $$

(for $k>0$ or $k=0$ and $|z|<1$) and the very well-poised hypergeometric series

$$ {}_{r+3}\phi_{r+2}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a_1,\dots,a_r \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{a_1},\dots,\frac{aq}{a_r}\end{matrix};q,z\right] $$

is abbreviated as ${}_{r+3}w_{r+2}[a;a_1,a_2,\dots,a_r;q,z]$. In a subsequent paper they [**7**] obtained another extension of (1.2) as a q-analogue of the integral representation of Appell's function F_1:

If $\max(|q|,|a|,|b|,|c|,|d|,|f|,|g|)) < 1$, $\left|\frac{\lambda\mu q}{abcdfg}\right| < 1$, then

$$ \begin{aligned} (1.3)\quad S(a,b,c,d,f,g;\lambda,\mu) &\equiv \int_{-1}^{1} w(z;a,b,c,d)\frac{h(z;\lambda)h(z;\mu)}{h(z;f)h(z;g)}\,dz \\ &= K\frac{\left(\frac{q}{abcd},\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\lambda a,\frac{\lambda}{a},\mu a,\frac{\mu}{a};q\right)_\infty}{\left(a^2q,\frac{q}{bc},\frac{q}{bd},\frac{q}{cd},af,f/a,ag,g/a;q\right)_\infty} \\ &\quad\cdot {}_{10}W_9[a^2;ab,ac,ad,af,ag,aq/\lambda,aq/\mu;q,\lambda\mu q/abcdfg] \\ &\quad + \text{idem}\,(a;f,g), \end{aligned} $$

where

$$ f(a_1) + \text{idem}\,(a_1;a_2,\dots,a_r) \quad \text{means} \quad \sum_{j=1}^{r} f(a_j). $$

Nassrallah-Rahman's proof was based on using the integral representation of the sum of two nonterminating balanced ${}_3\phi_2$, [**2**], viz.,

$$ (1.4)\quad \int_a^b \frac{(qu/a,qu/b,cu;q)_\infty}{(du,eu,fu;q)_\infty}d_qu = \frac{b(1-q)\left(q,\frac{bq}{a},\frac{a}{b},\frac{c}{d},\frac{c}{e},\frac{c}{f};q\right)_\infty}{(ad,ae,af,bd,be,bf;q)_\infty}, $$

where $c = abdef$ and the q-integrals $\int_a^b f(x)d_qx$ is defined as

$$ (1.5)\quad \begin{aligned} \int_0^a f(x)d_qx &= a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n, \\ \int_a^b f(x)d_q(x) &= \int_0^b f(x)d_qx - \int_0^a f(x)d_qx, \end{aligned} $$

to rewrite, the left hand side of (1.3) as:

(1.6)

$$
\begin{aligned}
S(a, b, c, d, f, g; \lambda, \mu) \\
&= \frac{(\mu/f, \mu/g; q)_\infty}{g(1-q)(q, gq/f, f/g, fg; q)_\infty} \int_f^g \frac{(uq/f; uq/g, \mu u; q)_\infty}{(\mu u/fg; q)_\infty} d_q u \\
&\quad \cdot \int_{-1}^{1} w(z; a, b, c, d) \frac{h(z; \lambda)}{h(z; u)}\, dz.
\end{aligned}
$$

At this stage they remarked, "It would be nice if we could find a transformation that would lead directly to (1.3). Such a short cut does not seem to exist at the moment, so we are forced into a rather long and tedious computation." It may, however, also be noted that it is not possible to extend Nassrallah-Rahman's proof to obtain integral representation of higher order basic hypergeometric series. In this paper we develop an alternative proof of (1.3) in Section 2 and show in Section 3 that our method of proof could be used with advantage to obtain

$$
\begin{aligned}
S(a, b, c, d, (f_n); (\lambda_n)) &\equiv \int_{-1}^{1} w(z; a, b, c, d) \prod_{j=1}^{n} \left\{ \frac{h(z; \lambda_j)}{h(z; f_j)} \right\} dz \\
&= K \frac{(q/abcd, aq/b, aq/c, aq/d; q)_\infty}{(a^2 q, q/bc, q/bd, q/cd; q)_\infty} \prod_{j=1}^{n} \left\{ \frac{(a\lambda_j, \lambda_j/a; q)_\infty}{(af_j, f_j/a; q)_\infty} \right\} \\
&\quad \cdot {}_{2n+6}W_{2n+5} \left[a^2; ab, ac, ad, a(f_n), aq/(\lambda_n); q; \frac{\lambda_1 \lambda_2 \ldots \lambda_n q}{abcdf_1 \ldots f_n} \right] \\
&\quad + \text{idem}\,(a; f_1, f_2, \ldots, f_n),
\end{aligned} \tag{1.7}
$$

provided $\max(|q|, |a|, |b|, |c|, |d|, |(f_n)|) < 1$, $\left| \frac{\lambda_1 \lambda_2 \ldots \lambda_n q}{abcdf_1 \ldots f_n} \right| < 1$, which for $n = 2$ reduces to (1.3). A complete combinatorial theory and interpretations for integrals of the form (1.7) with $\lambda_j = 0$, $j = 1, 2, \ldots, n$, was developed by Ismail, Stanton and Viennot [**5**]. Interesting quadratic transformations implied by (1.7) are discussed in Section 4.

2. We begin this section by first proving the transformation:

(2.1)

$$
\begin{aligned}
&{}_{2n+6}W_{2n+5}\left[t^2; tb, tc, td, tx_1, \dots, tx_n, \frac{tq}{y_1}, \dots, \frac{tq}{y_n}; q, \frac{y_1 \dots y_n q}{bcdtx_1 \dots x_n}\right] \\
&= \frac{(q/bc, q/bd, q/cd, t^2q; q)_\infty}{(tq/b, tq/c, tq/d, q/bcdt; q)_\infty} \sum_{j=0}^{\infty} \frac{(bt, ct, dt; q)_j q^j}{(q, bcdt, t^2q; q)_j} \\
&\cdot {}_{2n+4}W_{2n+3}\left[t^2; tx_1, \dots, tx_n, \frac{tq}{y_1}, \dots, \frac{tq}{y_n}, q^{-j}; q, \frac{y_1 \dots y_n}{x_1 \dots x_n} q^j\right] \\
&\quad + \frac{(bt, ct, dt, tq^2/bcd; q)_\infty}{(tq/b, tq/c, tq/d, bcdt/q; q)_\infty} \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \frac{(q/bc, q/bd, q/cd; q)_j q^j}{(q;q)_j \left(\frac{tq^2}{bcd}; q\right)_{j+r} \left(\frac{q^2}{bcdt}; q\right)_{j-r}} \\
&\cdot \frac{\left(t^2, tq, -tq, tx_1, \dots, tx_n, \frac{tq}{y_1}, \dots, \frac{tq}{y_n}; q\right)_r (-1)^r q^{\frac{r}{2}(r-1)} (y_1 \dots y_n)^r}{(q, t, -t, tq/x_1, \dots, tq/x_n, ty_1, \dots, ty_n; q)_r \; (x_1 \dots x_n)^r}.
\end{aligned}
$$

Proof of (2.1). In the Watson's transformation [**4**; 8.5(3)], connecting nonterminating very well-poised ${}_8W_7$ with balanced ${}_4\phi_3$'s letting $e \to 0$, we have

$$
\begin{aligned}
\frac{(aq/f, aq/g, aq/h; q)_\infty}{(f, g, h; q)_\infty} &= \frac{(aq, aq/fg, aq/fh, aq/gh; q)_\infty}{(f, g, h, aq/fgh; q)_\infty} \\
&\cdot {}_3\phi_2\left[\begin{matrix} f, g, h \\ aq, fgh/a \end{matrix}; q, q\right] \\
&+ \frac{(a^2q^2/fgh; q)_\infty}{(fgh/aq; q)_\infty} {}_3\phi_2\left[\begin{matrix} aq/fg, aq/gh, aq/fh \\ aq^2/fgh, a^2q^2/fgh \end{matrix}; q, q\right].
\end{aligned} \tag{2.2}
$$

In (2.2) replacing f, g, h and a by btq^r, ctq^r, dtq^r and t^2q^{2r}, respectively, and then multiplying by

$$
\frac{(t^2, tq, -tq, tx_1, tx_2, \dots, tx_n, tq/y_1, \dots, tq/y_n; q)_r (y_1 \dots y_n)^r}{(q, t, -t, tq/x_1, \dots, tq/x_n, ty_1, \dots, ty_n; q)_r \; (bcdtx_1 \dots x_n)^r}
$$

and summing with respect to r from 0 to ∞, we get (2.1) on rearranging the series on the right hand side. □

Proof of (1.3). Transforming the ${}_8W_7$ on the right hand side of (1.2) by [**1**; 5.1], we get

(2.3)

$$\int_{-1}^{1} w(z;a,b,c,d)\frac{h(z;\lambda)}{h(z;f)}\,dz = K\frac{(q/abcd, aq/b, aq/c, aq/d, \lambda a, \lambda/a;q)_\infty}{(a^2q, q/bc, q/bd, q/cd, af, f/a;q)_\infty}$$

$$\cdot\, {}_8W_7[a^2; ab, ac, ad, af, aq/\lambda; q, \lambda q/abcdf] + \text{idem}\ \ (a;f).$$

Next, using (2.3) in (1.6), we get

(2.4)

$$S(a,b,c,d,f,g;\lambda,\mu)$$

$$= \frac{2\pi(\mu/f,\mu/g;q)_\infty}{g(1-q)(q,gq/f,f/g,fg;q)_\infty}\int_f^g \frac{(uq/f,uq/g,\mu u;q)_\infty}{(\mu u/fg;q)_\infty}d_qu$$

$$\cdot\left\{\frac{(abcd, q/abcd, aq/b, aq/c, aq/d, \lambda a, \lambda/a;q)_\infty}{(q,ab,ac,ad,bc,bd,cd,a^2q,q/bc,q/bd,q/cd,au,u/a;q)_\infty}\right.$$

$$\left.\cdot\, {}_8W_7[a^2;ab,ac,ad,au,aq/\lambda;q,\lambda q/abcdu] + \text{idem}\ \ (a;u)\right\}$$

$$= K\frac{\left(q/abcd, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \lambda a, \frac{\lambda}{a}, \frac{\mu}{f}, \frac{\mu}{g};q\right)_\infty}{(a^2q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, q, \frac{gq}{f}, \frac{f}{g}, fg;q)_\infty}$$

$$\cdot\sum_{j=0}^{\infty}\frac{\left(a^2,aq,-aq,ab,ac,ad,\frac{ad}{\lambda};q\right)_j\lambda^jq^j}{\left(q,a,-a,\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},a\lambda;q\right)_j(abcd)^j}$$

$$\cdot\int_f^g\frac{(uq/f,uq/g,\mu u,aq^{1+j}/u;q)_\infty}{(\mu u/fg,u/a,aq/u,auq^j;q)_\infty u^j}d_qu$$

$$+\frac{2\pi(\mu/f,\mu/g;q)_\infty}{g(1-q)(q,gq/f,f/g,fg;q)_\infty}$$

$$\cdot\int_f^g\frac{\left(\frac{uq}{f},\frac{uq}{g},\mu u, ubcd, \frac{q}{ubcd}, \frac{uq}{b},\frac{uq}{c},\frac{uq}{d},\lambda u,\frac{\lambda}{u};q\right)_\infty}{\left(\frac{\mu u}{fg},q,ub,uc,ud,bc,bd,cd,u^2q,q/bc;q\right)_\infty}$$

$$\cdot\frac{1}{(q/bd,q/cd,au,a/u;q)_\infty}{}_8W_7[u^2;ub,uc,ud,au,uq/\lambda;q,\lambda q/abcdu]d_qu$$

$$\equiv I_1 + I_2 \text{ (say)}$$

$$I_1 = \frac{K\left(\frac{q}{abcd}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \lambda a, \frac{\lambda}{a}, \mu g, \frac{\mu}{g}; q\right)_\infty}{(a^2q, q/bc, q/bd, q/cd, f/g, fg, \frac{q}{a}, ag; q)_\infty}$$

$$\cdot \sum_{j=0}^{\infty} \frac{(a^2, aq, -aq, ab, ac, ad, ag, ag/\lambda; q)_j \lambda^j q^j}{\left(q, a, -a, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{g}, a\lambda; q\right)_j (abcdg)^j}$$

$$\cdot \Bigg\{ {}_3\phi_2 \left[\begin{matrix} \mu/f, agq^j, (g/a)q^{-j} \\ gq/f, \mu g \end{matrix} ; q, q \right] \tag{2.5}$$

$$- \frac{g^{j-1}\left(\frac{fg}{g}, \mu f, \frac{\mu}{f}, g/a, \frac{aq}{g}, agq^j, \frac{aq^{1+j}}{f}; q\right)_\infty}{f^{j-1}\left(\mu g, \frac{\mu}{g}, \frac{aq}{f}, \frac{f}{a}, \frac{gq}{f}, afq^j, aq^{1+j}/g; q\right)_\infty}$$

$$\cdot {}_3\phi_2 \left[\begin{matrix} \mu/g, afq^j, (f/a)q^{-j} \\ fq/g, \mu f \end{matrix} ; q, q \right] \Bigg\}.$$

Next, using (2.2) (with f, g, h, a replaced by μ/f, agq^j, $\frac{g}{a}q^{-j}$, $\mu g/q$, respectively) in (2.5), we have

$$I_1 = \frac{K(q/abcd, aq/b, aq/c, aq/d, \lambda a, \lambda/a, \mu a, \mu/a; q)_\infty}{(a^2q, q/bc, q/bd, q/cd, af, f/a, ag, g/a; q)_\infty} \tag{2.6}$$

$$\cdot {}_{10}W_9[a^2, ab, ac, ad, ag, af, aq/\lambda, aq/\mu; q, \lambda\mu q/abcdfg].$$

For evaluating I_2, transforming the inner ${}_8W_7$ by (2.1) (with $n = 1$, $x_1 = a$, $y_1 = \lambda$, $t = u$), we get

(2.7)

$$
\begin{aligned}
I_2 = {} & \frac{2\pi(\mu/g, \mu/f; q)_\infty}{g(1-q)(q, gq/f, f/g, fg, q, q/bc, q/bd, q/cd, bc, bd, cd; q)_\infty} \\
& \cdot \int_f^g \frac{(uq/f, uq/g, \mu u, ubcd, q/ubcd, uq/b, uq/c, uq/d, \lambda u, \lambda/u; q)_\infty}{(\mu u/fg, ub, uc, ud, u^2q, au, a/u; q)_\infty} \\
& \cdot \Bigg\{ \frac{\left(\frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, u^2q; q\right)_\infty}{\left(\frac{uq}{b}, \frac{uq}{c}, \frac{uq}{d}, \frac{q}{bcdu}; q\right)_\infty} \sum_{j=0}^{\infty} \frac{(bu, cu, du; q)_j q^j}{(q, u^2q, bcdu; q)_j} \\
& \cdot {}_6W_5\left[u^2; ua, \frac{uq}{\lambda}, q^{-j}; q, \frac{\lambda q^j}{a}\right] \\
& \qquad + \frac{(bu, cu, du, uq^2/bcd; q)_\infty}{\left(\frac{uq}{b}, \frac{uq}{c}, \frac{uq}{d}, \frac{bcdu}{q}; q\right)_\infty} \\
& \cdot \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}; q\right)_j \left(u^2, uq, -uq, ua, \frac{uq}{\lambda}; q\right)_r q^j}{(q;q)_j (uq^2/bcd; q)_{j+r} (q^2/bcdu; q)_{j-r}} \\
& \cdot \frac{\lambda^r (-1)^r q^{\frac{r}{2}(r-1)}}{(q, u, -u, uq/a, u\lambda; q)_r a^r} \Bigg\} d_q u \\
& = [\ldots] \int_0^g [\ldots] d_q u - [\ldots] \int_0^f [\ldots] d_q u \\
& \equiv I_2' + I_2''
\end{aligned}
$$

(2.8)

$$I_2' = \frac{2\pi(gbcd, q/gbcd, gq/b, gq/c, gq/d, \lambda g, \lambda/g, \mu g, \mu/g; q)_\infty}{(q, g^2q, gb, gc, gd, bc, bd, cd, q/bc, q/cd, q/bd, fg, f/g, ag, a/g; q)_\infty}$$

$$\cdot \left\{ \frac{\left(\frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, g^2q; q\right)_\infty}{\left(\frac{gq}{b}, \frac{gq}{c}, \frac{gq}{d}, \frac{q}{bcdg}; q\right)_\infty} \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^{j} \frac{(\mu/f; q)_s (bg, cg, dg; q)_{j+s}}{(q, \mu g, gq/f; q)_s (bcdg; q)_{j+s}} \right.$$

$$\cdot \frac{(gq, -gq, ga, gq/\lambda; q)_{r+s} (g^2; q)_{r+2s} q^{j+s} \lambda^{r+s} (-1)^r q^{\frac{r}{2}(r-1)}}{(g, -g, gq/a, \lambda g; q)_{r+s} (g^2q; q)_{j+r+2s} (q; q)_r (q; q)_{j-r} a^{r+s}}$$

$$+ \frac{(bg, cg, dg, gq^2/bcd; q)_\infty}{\left(\frac{gq}{b}, \frac{gq}{c}, \frac{gq}{d}, \frac{bcdg}{q}; q\right)_\infty}$$

$$\cdot \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\mu/f; q)_s \left(\frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}; q\right)_j (g^2; q)_{r+2s}}{(q, \mu g, gq/f; q)_s (q; q)_j (q; q)_r}$$

$$\left. \cdot \frac{(gq, -gq, ga, gq/\lambda; q)_{r+s} (-1)^r \lambda^{r+s} q^{j+r(r-1)/2}}{(g, -g, gq/a, g\lambda; q)_{r+s} (gq^2/bcd; q)_{j+r+s} (q^2/bcdg; q)_{j-r-s} a^{r+s}} \right\}$$

$$= \frac{2\pi(gbcd, q/gbcd, gq/b, gq/c, gq/d, \lambda g, \lambda/g, \mu g, \mu/g; q)_\infty}{(q, g^2q, gb, gc, gd, bc, bd, cd, q/bc, q/cd, q/bd, fg, f/g, ag, a/g; q)_\infty}$$

$$\cdot \left\{ \frac{(q/bc, q/bd, q/cd, g^2q; q)_\infty}{(gq/b, gq/c, gq/d, q/bcdg; q)_\infty} \right.$$

$$\cdot \sum_{j=0}^{\infty} \sum_{r=0}^{j} \frac{(bg, cg, dg; q)_j (g^2, gq, -gq, ga, \frac{gq}{\lambda}, q)_r}{(q, bcdg; q)_j (q, g, -g, gq/a, \lambda g; q)_r}$$

$$\cdot \frac{(q^{-j}; q)_r \lambda^r q^{jr}}{(g^2q; q)_{j+r} a^r} {}_3\phi_2 \left[\begin{matrix} \frac{\mu}{f}, g^2q^r, q^{-r} \\ \frac{gq}{f}, \mu g \end{matrix} ; q, q \right] + \frac{(bg, cg, dg, gq^2/bcd; q)_\infty}{(gq/b, gq/c, gq/d, bcdg/q; q)_\infty}$$

$$\cdot \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}; q\right)_j (g^2, gq, -gq, ag, gq/\lambda; q)_r \lambda^r (-1)^r q^{j+r(r-1)/2}}{(q; q)_j (q, g, -g, gq/a, g\lambda; q)_r (gq^2/bcd; q)_{j+r} (q^2/bcdg; q)_{j-r} a^r}$$

$$\left. \cdot {}_3\phi_2 \left[\begin{matrix} \frac{\mu}{f}, g^2q^r, q^{-r} \\ \frac{gq}{f}, \mu g \end{matrix} ; q, q \right] \right\}.$$

Summing the ${}_3\phi_2$'s on the right hand side by the q-analogue of Saalschütz's summation theorem [**4**; 8.4(1)] and then transforming the resulting right hand side by (2.1) (with $n = 2$, $t = g$, $x_1 = a$, $x_2 = f$, $y_1 = \lambda$, $y_2 = \mu$), we get

$$(2.9)\quad I_2' = \frac{2\pi(gbcd, q/gbcd, gq/b, gq/c, gq/d, \lambda g, \lambda/g, \mu g, \mu/g; q)_\infty}{\left(q, g^2q, gb, gc, gd, bc, bd, cd, \frac{q}{bc}, \frac{q}{cd}, \frac{q}{bd}, fg, f/g, ag, a/g; q\right)_\infty} \cdot {}_{10}W_9[g^2; gb, gc, gd, ga, gf, gq/\lambda, gq/\mu; q, \lambda\mu q/abcdgf].$$

Similarly, we have

(2.10)

$$I_2'' = \frac{2\pi(fbcd, q/fbcd, fq/b, fq/c, fq/d, \lambda f, \lambda/f, \mu f, \mu/f; q)_\infty}{(q, f^2q, fb, fc, fd, bc, bd, cd, q/bc, q/cd, q/bd, fg; g/f, af, a/f; q)_\infty} \cdot {}_{10}W_9[f^2; fb, fc, fd, fa, fg, fq/\lambda, fq/\mu; q, \lambda\mu q/abcdfg].$$

(2.4) gives (1.3) on substituting I_1 and I_2 from (2.6), (2.7), (2.9) and (2.10). □

3. Proof of (1.7). The proof is by induction (1.7) for $n = 1$ reduces to (2.3) and for $n = 2$ it yields (1.3). Next let us assume it to be true for $n = m$, so that on setting $f_m = f$, $f_{m+1} = g$, $\lambda_m = \lambda$ and $\lambda_{m+1} = \mu$, we get

(3.1)

$$S = S(a, b, c, d, f, g, (f_{m-1}); \lambda, \mu, (\lambda_{m-1})) = \int_{-1}^{1} w(z; a, b, c, d\frac{h(z;\lambda)h(z;\mu)}{h(z;f)h(z;g)} \prod_{j=1}^{m-1} \left\{\frac{h(z;\lambda_j)}{h(z;f_j)}\right\} dz$$

(3.2)

$$S = \frac{(\mu/f, \mu/g; q)_\infty}{g(1-q)(q, gq/f, f/g, fg; q)_\infty} \int_f^g d_q u \frac{(uq/f, uq/g, \mu u; q)_\infty}{(\mu u/fg; q)_\infty} \cdot \int_{-1}^{1} w(z; a, b, c, d) \frac{h(z;\lambda)}{h(z;u)} \prod_{j=1}^{m-1} \left\{\frac{h(z;\lambda_j)}{h(z;f_j)}\right\} dz \qquad \text{(using (1.6))}$$

$$= \frac{(\mu/f, \mu/g; q)_\infty}{g(1-q)\left(q, \frac{gq}{f}, \frac{f}{g}, fg; q\right)_\infty} \int_f^g \frac{\left(\frac{uq}{f}, \frac{uq}{g}, \mu u; q\right)_\infty}{(\mu u/fg; q)_\infty}$$

$$\cdot \left\{ K \frac{\left(\frac{q}{abcd}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}; q\right)_\infty}{\left(a^2q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}; q\right)_\infty} \right.$$

$$\cdot \frac{(a\lambda, \lambda/a; q)_\infty}{(au, u/a; q)_\infty} \prod_{j=1}^{m-1} \frac{(a\lambda_j, \lambda_j/a; q)_\infty}{(af_j, f_j/a; q)_\infty} {}_{2m+6}W_{2m+5}[a^2; ab, ac, ad, au,$$

$$aq/\lambda, a(f_{m-1}), aq/(\lambda_{m-1}); q, q\lambda\lambda_1 \ldots \lambda_{m-1}/abcduf_1 \ldots f_{m-1}]$$

(3.3)

$$\left. + \text{idem}\,(a; f_1, \ldots, f_{m-1}, u) \right\} d_q u$$

(3.4)

$$= I + I' \text{ (say)}$$

where

$$I = \frac{(\mu/f, \mu/g; q)_\infty}{g(1-q)\left(q, \frac{gq}{f}, \frac{f}{g}, fg; q\right)_\infty} \int_f^g \frac{\left(\frac{uq}{f}, \frac{uq}{g}, \mu u; q\right)_\infty}{(\mu u/fg; q)_\infty} \left\{ K \frac{\left(\frac{q}{abcd}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, ; q\right)_\infty}{\left(a^2q, \frac{q}{bc}, \frac{q}{bc}, \frac{q}{cd}; q\right)_\infty} \right.$$

$$\cdot \frac{(a\lambda, \lambda/a; q)_\infty}{(au, u/a; q)_\infty} \prod_{j=1}^{m-1} \frac{(a\lambda_j, \lambda_j/a; q)_\infty}{(af_j, f_j/a; q)_\infty}$$

$${}_{2m+6}W_{2m+5}\left[a^2; ab, ac, ad, au, \frac{aq}{\lambda}, a(f_{m-1}), \frac{aq}{(\lambda_{m-1})}; q, \frac{q\lambda\lambda_1 \ldots \lambda_{m-1}}{abcduf_1 \ldots f_{m-1}}\right]$$

(3.5)

$$\left. + \text{idem}\,(a; f_1, \ldots, f_{m-1}) \right\} d_q u$$

(3.6)

$$= I_1 + I_2 + \cdots + I_m \quad \text{(say)}$$

and

(3.7)

$$I' = \frac{2\pi(\mu/f, \mu/g; q)_\infty}{g(1-q)\left(q, \frac{gq}{f}, \frac{f}{g}, fg; q\right)_\infty}$$

$$\cdot \int_f^g \frac{\left(\frac{uq}{f}, \frac{uq}{g}, \mu u, ubcd, \frac{q}{ubcd}, \frac{uq}{b}, \frac{uq}{c}, \frac{uq}{d}, u\lambda, \frac{\lambda}{u}; q\right)_\infty}{\left(\frac{\mu u}{fg}, q, ub, uc, ud, bc, bd, cd, u^2q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, au, \frac{a}{u}; q\right)_\infty}$$

$$\cdot \prod_{j=1}^{m-1} \frac{(u\lambda_j, \lambda_j/u; q)_\infty}{(uf_j, f_j/u; q)_\infty} {}_{2m+6}W_{2m+5}\Big[u^2; ub, uc, ud, ua, uq/\lambda, u(f_{m-1}),$$

$$uq/(\lambda_{m-1}); q, q\lambda\lambda_1 \dots \lambda_{m-1}/abcduf_1 \dots f_{m-1}\Big] d_q u$$

(3.8)

$$I_1 = \frac{K\left(\frac{q}{abcd}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, a\lambda, \lambda/a, \mu g, \mu/g; q\right)_\infty}{\left(a^2q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, ag, \frac{g}{a}, fg, f/g; q\right)_\infty} \prod_{j=1}^{m-1} \left\{ \frac{(a\lambda_j, \lambda_j/a; q)_\infty}{(af_j, f_j/a; q)_\infty} \right\}$$

$$\cdot \sum_{j=0}^{\infty} \frac{\left(a^2, aq, -aq, ab, ac, ad, ag, aq/\lambda, a(f_{m-1}), \frac{aq}{(\lambda_{m-1})}; q\right)_j (q\lambda\lambda_1 \dots \lambda_{m-1})^j}{\left(q, a, -a, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \lambda a, \frac{aq}{(f_{m-1})}, \frac{aq}{g}, a(\lambda_{m-1}); q\right)_j (abcdgf_1 \dots f_{m-1})^j}$$

$$\cdot \Bigg\{ {}_3\phi_2 \left[\begin{matrix} \frac{\mu}{f}, \frac{g}{a}q^{-j}, agq^j \\ \frac{gq}{f}, g\mu \end{matrix} ; q, q \right]$$

$$- \left(\frac{g}{f}\right)^{j-1} \frac{(ag, f\mu, fq/g, \mu/f, g/a; q)_\infty (af, aq/g; q)_j}{\left(af, g\mu, gq/f, f/a, \frac{\mu}{g}; q\right)_\infty (ag, aq/f; q)_j}$$

$$\cdot {}_3\phi_2 \left[\begin{matrix} \mu/g, \frac{f}{a}q^{-j}, afq^j \\ gq/f, f\mu \end{matrix} ; q, q \right] \Bigg\}$$

$$= \frac{K(q/abcd, aq/b, aq/c, aq/d, a\lambda, \lambda/a, a\mu, \mu/a; q)_\infty}{(a^2q, q/bc, q/bd, q/cd, ag, g/a, af, f/a; q)_\infty} \prod_{j=1}^{m-1} \left\{ \frac{(a\lambda_j, \lambda_j/a; q)_\infty}{(af_j, f_j/a; q)_\infty} \right\}$$

$$\cdot {}_{2m+8}W_{2m+7}\Bigg[a^2; ab, ac, ad, ag, af, aq/\lambda, aq/\mu, a(f_{m-1}), \frac{aq}{(\lambda_{m-1})};$$

$$q, \frac{q\lambda\mu\lambda_1 \dots \lambda_{m-1}}{abcdfgf_1 \dots f_{m-1}}\Bigg]$$

Similarly, $I_2, \dots, I_m$ are also evaluated and we have

(3.9)

$$I = \frac{K(q/abcd, aq/b, aq/c, aq/d, a\lambda, \lambda/a, a\mu, \mu/a; q)_\infty}{(a^2q, q/bc, q/bd, q/cd, ag, g/a, af, f/a; q)_\infty} \cdot \prod_{j=1}^{m-1} \left\{ \frac{(a\lambda_j, \lambda_j/a; q)_\infty}{(af_j, f_j/a; q)_\infty} \right\}$$

$${}_{2m+8}W_{2m+7}\left[a^2; ab, ac, ad, ag, af, \frac{aq}{\lambda}, \frac{aq}{\mu}, a(f_{m-1}), \frac{aq}{(\lambda_{m-1})}; q, \frac{q\lambda\mu\lambda_1 \dots \lambda_{m-1}}{abcdfgf_1 \dots f_{m-1}}\right] + \text{idem}\,(a; f_1, \dots, f_{m-1})$$

For evaluation of I', transform inner ${}_{2m+6}W_{2m+5}$ by (2.1) (with $n = m$, $t = u$, $x_1 = a$, $x_2 = f_1, \dots, x_m = f_{m-1}$, $y_1 = \lambda$, $y_2 = \lambda_1, \dots, y_m = \lambda_{m-1}$) to obtain

$$I' = \frac{2\pi(\mu/f, \mu/g; q)_\infty}{g(1-q)\left(q, \frac{gq}{f}, \frac{f}{g}, fg; q\right)_\infty} \cdot \int_f^g \frac{\left(\frac{uq}{f}, \frac{uq}{g}, \mu u, ubcd, \frac{q}{ubcd}, \frac{uq}{b}, \frac{uq}{c}, \frac{uq}{d}, u\lambda, \frac{\lambda}{u}; q\right)_\infty}{\left(\frac{\mu u}{fg}, q, ub, uc, ud, bc, bd, cd, u^2q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, au, \frac{a}{u}; q\right)_\infty}$$

$$\cdot \prod_{j=1}^{m-1} \frac{\left(u\lambda_j, \frac{\lambda_j}{u}; q\right)_\infty}{\left(uf_j, \frac{f_j}{u}; q\right)_\infty} \left\{ \frac{(q/bc, q/bd, q/cd, u^2q; q)_\infty}{(uq/b, uq/c, uq/d, q/bcdu; q)_\infty} \sum_{j=0}^\infty \frac{(bu, cu, du; q)_j q^j}{(q, bcdu, u^2q; q)_j} \right.$$

$$\cdot {}_{2m+4}W_{2m+3}\left[u^2; ua, \frac{uq}{\lambda}, u(f_{m-1}), \frac{uq}{(\lambda_{m-1})}, q^{-j}; q, \frac{\lambda\lambda_1 \dots \lambda_{m-1} q^j}{af_1 \dots f_{m-1}}\right]$$

$$+ \frac{(bu, cu, du, uq^2/bcd; q)_\infty}{\left(\frac{uq}{b}, \frac{uq}{c}, \frac{uq}{d}, \frac{bcdu}{q}; q\right)_\infty} \sum_{j=0}^\infty \sum_{r=0}^\infty \frac{(q/bc, q/bd, q/cd; q)_j \; q^j (-1)^r q^{r(r-1)/2}}{(q; q)_j (uq^2/bcd; q)_{j+r} (q^2/bcdu; q)_{j-r}}$$

$$\left. \cdot \frac{(u^2, uq, -uq, ua, uq/\lambda, u(f_{m-1}), uq/(\lambda_{m-1}); q)_r \; (\lambda\lambda_1 \dots \lambda_{m-1})^r}{(q, u, -u, uq/a, \lambda u, uq/(f_{m-1}), u(\lambda_{m-1}); q)_r \; (af_1 \dots f_{m-1})^r} \right\}$$

(3.10)

$$= [\dots] \int_0^g [\dots] d_q u - [\dots] \int_0^f [\dots] d_q u$$

(3.11)
$\equiv I_1' + I_2'$ (say)

Then

$$I_1' = \frac{2\pi\left(gbcd, \frac{q}{gbcd}, \frac{gq}{b}, \frac{gq}{c}, \frac{gq}{d}, g\lambda, \frac{\lambda}{g}, g\mu, \mu/g; q\right)_\infty}{\left(q, gb, gc, gd, bc, bd, cd, g^2q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, ga, \frac{a}{g}, gf, \frac{f}{g}; q\right)_\infty}$$

$$\cdot \prod_{j=1}^{m-1}\left\{\frac{(g\lambda_j, \lambda_j/g; q)_\infty}{(gf_j, f_j/g; q)_\infty}\right\}\left\{\frac{(q/bc, q/bd, q/cd, g^2q; q)_\infty}{(gq/b, gq/c, gq/d, q/bcdg; q)_\infty}\right.$$

$$\cdot \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{j}\frac{(\mu/f; q)_s(bg, cg, dg; q)_{j+s}(-1)^r q^{j+s+r(r-1)/2}}{(q, \mu g, gq/f; q)_s\ (bcdg; q)_{j+s}(q; q)_{j-r}}$$

$$\cdot \frac{(g^2; q)_{r+2s}(gq, -gq, ga, g(f_{m-1}), gq/\lambda, gq/(\lambda_{m-1}); q)_{r+s}}{(q; q)_r(g, -g, gq/a, g\lambda, gq/(f_{m-1}), g(\lambda_{m-1}); q)_{r+s}(g^2q; q)_{j+r+2s}}$$

$$\cdot \frac{(\lambda\lambda_1\ldots\lambda_{m-1})^{r+s}}{(af_1\ldots f_{m-1})^{r+s}} + \frac{(bg, cg, dg, gq^2/bcd; q)_\infty}{\left(gq/b, gq/c, gq/d, \frac{bcdg}{q}; q\right)_\infty}$$

$$\cdot \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{\left(\frac{\mu}{f}; q\right)_s (q/bc, q/bd, q/cd; q)_j q^j}{\left(q, \mu g, \frac{gq}{f}; q\right)_s (q; q)_j \left(\frac{gq^2}{bcd}, q\right)_{j+r+s}}$$

$$\cdot \frac{(g^2; q)_{r+2s}(gq, -gq, ga, gq/\lambda, g(f_{m-1}), gq/(\lambda_{m-1}); q)_{r+s}}{\left(\frac{q^2}{bcdg}; q\right)_{j-r-s}(q; q)_r\left(g, -g, \frac{gq}{a}, \lambda g, \frac{gq}{(f_{m-1})}, g(\lambda_{m-1}); q\right)_{r+s}}$$

$$\left.\cdot \frac{(\lambda\lambda_1\ldots\lambda_{m-1})^{r+s}(-1)^r q^{r(r-1)/2}}{(af_1\ldots f_{m-1})^{r+s}}\right\}$$

$$= \frac{2\pi\left(gbcd, q/gbcd, \frac{gq}{b}, \frac{gq}{c}, \frac{gq}{d}, g\lambda, \frac{\lambda}{g}, g\mu, \mu/g; q\right)_\infty}{\left(q, gb, gc, gd, bc, bd, cd, g^2q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, ga, \frac{a}{g}, gf, \frac{f}{g}; q\right)_\infty}$$

$$\cdot \prod_{j=1}^{m-1}\left\{\frac{(g\lambda_j, \lambda_j/g; q)_\infty}{(gf_j, f_j/g; q)_\infty}\right\}$$

$$\cdot \left\{ \frac{(q/bc, q/bd, q/cd, g^2q; q)_\infty}{(gq/b, gq/c, gq/d, q/bcdg; q)_\infty} \sum_{j=0}^{\infty} \sum_{r=0}^{j} \frac{(bg, cg, dg; q)_j \; q^{j+jr}}{(q;q)_j (bcdg, g^2q; q)_j} \right.$$

$$\cdot \frac{(g^2, gq, -gq, ga, gq/\lambda, g(f_{m-1}), gq/(\lambda_{m-1}), q^{-j}; q)_r (\lambda\lambda_1 \ldots \lambda_{m-1})^r}{\left(q, g, -g, gq/a, g\lambda, \frac{gq}{(f_{m-1})}, g(\lambda_{m-1}), g^2q^{1+j}; q\right)_r (af_1 \ldots f_{m-1})^r}$$

$$\cdot {}_3\phi_2 \left[\begin{matrix} \frac{\mu}{f}, g^2q^r, q^{-r} \\ \frac{gq}{f}, \mu g \end{matrix} ; q, q \right]$$

$$+ \frac{(bg, cg, dg, gq^2/bcd; q)_\infty}{\left(gq/b, gq/c, gq/d, \frac{bcdg}{q}; q\right)_\infty} \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}; q\right)_j (-1)^r q^{j+r(r-1)/2}}{(q;q)_j \left(\frac{gq^2}{bcd}; q\right)_{j+r} \left(\frac{q^2}{bcdg}; q\right)_{j-r}}$$

$$\cdot \frac{(g^2, gq, -gq, ga, gq/\lambda, g(f_{m-1}), gq/(\lambda_{m-1}); q)_r (\lambda\lambda_1 \ldots \lambda_{m-1})^r}{(q, g, -g, gq/a, g\lambda, gq/(f_{m-1}), g(\lambda_{m-1}); q)_r (af_1 \ldots f_{m-1})^r}$$

$$\left. \cdot {}_3\phi_2 \left[\begin{matrix} \frac{\mu}{f}, g^2q^r, q^{-r} \\ \frac{gq}{f}, \mu g \end{matrix} ; q, q \right] \right\}$$

Summing the ${}_3\phi_2$'s on the right hand side by the q-analogue of Saalschutz summation theorem [**4**; 8.4(1)] and then transforming the resulting right hand side by (2.1) (with $n = m+1$, $t = g$, $x_1 = f_1$, $x_2 = f_2, \ldots, x_{m-1} = f_{m-1}$, $x_m = a$, $x_{m+1} = f$, $y_1 = \lambda_1$, $y_2 = \lambda_2, \ldots, y_{m-1} = \lambda_{m-1}$, $y_m = \lambda$, $y_{m+1} = \mu$), we get

(3.12)

$$I_1' = \frac{2\pi \left(gbcd, q/gbcd, gq/b, gq/c, gq/d, g\lambda, \frac{\lambda}{g}, g\mu, \frac{\mu}{g}; q\right)_\infty}{\left(q, gb, gc, gd, bc, bd, cd, g^2q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, ga, \frac{a}{g}, gf, \frac{f}{g}; q\right)_\infty}$$

$$\cdot \prod_{j=1}^{m-1} \left\{ \frac{\left(g\lambda_j, \frac{\lambda_j}{g}; q\right)_\infty}{\left(gf_j, \frac{f_j}{g}; q\right)_\infty} \right\}$$

$$\cdot {}_{2m+8}W_{2m+7}\left[g^2; gb, gc, gd, ga, gf, \frac{gq}{\lambda}, \frac{gq}{\mu}, g(f_{m-1}), \frac{gq}{(\lambda_{m-1})}; \right.$$

$$\left. q, \frac{\mu\lambda\lambda_1 \ldots \lambda_{m-1} q}{abcdfgf_1, \ldots f_{m-1}} \right].$$

Similarly, we have

(3.13)

$$I_2' = \frac{2\pi(fbcd, q/fbcd, fq/b, fq/c, fq/d, f\lambda, \lambda/f, f\mu, \mu/f; q)_\infty}{\left(q, fb, fc, fd, bc, bd, cd, f^2q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, fa, \frac{a}{f}, fg, \frac{g}{f}; q\right)_\infty}$$
$$\cdot \prod_{j=1}^{m-1} \left\{ \frac{\left(f\lambda_j, \frac{\lambda_j}{f}; q\right)_\infty}{(ff_j, f_j/f; q)_\infty} \right\}$$
$$\cdot {}_{2m+8}W_{2m+7}\Bigg[f^2; fb, fc, fd, fa, fg, \frac{fq}{\lambda}, \frac{fq}{\mu}, f(f_{m-1}), \frac{fq}{(\lambda_{m-1})}; q, \frac{\mu\lambda\lambda_1\ldots\lambda_{m-1}q}{abcdfgf_1\ldots f_{m-1}}\Bigg].$$

Combination of (3.4), (3.9), (3.11), (3.12) and (3.13) completes the proof of (1.7) by induction. □

4. In this section we obtain some quadratic transformations as special cases of (1.7).

Case I. Bailey showed [**4**; p. 54] that the nonterminating version of Whipple's transformation [**4**; 4.5(1)] connecting terminating nearly-poised ${}_4F_3$ with terminating balanced ${}_5F_4$ is a five term relation between two nearly-poised ${}_4F_3$ of the second kind, one nearly-poised ${}_4F_3$ of the first kind and the other two are balanced ${}_5F_4$. Recently, Nassrallah-Rahman [**7**;3.17] obtained a different nonterminating version of Whipple's transformation connecting three nearly-posed ${}_4F_3$'s of the second kind and the remaining two are balanced ${}_5F_4$ and gave its q-analogue. We obtain below a five term relation which for $q \to 1^-$ reduces to the aforesaid result of Bailey [**4** ;p. 54].

$$\frac{(-abc^2, -q/abc^2, aq/b; q)_\infty (a^2q^2/c^2, a^2\lambda^2, \lambda^2/a^2; q^2)_\infty}{(ab, a^2q, -c^2, -q/c^2; q)_\infty (a^2c^2, b^2c^2, a^2f^2, f^2/a^2, q^2/b^2c^2; q^2)_\infty}$$
$$\cdot {}_{10}W_9[a^2; ab, ac, -ac, af, -af, aq/\lambda, -aq/\lambda; q, -\lambda^2q/abc^2f^2]$$
$$+ \frac{(-fbc^2, -q/fbc^2, fq/b; q)_\infty (f^2q^2/c^2, f^2\lambda^2, \lambda^2/f^2; q^2)_\infty}{(fb, f^2q, -c^2, -q/c^2, af, a/f, -f^2, -1; q)_\infty (f^2c^2, b^2c^2, q^2/b^2c^2; q^2)_\infty}$$
$$\cdot {}_{10}W_9[f^2; fb, fc, -fc, af, -f^2, fq/\lambda, -fq/\lambda; q, -\lambda^2q/abc^2f^2]$$

$$+\frac{(fbc^2, q/fbc^2, -fq/b; q)_\infty (f^2q^2/c^2, f^2\lambda^2, \lambda^2/f^2; q^2)_\infty}{(-fb, f^2q, -c^2, -q/c^2, -af, -a/f, -f^2, -1; q)_\infty (f^2c^2, b^2c^2, q^2/b^2c^2; q^2)_\infty}$$

$$\cdot {}_{10}W_9\left[f^2; -fb, fc, -fc, -f^2, -af, \frac{fq}{\lambda}, \frac{-fq}{\lambda}; q, -\lambda^2 q/abc^2f^2\right]$$

(4.1)

$$= \frac{(\lambda^2 f^2, \lambda^2/f^2; q^2)_\infty (-abf^2; q)_\infty}{(a^2f^2, b^2f^2, c^2f^2, c^2/f^2; q^2)_\infty (ab, -f^2; q)_\infty}$$

$$\cdot {}_5\phi_4\left[\begin{matrix} a^2f^2, b^2f^2, \lambda^2/c^2, -f^2, -f^2q \\ \frac{f^2q^2}{c^2}, \lambda^2 f^2, -abf^2, -abf^2q \end{matrix}; q^2, q^2\right]$$

$$+ \frac{(\lambda^2 c^2, \lambda^2/c^2; q^2)_\infty (-abc^2; q)_\infty}{(a^2c^2, b^2c^2, f^2c^2, f^2/c^2; q^2)_\infty (ab, -c^2; q)_\infty}$$

$$\cdot {}_5\phi_4\left[\begin{matrix} a^2c^2, b^2c^2, \lambda^2/f^2, -c^2, -c^2q \\ \frac{c^2q^2}{f^2}, \lambda^2 c^2, -abc^2, -abc^2q \end{matrix}; q^2, q^2\right],$$

$$|\lambda^2 q/abc^2f^2| < 1.$$

Proof of (4.1). If we denote the left hand side of (4.1) by I, replacing d, g, and μ in (1.3) by $-c$, $-f$ and $-\lambda$, respectively, we get

$$I = \frac{(q;q)_\infty}{2\pi}\int_{-1}^{1} w(z; a, b, c, -c)\frac{h(z;\lambda)h(z;-\lambda)}{h(z;f)h(z;-f)}\,dz$$

$$= \frac{(q;q)_\infty (\lambda^2/c^2, \lambda^2/f^2; q^2)_\infty}{2\pi f^2(1-q^2)(q^2, f^2q^2/c^2, c^2/f^2, c^2f^2; q^2)_\infty}$$

$$\cdot \int_{c^2}^{f^2} \frac{(uq^2/c^2, uq^2/f^2, \lambda^2 u; q^2)_\infty}{(\lambda^2 u/c^2f^2; q^2)_\infty} d_p u$$

$$\cdot \int_{-1}^{1} w(z; a, b, \sqrt{u}, -\sqrt{u})\,dz, \qquad p = q^2 \text{ (on using (1.6))} \tag{4.2}$$

$$= \frac{(\lambda^2/c^2, \lambda^2/f^2; q^2)_\infty}{f^2(1-q^2)\left(q^2, \frac{f^2q^2}{c^2}, \frac{c^2}{f^2}, c^2f^2; q^2\right)_\infty (ab;q)_\infty}$$

$$\cdot \int_{c^2}^{f^2} d_p u \frac{\left(\frac{uq^2}{c^2}, \frac{uq^2}{f^2}, \lambda^2 u, -abu, -abuq; q^2\right)_\infty}{\left(\frac{\lambda^2 u}{c^2f^2}, -u, -uq, a^2u, b^2u; q^2\right)_\infty}$$

(on using (1.1))

(4.2) gives (4.1) on using (1.5). □

Next, in (4.1) replacing a, b, c, f and λ by $-iq^a, iq^b, iq^c, iq^f$ and iq^λ, respectively, and letting $q \to 1^-$, we get the following formula of Bailey [**4**]:

$$
\begin{aligned}
&\Gamma\left[\begin{matrix} b+c, 1-b-c, 1-2c, a+f, f-a, a+b, a+c, 2c; \\ a+b+2c, 1-a-b-2c, 1+a-c, a+\lambda, \lambda-a \end{matrix}\right] \\
&\cdot {}_4F_3\left[\begin{matrix} a+b, a+c, a+f, 1+a-\lambda; \\ 1+a-c, 1+a-f, a+\lambda \end{matrix}\right] \\
&\quad + \Gamma\left[\begin{matrix} b+c, 1-b-c, 2c, 1-2c, a+f, c+f, 2f; \\ 1+f-b, 1+f-c, \lambda+f, \lambda-f \end{matrix}\right] \\
&\cdot {}_4F_3\left[\begin{matrix} 2f, f+c, 1+f-\lambda, f+a; \\ 1+f-c, f+\lambda, 1+f-b \end{matrix}\right] \\
&\quad + \Gamma\left[\begin{matrix} b+c, 1-b-c, 2c, 1-2c, a-f, b+f, c+f, 2f; \\ f+b+2c, 1-f-b-2c, 1+f-c, \lambda+f, \lambda-f \end{matrix}\right] \\
&\cdot {}_4F_3\left[\begin{matrix} 2f, f+c, 1+f-\lambda, f+b; \\ 1+f-c, f+\lambda, 1+f-a \end{matrix}\right] \\
&= \Gamma\left[\begin{matrix} a+f, b+f, c+f, a+b, c-f, 2f; \\ \lambda+f, \lambda-f, a+b+2f \end{matrix}\right] \\
&\cdot {}_5F_4\left[\begin{matrix} \lambda-c, a+f, b+f, f, f+1/2; \\ 1+f-c, f+\lambda, \frac{(a+b+2f)}{2}, \frac{(1+a+b+2f)}{2} \end{matrix}\right] \\
&\quad + \Gamma\left[\begin{matrix} f+c, f-c, a+c, b+c, a+b, 2c; \\ \lambda+c, \lambda-c, a+b+2c \end{matrix}\right] \\
&\cdot {}_5F_4\left[\begin{matrix} \lambda-f, a+c, b+c, c, c+\frac{1}{2}; \\ 1+c-f, c+\lambda, \frac{(a+b+2c)}{2}, \frac{(1+a+b+2c)}{2} \end{matrix}\right]
\end{aligned}
\tag{4.3}
$$

Case II. Bailey [**4**; p. 54] remarks that the generalization of [**4**; 4.5(2)] contains five series in which two of the series are nearly-poised ${}_5F_4$ of the second kind, one is nearly-poised ${}_5F_4$ of the first kind and the other two are balanced ${}_5F_4$. Following Bailey's remark, we obtain the following formula:

(4.4)

$$\frac{(-abc^2, -q/abc^2, aq/b, -a^2; q)_\infty \left(\frac{a^2q^2}{c^2}, a^2\lambda^2, \lambda^2/a^2; q^2\right)_\infty}{(-q/c^2, -c^2, ab; q)_\infty (a^4q^2, q^2/b^2c^2, b^2c^2, a^2c^2, a^2f^2, f^2/a^2; q^2)_\infty}$$

$$\cdot {}_{12}W_{11}\left[a^2; ab, ac, -ac, af, -af, iaq, -iaq, \frac{aq}{\lambda}, -\frac{aq}{\lambda}; q, -\frac{\lambda^2}{abc^2f^2}\right]$$

$$+ \frac{a}{f} \frac{(-fbc^2, -q/fbc^2, fq/b; q)_\infty (f^2q^2/c^2, f^2\lambda^2, \lambda^2/f^2; q^2)_\infty}{(-1, -q/c^2, -c^2, fb, af, a/f; q)_\infty (f^4q^2, q^2/b^2c^2, b^2c^2, f^2c^2; q^2)_\infty}$$

$$\cdot {}_{12}W_{11}\left[f^2; fb, fc, -fc, af, -f^2, ifq, -ifq, fq/\lambda, -fq/\lambda; q, \frac{-\lambda^2}{abc^2f^2}\right]$$

$$- \frac{a}{f} \frac{(fbc^2, q/fbc^2, -fq/b, ab; q)_\infty \left(\frac{f^2q^2}{c^2}, f^2\lambda^2, \lambda^2/f^2; q^2\right)_\infty}{(-1, -q/c^2, -c^2, -bf, -af, -a/f; q)_\infty (f^4q^2, \frac{q^2}{b^2c^2}, b^2c^2, c^2f^2; q^2)_\infty}$$

$$\cdot {}_{12}W_{11}[f^2; -fb, -fc, fc, -f^2, -af, ifq, -ifq, \frac{fq}{\lambda}, -\frac{fq}{\lambda}; q, -\lambda^2/abc^2f^2]$$

$$= a^2 \frac{(\lambda^2f^2, \lambda^2/f^2; q^2)_\infty (-abf^2q, -b/a; q)_\infty}{(a^2f^2, b^2f^2, c^2f^2, c^2/f^2; q^2)_\infty (ab, -f^2q, -bq/a; q)_\infty}$$

$$\cdot {}_5\phi_4 \left[\begin{matrix} a^2f^2, b^2f^2, \lambda^2/c^2, -f^2q, -f^2q^2 \\ f^2q^2/c^2, \lambda^2f^2, -abf^2q, -abf^2q^2 \end{matrix}; q^2, q^2\right]$$

$$+ a^2 \frac{(\lambda^2c^2, \lambda^2/c^2; q^2)_\infty (-abc^2q, -b/a; q)_\infty}{(a^2c^2, b^2c^2, c^2f^2, f^2/c^2; q^2)_\infty (ab, -c^2q, -bq/a; q)_\infty}$$

$$\cdot {}_5\phi_4 \left[\begin{matrix} a^2c^2, b^2c^2, \lambda^2/f^2, -c^2q, -c^2q^2 \\ c^2q^2/f^2, \lambda^2c^2, -abc^2q, -abc^2q^2 \end{matrix}; q^2, q^2\right], \left|\frac{\lambda^2}{abc^2f^2}\right| < 1.$$

Proof of (4.4). If we denote the left hand side of (4.4) by I, setting $n = 3$, $d = -c$, $f_1 = f$, $f_2 = -f$, $f_3 = iq$, $\lambda_1 = \lambda$, $\lambda_2 = -\lambda$, $\lambda_3 = i$ in (1.7), we get

(4.5)

$$I = (q;q)_\infty \frac{ia}{2\pi} \int_{-1}^{1} w(z;a,b,c,-c) \frac{h(z;\lambda)h(z;-\lambda)h(z;i)}{h(z;f)h(z;-f)h(z;iq)}\, dz$$

$$= (q;q)_\infty \frac{ia\ (\lambda^2/c^2, \lambda^2/f^2; q^2)_\infty}{2\pi f^2(1-q^2)(q^2, f^2q^2/c^2, c^2/f^2, c^2f^2; q^2)_\infty}$$

$$\cdot \int_{c^2}^{f^2} d_p u \frac{\left(\frac{uq^2}{c^2}, \frac{uq^2}{f^2}, \lambda^2 u; q^2\right)_\infty}{(\lambda^2 u/c^2 f^2; q^2)_\infty}$$

$$\cdot \int_{-1}^{1} w(z;a,b,\sqrt{u},-\sqrt{u}) \frac{h(z;i)}{h(z;iq)}\, dz, \quad p = q^2 \text{ (on using (1.6))}.$$

However, (1.2) on specialization yields

(4.6)

$$\int_{-1}^{1} w(z;a,b,x,-x) \frac{h(z;i)}{h(z;iq)}\, dz = \frac{-2\pi i(a+b)(-abx^2q;q)_\infty}{(a^2x^2, b^2x^2; q^2)_\infty (q, ab, -x^2q; q)_\infty}$$

using (4.6) in (4.5), we have

(4.7)

$$I = \frac{a(a+b)(\lambda^2/c^2, \lambda^2/f^2; q^2)_\infty}{f^2(1-q^2)\left(q^2, \frac{f^2q^2}{c^2}, \frac{c^2}{f^2}, c^2f^2; q^2\right)_\infty (ab;q)_\infty}$$

$$\cdot \int_{c^2}^{f^2} d_p u \frac{(uq^2/c^2, uq^2/f^2, u\lambda^2, -abuq, -abuq^2; q^2)_\infty}{(\lambda^2 u/c^2 f^2, a^2u, b^2u, -uq, -uq^2; q^2)_\infty}$$

(4.7) on simplification in view of (1.5) yields (4.4).

Setting $a \to -iq^a$, $b \to iq^b$, $c \to iq^c$, $\lambda \to iq^\lambda$, $f \to iq^f$ in (4.4) and then letting $q \to 1^-$, we get the following formula of Bailey [**4**]:

$$\Gamma\left[\begin{matrix}1+2a,1-b-c,1-2c,b+c,2c,a+c,a+f,f-a,a+b;\\ a+b+2c,1-a-b-2c,1+a-c,a+\lambda,\lambda-a,2a\end{matrix}\right]$$

$$\cdot\,{}_5F_4\left[\begin{matrix}a+b,1+a,a+c,a+f,1+a-\lambda;\\ a,1+a-c,1+a-f,a+\lambda\end{matrix}\right]$$

$$+\,\Gamma\left[\begin{matrix}1-b-c,1-2c,b+c,2c,f+c,a+f,1+2f;\\ 1+f-b,1+f-c,f+\lambda,\lambda-f\end{matrix}\right]$$

$$\cdot\,{}_5F_4\left[\begin{matrix}2f,1+f,f+c,1+f-\lambda,a+f;\\ f,1+f-c,f+\lambda,1+f-b\end{matrix}\right]$$

$$-\,\Gamma\left[\begin{matrix}1-b-c,1-2c,b+c,2c,b+f,c+f,1+2f,a-f;\\ f+b+2c,1-f-b-2c,1+f-c,\lambda+f,\lambda-f\end{matrix}\right]$$

(4.8)

$$\cdot\,{}_5F_4\left[\begin{matrix}2f,1+f,f+c,1+f-\lambda,b+f;\\ f,1+f-c,f+\lambda,1+f-a\end{matrix}\right]$$

$$=\Gamma\left[\begin{matrix}c-f,c+f,a+f,b+f,a+b,1+2f,1+b-a;\\ \lambda-f,\lambda+f,1+a+b+2f,b-a\end{matrix}\right]$$

$$\cdot\,{}_5F_4\left[\begin{matrix}\lambda-c,f+1,f+1/2,a+f,b+f;\\ 1+f-c,\lambda+f,\frac{(1+a+b+2f)}{2},\frac{(2+a+b+2f)}{2}\end{matrix}\right]$$

$$+\,\Gamma\left[\begin{matrix}f-c,f+c,a+c,b+c,a+b,1+2c,1+b-a;\\ \lambda-c,\lambda+c,1+a+b+2c,b-a\end{matrix}\right]$$

$$\cdot\,{}_5F_4\left[\begin{matrix}\lambda-f,c+1,c+1/2,a+c,b+c;\\ 1+c-f,\lambda+c,\frac{(1+a+b+2c)}{2},\frac{(2+a+b+2c)}{2}\end{matrix}\right].$$

Case III. Next, we prove the following interesting quadratic transformation

(4.9)

$$\frac{(-cda^2,-q/cda^2,-q,aq/c,aq/d;q)_\infty(a^2\lambda^2,\lambda^2/a^2;q^2)_\infty}{(cd,-q/ac,-q/ad,q/cd;q)_\infty(a^4q^2,a^2c^2,a^2d^2,a^2f^2,f^2/a^2;q^2)_\infty}$$

$$\cdot\,{}_{12}W_{11}[a^2;-a^2,ac,ad,af,-af,iaq,-iaq,aq/\lambda,-aq/\lambda;q,-\lambda^2/a^2cdf^2]$$

$$+\frac{a}{f}\frac{(-facd,\frac{-q}{acdf},\frac{-fq}{a},\frac{fq}{c},fq/d;q)_\infty(f^2\lambda^2,\frac{\lambda^2}{f^2};q^2)_\infty}{(-1,-q/ac,\frac{-q}{ad},\frac{q}{cd},-ac,-ad,cd,cf,df,\frac{a}{f};q)_\infty(f^4q^2,a^2f^2;q^2)_\infty}$$

$$\cdot\,{}_{12}W_{11}[f^2;-f^2,-af,af,cf,df,ifq,-ifq,fq/\lambda,-fq/\lambda;q,-\lambda^2/a^2cdf^2]$$

$$-\frac{a}{f}\frac{(acdf,\frac{q}{acdf},\frac{fq}{a},-fq/c,-fq/d;q)_\infty(f^2\lambda^2,\lambda^2/f^2;q^2)_\infty}{\left(-1,-\frac{q}{ac},-\frac{q}{ad},\frac{q}{cd},-ac,-ad,cd,-cf,-df,-\frac{a}{f};q\right)_\infty(f^4q^2,a^2f^2;q^2)_\infty}$$

$$\cdot\,{}_{12}W_{11}[f^2;-f^2,af,-af,-cf,-df,ifq,-ifq,fq/\lambda,\frac{-fq}{\lambda};q,\frac{-\lambda^2}{a^2cdf^2}]$$

$$=\frac{c}{a}\frac{(\lambda^2/f^2,\lambda^2f^2;q^2)_\infty(-cdf^2q,-d/c;q)_\infty}{(a^2f^2,c^2f^2,d^2f^2,a^2/f^2;q^2)_\infty\left(cd,-f^2q,-\frac{dq}{c};q\right)_\infty}$$

$$\cdot\,{}_5\phi_4\left[\begin{matrix}c^2f^2,d^2f^2,\lambda^2/a^2,-f^2q,-f^2q^2\\ f^2q^2/a^2,f^2\lambda^2,-cdf^2q,-cdf^2q^2\end{matrix};q^2,q^2\right]$$

$$+\frac{c}{a}\frac{(\lambda^2/a^2,\lambda^2a^2;q^2)_\infty(-a^2cdq,-d/c;q)_\infty}{(a^2f^2,a^2c^2,a^2d^2,f^2/a^2;q^2)_\infty(cd,-a^2q,-dq/c;q)_\infty}$$

$$\cdot\,{}_5\phi_4\left[\begin{matrix}a^2c^2,a^2d^2,\lambda^2/f^2,-a^2q,-a^2q^2\\ a^2q^2/f^2,a^2\lambda^2,-a^2cdq,-a^2cdq^2\end{matrix};q^2,q^2\right],$$

$$|\lambda^2/a^2cdf^2|<1.$$

Proof of (4.9). Setting $n=3$, $b=-a$, $f_1=f$, $f_2=-f$, $f_3=iq$, $\lambda_1=\lambda$, $\lambda_2=-\lambda$, $\lambda_3=i$ in (1.7), and following the method of proof of (4.4) the proof of (4.9) can be completed.

Furthermore, it may be noted that (4.9) for $q\to 1^-$ reduces to the transformation which contains five series in which three are nearly poised of second kind and the remaining two are Saalschutzian and for $ac = q^{-N}$, it reduces to a q-analogue of Bailey's transformation [**4**; 4.5(2)] (cf. [**7**; 3.17]).

Case IV. Transformation connecting terminating very well poised ${}_{12}W_{11}$ on base q (base q^3) into terminating balanced ${}_6\phi_5$ on base q^3 (base q) were obtained in [**8**; 1.5,1.6]. Later on, their nonterminating versions connecting a nonterminating very well-poised ${}_{12}W_{11}$ on base q (base q^3) with two nonterminating balanced ${}_6\phi_5$ on base q^3 (base q) were given in [**9**; 5.1, 6.1]. Here, we present the following nonterminating versions of [**8**; 1.5, 1.6] different from the known ones:

(4.10)

$$\frac{(a\lambda, \lambda/a, aq/b; q)_\infty (ab^3q^3, 1/ab^3; q^3)_\infty}{(ab, af, b^2q, 1/b^2, f/a; q)_\infty (a^2q^3; q^3)_\infty}$$

$$\cdot {}_{12}W_{11}[a^2; ab, abq, abq^2, af, afq, afq^2, aq/\lambda, aq^2/\lambda, aq^3/\lambda; q^3, \lambda^3/ab^3f^3]$$

$$+ \frac{(f\lambda, \lambda/f, fq/b; q)_\infty (fb^3q^3, 1/fb^3; q^3)_\infty}{(fb, f^2q, b^2q, 1/b^2; q)_\infty (q, q^2, af, a/f; q)_\infty}$$

$$\cdot {}_{12}W_{11}[f^2; fb, fbq, fbq^2, af, f^2q, f^2q^2, fq/\lambda, fq^2/\lambda, fq^3/\lambda; q^3, \lambda^3/ab^3f^3]$$

$$+ \frac{(f\lambda q, \lambda/fq, fq^2/b; q)_\infty (f^2q^2, fb^3q^4, 1/fb^3q; q^3)_\infty}{(fbq, f^2q, b^2q, 1/b^2; q)_\infty (q^{-1}, q, afq, f^2q^5, a/fq; q^3)_\infty}$$

$$\cdot {}_{12}W_{11}\left[f^2q^2; fbq, fbq^2, fbq^3, f^2q, afq, f^2q^3, \frac{fq^2}{\lambda}, \frac{fq^3}{\lambda}, \frac{fq^4}{\lambda}; q^3, \frac{\lambda^3}{ab^3f^3}\right]$$

$$+ \frac{(f\lambda q^2, \lambda/fq^2, fq^3/b; q)_\infty (f^2q^4, fb^3q^5, 1/fb^3q^2; q^3)_\infty}{(f^2q^2, fbq^2, b^2q, 1/b^2; q)_\infty (q^{-1}, q^{-2}, f^2q^7, afq^2, a/fq^2; q^3)_\infty}$$

$$\cdot {}_{12}W_{11}\left[f^2q^4; fbq^2, fbq^3, fbq^4, f^2q^2, f^2q^3, afq^2, \frac{fq^3}{\lambda}, \frac{fq^4}{\lambda}, \frac{fq^5}{\lambda}; q^3, \frac{\lambda^3}{ab^3f^3}\right]$$

$$= \frac{(\lambda f, \frac{\lambda}{f}; q)_\infty (af^3q^3; q^3)_\infty}{(af, bf, f^2q, \frac{b}{f}; q)_\infty} {}_6\phi_5\left[\begin{matrix} \lambda/b, af, fq, -fq, f\sqrt{q}, -f\sqrt{q} \\ fq/b, \lambda f, fqa^{1/3}, fwqa^{1/3}, fw^2qa^{1/3} \end{matrix}; q, q\right]$$

$$+ \frac{(\lambda b, \lambda/b; q)_\infty (ab^3q^3; q^3)_\infty}{(ab, bf, b^2q, f/b; q)_\infty} {}_6\phi_5\left[\begin{matrix} \lambda/f, ab, b\sqrt{q}, -b\sqrt{q}, bq, -bq \\ bq/f, \lambda b, bqa^{1/3}, bwqa^{1/3}, bw^2qa^{1/3} \end{matrix}; q, q\right]$$

$$|\lambda^3/ab^3f^3| < 1.$$

(4.11)

$$\frac{(ab^3, q/ab^3; q)_\infty (a^3\lambda^3, \lambda^3/a^3, a^3q^3/b^3; q^3)_\infty}{(a^2q; q)_\infty (b^6, a^3b^3, a^3f^3, f^3/a^3, q^3/b^6; q^3)_\infty}$$

$$\cdot {}_{12}W_{11}[a^2; ab, abw, abw^2, af, afw, afw^2, \frac{aq}{\lambda}, \frac{aqw}{\lambda}, \frac{aqw^2}{\lambda}; q, \frac{\lambda^3}{ab^3f^3}]$$

$$+ \frac{(f^2, b^3f, q/b^3f; q)_\infty (f^3\lambda^3, \lambda^3/f^3, f^3q^3/b^3; q^3)_\infty}{(w, w^2, af, f^2q, a/f; q)_\infty (f^6, b^6, b^3f^3, q^3/b^6; q^3)_\infty}$$

$$\cdot {}_{12}W_{11}[f^2; bf, bfw, bfw^2, af, f^2w, f^2w^2, \frac{fq}{\lambda}, \frac{fqw}{\lambda}, \frac{fqw^2}{\lambda}; q, \frac{\lambda^3}{ab^3f^3}]$$

$$+ \frac{(f^2w^2, b^3fw, w^2q/b^3f; q)_\infty (f^3\lambda^3, \lambda^3/f^3, f^3q^3/b^3; q^3)_\infty}{(w, w^2, afw, f^2qw^2, aw^2/f; q)_\infty (f^6, b^6, b^3f^3, q^3/b^6; q^3)_\infty}$$

$$\cdot {}_{12}W_{11}[f^2w^2; bf, bfw, bfw^2, f^2, f^2w, afw, \frac{fq}{\lambda}, \frac{fqw}{\lambda}, \frac{fqw^2}{\lambda}; q, \frac{\lambda^3}{ab^3f^3}]$$

$$+ \frac{(f^2w, b^3fw^2, aw/b^3f; q)_\infty (f^3\lambda^3, \lambda^3/f^3, f^3q^3/b^3; q^3)_\infty}{(w, w^2, afw^2, aw/f, f^2qw; q)_\infty (f^6, b^6, b^3f^3, q^3/b^6; q^3)_\infty}$$

$$\cdot {}_{12}W_{11}[f^2w; bf, bfw, bfw^2, f^2, f^2w^2, afw^2, \frac{fq}{\lambda}, \frac{fqw}{\lambda}, \frac{fqw^2}{\lambda}; q, \frac{\lambda^3}{ab^3f^3}]$$

$$= \frac{(af^3; q)_\infty (\lambda^3f^3, \lambda^3/f^3; q^3)_\infty}{(f^6, a^3f^3, b^3f^3, b^3/f^3; q^3)_\infty}$$

$$\cdot {}_6\phi_5 \left[\begin{matrix} \frac{\lambda^3}{b^3}, a^3f^3, f^3, -f^3, f^3q^{3/2}, -f^3q^{3/2} \\ f^3q^3/b^3, \lambda^3f^3, af^3, af^3q, af^3q^2 \end{matrix} ; q^3, q^3 \right]$$

$$+ \frac{(ab^3; q)_\infty (\lambda^3b^3, \lambda^3/b^3; q^3)_\infty}{(b^6, b^3f^3, a^3b^3, f^3; b^3; q^3)_\infty}$$

$$\cdot {}_6\phi_5 \left[\begin{matrix} \frac{\lambda^3}{f^3}, a^3b^3, b^3, -b^3, b^3q^{3/2}, -b^3q^{3/2} \\ b^3q^3/f^3, \lambda^3b^3, ab^3, ab^3q, ab^3q^2 \end{matrix} ; q^3, q^3 \right],$$

$$|\lambda^3/ab^3f^3| < 1.$$

Proof of (4.10). In (1.7) first replacing q by q^3 and then setting $n = 3$, $c = bq$, $d = bq^2$, $f_1 = f$, $f_2 = fq$, $f_3 = fq^2$, $\lambda_1 = \lambda$, $\lambda_2 = \lambda q$, $\lambda_3 = \lambda q^2$, and using (1.1) we get (if we denote the left hand side of (4.10) by I)

$$I = \frac{(\lambda/b, \lambda/f; q)_\infty}{f(1-q)(q, bf, b/f, fq/b; q)_\infty} \int_b^f \frac{(uq/b, uq/f, \lambda u; q)_\infty (au^3q^3; q^3)_\infty}{(\lambda u/bf, au, u^2q; q)_\infty} d_q u$$

evaluating the right hand side by (1.5), we get (4.10). □

The proof of (4.11) follows on the lines of (4.4) and hence the details are omitted.

In this sequel it may be noted that Nassrallah-Rahman's quadratic transformation [**7**; 4.5] is in fact a nonterminating extension of [**8**; 1.4].

REFERENCES

1. R.P. Agarwal, *Some transformations of well-poised basic hypergeometric series of the type* ${}_8\phi_7$, Proc. Amer. Math. Soc. **4** (1953), 678–685.

2. W.A. Al Salam and A. Verma, *Some remarks on q-beta integral*, Proc. Amer. Math. Soc. **85** (1982), 360–362.

3. R. Askey and J.A. Wilson, *Some basic hypergeometric polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **319** (1985).

4. W.N. Bailey, *Generalized hypergeometric series*, Stechert-Hafner Service Agency, New York and London, 1964.

5. M.E.H. Ismail, D. Stanton and G. Viennot, *The combinatorics of q-Hermite polynomials and the Askey-Wilson integral*, European J. Combinatorics **8** (1987), 379–392.

6. B. Nassrallah and Mizan Rahman, *Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials*, SIAM J. Math. Anal. **16** (1985), 186–197.

7. ———, *A q-analogue of Appell's* F_1 *function and some quadratic transformation formulas for nonterminating basic hypergeometric series*, Rocky Mountain J. Math. **16** (1986), 63–82.

8. A. Verma and V.K. Jain, *Transformations between basic hypergeometric series on different bases and identities of Rogers-Ramanujan type*, J. Math. Anal. Appl. **76** (1980), 230–269.

9. ———, *Transformations of nonterminating basic hypergeometric series, their contour integrals and applications to Rogers-Ramanujan identities*, J. Math. Anal. Appl. **87** (1982), 9–44.

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