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SOME NONCOMPACT HYPERSPACES WITH THE ALMOST FIXED POINT PROPERTY

SAM B. NADLER, JR.

ABSTRACT. It is shown that for certain types of subsets Z of arc-like continua, the space of all nonempty subcontinua of Z has the almost fixed point property. Our results generalize the well-known theorem that the space of all nonempty subcontinua of an arc-like continuum has the fixed point property. An example is given to show that our results do not extend to circle-like continua. Proofs are based on the use of the relatively new notion of almost universal mappings.

1. Introduction. A metric space (Y, d) is said to have the *almost fixed point property* (AFPP) provided that if $f : Y \to Y$ is a mapping (= continuous function), then, for each $\varepsilon > 0$, there exists $y_\varepsilon \in Y$ such that $d(f(y_\varepsilon), y_\varepsilon) < \varepsilon$. We note that for compact metric spaces, AFPP and the fixed point property are equivalent. Also, AFPP is preserved by uniformly continuous homeomorphisms (but not by homeomorphisms in general).

In [**1, 2, 15**] certain metric spaces are shown to have AFPP. We remark that the techniques in this paper are different than those in [**2, 15**] in that we use the notion of almost universal mappings which was defined and studied in [**1**].

A *continuum* is a nonempty compact connected metric space. For a continuum X, $C(X)$ denotes the space of all (nonempty) subcontinua of X with the Hausdorff metric [**9**]; if $Z \subset X$, then $C(Z) = \{A \in C(X) : A \subset Z\}$. An *arc-like* (*chainable*) *continuum* is a continuum X such that for each $\varepsilon > 0$, there is an ε-mapping from X onto $[0, 1]$. In [**14**] it was shown that if X is an arc-like continuum, then $C(X)$ has the fixed point property. In this paper we generalize this result by showing that $C(Z)$ has AFPP for certain types of subsets Z of arc-like continua. If X is a circle-like continuum, then $C(X)$ has the fixed point

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property [**5, 13**]. However, as we show in Example 2.6, the analogues of our results for subsets of circle-like continua are false. We remark that there do not seem to be any previous results concerning the almost fixed point property for noncompact hyperspaces.

We now define the notions of universal and almost universal mappings. Let (X, d) and (Y, p) be metric spaces. A mapping $f : X \to Y$ is *universal* [**4**] provided that given any mapping $g : X \to Y$, there exists $x_0 \in X$ such that $f(x_0) = g(x_0)$. A mapping $f : X \to Y$ is *almost universal* [**1**] provided that given any mapping $g : X \to Y$ and any $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that $p(f(x_\varepsilon), g(x_\varepsilon)) < \varepsilon$.

Recall that if f is a mapping from a continuum X into a continuum Y, then the *induced mapping* $\hat{f} : C(X) \to C(Y)$ is defined by letting $\hat{f}(A) = \{f(a) : a \in A\}$ for each $A \in C(X)$.

2. Results. The following lemma is essentially contained in a proof in [**5**].

Lemma 2.1. *If f is any mapping from a continuum X onto $[0,1]$, then $\hat{f} : C(X) \to C([0,1])$ is universal.*

Proof. The proof of 4.1 of [**5**] shows that there is a subcontinuum Λ of $C(X)$ such that $\hat{f}|\Lambda$ maps Λ essentially onto the simple closed curve Γ which is the manifold boundary of the two-cell $C([0,1])$. Thus, since $C(X)$ is contractible with respect to Γ (1.181 of [**9**, p. 179]), $\hat{f}|\Lambda : \Lambda \to \Gamma$ cannot be extended to a mapping of $C(X)$ to Γ. Hence, $\hat{f}$ is an AH-essential mapping [**5**, p. 156] from $C(X)$ onto $C([0,1])$. Therefore, since AH-essential mappings onto n-cells are universal [**8**], $\hat{f}$ is universal.

In [**10**, 2.11] it was shown that if f is any mapping from a continuum X onto an arc-like continuum Y, then $\hat{f} : C(X) \to C(Y)$ is universal. This result implies the fixed point theorem in [**14**] and a theorem on weakly confluent mappings in [**11**, Theorem 4], see [**10**, 2.12, 2.13]. The following Proposition is a generalization of [**10**, 2.11] and is used to prove our results about AFPP.

Proposition 2.2. *Let X be a continuum and let Z be a subset of X such that there is a sequence of subcontinua of Z converging to X.*

Let f be a mapping from X onto an arc-like continuum M. Then, the mapping

$$\hat{f}|C(Z) : C(Z) \to C(M)$$

is almost universal.

Proof. Let $\varepsilon > 0$ and let $g : C(Z) \to C(M)$ be a mapping. Since M is an arc-like continuum, there is an ε-mapping h_ε from M onto $I = [0, 1]$. By [**5**, 2.5], $\hat{h}_\varepsilon : C(M) \to C(I)$ is an ε-mapping. Hence, using the compactness of $C(M)$ and $C(I)$, an easy sequence argument shows that there exists a $\delta > 0$ such that if $S \subset C(I)$ is of diameter less than δ, then $\hat{h}_\varepsilon^{-1}(S)$ is of diameter less than ε. Let H denote the Hausdorff metric for $C(I)$. Since X is compact, $h_\varepsilon \circ f$ is a uniformly continuous mapping from X onto I. Thus, from the hypothesis on Z, there is a subcontinuum Y of Z such that

$$H(h_\varepsilon \circ f(Y), I) < \delta.$$

Hence, $h_\varepsilon \circ f(Y)$ is a subinterval $I_\delta = [s_\delta, t_\delta]$ of I such that $s_\delta < \delta$ and $t_\delta > 1 - \delta$. Define $r_\delta : I \to I_\delta$ by

$$r_\delta(t) = \begin{cases} s_\delta, & \text{if } 0 \le t \le s_\delta \\ t, & \text{if } s_\delta \le t \le t_\delta \\ t_\delta, & \text{if } t_\delta \le t \le 1. \end{cases}$$

We easily see that for all $t \in I$, $|r_\delta(t) - t| < \delta$. Hence,

$$(*) \qquad \forall J \in C(I), \qquad H(J, \hat{r}_\delta(J)) < \delta.$$

Let $k_\varepsilon = h_\varepsilon \circ f|Y$. Since Y is a continuum and k_ε maps Y onto I_δ, $\hat{k}_\varepsilon : C(Y) \to C(I_\delta)$ is universal by Lemma 2.1. Note that since $Y \subset Z$, $\hat{r}_\delta \circ \hat{h}_\varepsilon \circ g|C(Y)$ is a mapping defined on all of $C(Y)$ into $C(I_\delta)$. Thus, $\hat{k}_\varepsilon : C(Y) \to C(I_\delta)$ being universal, there exists $A \in C(Y)$ such that

$$\hat{k}_\varepsilon(A) = \hat{r}_\delta \circ \hat{h}_\varepsilon \circ g(A).$$

Hence, by $(*)$, $H(\hat{h}_\varepsilon \circ g(A), \hat{k}_\varepsilon(A)) < \delta$. Therefore,

$$H(\hat{h}_\varepsilon(g(A)), \hat{h}_\varepsilon(\hat{f}(A))) < \delta.$$

Thus, using the definition of δ, we see that $g(A)$ and $\hat{f}(A)$ are less than ε apart in the Hausdorff metric on $C(M)$. Therefore, since $A \in C(Z)$, we have proved Proposition 2.2. $\square$

Theorem 2.3. *Let X be an arc-like continuum. If Z is a subset of X such that there is a sequence of subcontinua of Z converging to a continuum $M \supset Z$, then $C(Z)$ has AFPP.*

Proof. Since M is a subcontinuum of X, M is an arc-like continuum or a point. Let $f : M \to M$ be the identity map. Then, by Proposition 2.2,

$$\hat{f}|C(Z) : C(Z) \to C(M)$$

is almost universal. Therefore, since $\hat{f}(A) = A$ for all $A \in C(Z)$, it follows easily that $C(Z)$ has AFPP. $\square$

A metric space Y is said to be *continuumwise connected* provided that given any two points $p, q \in Y$, there is a subcontinuum $S(p,q)$ of Y such that $p, q \in S(p,q)$. In the following theorem $\overline{Y}$ denotes the closure of Y in X.

Theorem 2.4. *Let Y be a nonempty continuumwise connected subset of an arc-like continuum X. Then, for any Z such that $Y \subset Z \subset \overline{Y}$, $C(Z)$ has* AFPP.

Proof. First note that since Y Is nonempty and connected, $\overline{Y}$ is a continuum. Next let $\{y_1, y_2, \dots\}$ be a countable dense subset of Y. For each $n = 1, 2, \dots$, there is a subcontinuum Y_n of Y such that $Y_n \supset \{y_1, y_2, \dots, y_n\}$. Clearly, $\{Y_n\}_{n=1}^{\infty}$ converges to $\overline{Y}$. Thus, since $Y \subset Z \subset \overline{Y}$, Z satisfies the hypotheses in Theorem 2.3. Therefore, by Theorem 2.3, $C(Z)$ has AFPP. $\square$

A *composant* of a point p in a continuum X is the union of all those proper subcontinua A of X such that $p \in A$ [**7**, p. 208] Any composant in a continuum X is clearly continuumwise connected and is dense in X [**7**, Theorem 2, p. 209]. Thus, the following result is a special case of Theorem 2.4.

Corollary 2.5. *Let X be an arc-like continuum. If Z is any subset of X such that Z contains a composant of X, then $C(Z)$ has* AFPP. *In particular, the space of all nonempty subcontinua of any composant of X has* AFPP.

The following example shows that the results above do not extend to circle-like continua.

Example 2.6. Let X be the dyadic solenoid, i.e., X is the inverse limit of countably many circles with the bonding maps $z \to z^2$ [**3**, p. 145]. It is known that X is a topological group and that the composant Z of the identity element e of X is an arc component of X. We will show that $C(Z)$ does not have AFPP. By [**12**, Proof of Theorem 2], or by [**6**, proof of 4.5], there is a homeomorphism h from $C(X)$ onto the cone $K(X)$ over X such that $h(X)$ is the vertex v of $K(X)$ and, for all $\{x\} \in C(X)$, $h(\{x\})$ is the point $(x, 0)$ in the base $B(X)$ of $K(X)$. Since h is uniformly continuous and AFPP is preserved by uniformly continuous homeomorphisms, it suffices to show that $h[C(Z)]$ does not have AFPP.

We will denote any point $(x, 0) \in B(X)$ by x. Let π denote the natural projection from $K(X) - \{v\}$ onto $B(X)$, i.e., $\pi(x, t) = x$ for all points $(x, t) \in K(X) - \{v\}$. Let $q \in Z$ be such that $q \neq e$. Let $\tau : X \to X$ be translation by q under the group multiplication on X, i.e., $\tau(x) = q \cdot x$ for all $x \in X$. Since $\tau(e) = q \cdot e = q$, $\tau(Z) \cap Z \neq \varnothing$. Thus, since Z is an arc component of X, $\tau|Z$ maps Z into Z. Since $q \neq e$, τ is fixed point free. Define f on $K(X) - \{v\}$ by letting $f = \tau \circ \pi$. Since τ is fixed point free and X is compact, there exists an $\varepsilon > 0$ such that $f(x, t)$ and (x, t) are at least ε apart for all $(x, t) \in K(X) - \{v\}$. By (2) of 1.52.1 of [**9**], $C(Z)$ is an arc component of $C(X) - \{X\}$. Thus, since $h(X) = v$, $h[C(Z)]$ is an arc component of $K(X) - \{v\}$. Hence, since $h(\{z\}) = z$ for all $\{z\} \in C(Z)$, we have that $h[C(Z)] = \pi^{-1}(Z)$. Since $\tau|Z$ maps Z into Z, $f|\pi^{-1}(Z)$ maps $\pi^{-1}(Z)$ into Z. Therefore, since $f(z, t)$ and (z, t) are at last ε apart for all $(z, t) \in \pi^{-1}(Z)$, it follows that $h[C(Z)]$ does not have AFPP.

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Department of Mathematics, West Virginia University, Morgantown, WV 26506