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HAUSDORFF METRIC ON THE SPACE OF UPPER SEMICONTINUOUS MULTIFUNCTIONS

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ABSTRACT. Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be metric spaces, and let h_ρ denote Hausdorff distance in $X \times Y$, induced by the metric ρ on $X \times Y$ given by $\rho[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$. Denote by $U(X, Y)$ the space of all upper semicontinuous multifunctions from X to Y with nonempty compact values. If X and Y are complete metric spaces and X is locally compact, then $\langle U(X, Y), h_\rho\rangle$ is also complete. Some applications on the space $C(X, Y)$ of continuous functions from X to Y are given.

1. Introduction. Let $\langle W, d\rangle$ be a metric space. If K is a subset of W and $\varepsilon > 0$, let $S[K, \varepsilon]$ denote the union of all open ε-balls whose centers run over K. If K_1 and K_2 are nonempty subsets of W and for some $\varepsilon > 0$, both $S[K_1, \varepsilon] \supset K_2$ and $S[K_2, \varepsilon] \supset K_1$, we define the *Hausdorff distance* h_d between them to be

$$h_d(K_1, K_2) = \inf\{\varepsilon : S[K_1, \varepsilon] \supset K_2 \quad \text{and} \quad S[K_2, \varepsilon] \supset K_1\}.$$

Otherwise, we write $h_d(K_1, K_2) = \infty$. It is easy to check that h_d defines an infinite valued pseudometric on the nonempty subsets of W, and that $h_d(K_1, K_2) = 0$ if and only if K_1 and K_2 have the same closure. Thus, if we restrict h_d to the closed subsets of W, then h_d defines an infinite valued metric on such sets.

In the sequel we shall denote the closed nonempty subsets of a metric space W by $\mathrm{CL}(W)$. We need the following result proved in [**10**] that we state in a lemma.

Lemma 1. *Let h_d be a Hausdorff distance for a complete metric space $\langle W, d\rangle$. Then $\langle \mathrm{CL}(W), h_d\rangle$ is complete.*

We shall also use the following notions and notation. Let X and Y be topological spaces, with $P(Y)$ denoting the power set of Y. A

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multifunction H from X to Y is a function $H : X \to P(Y)$. A multifunction H from X to Y is called *closed* if its graph $\{(x,y) : x \in X \text{ and } y \in H(x)\}$ is a closed subset of $X \times Y$. We shall denote the graph of a multifunction H by $G(H)$. If R is a nonempty subset of $X \times Y$, we shall use the following notation for the vertical section at x of R: $R(x) = \{y : (x,y) \in R\}$. Define the multifunction H_R induced by R by $H_R(x) = R(x)$. Then $G(H_R) = R$.

A multifunction H from X to Y is called *upper semicontinuous* at $z \in X$ if whenever V is an open subset of Y that contains $H(z)$, then the set $\{x : H(x) \subset V\}$ contains a neighborhood of z. It is called *usco* [**11**] if it is upper semicontinuous at every $z \in X$ and for each $z \in X$, the set $H(z)$ is a nonempty compact subset of Y. We will write $U(X,Y)$ for the space of all usco multifunctions from X to Y.

Now let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be metric spaces. Consider the product $X \times Y$ metrized in the following way:

$$\rho[(x_1,y_1),(x_2,y_2)] = \max\{d_x(x_1,x_2), d_y(y_1,y_2)\}.$$

If we identify members of $U(X,Y)$ with their graphs, then h_ρ defines a metric on $U(X,Y)$ (if $F \in U(X,Y)$, then F is closed [**9**]).

Let Y be a metric space and let χ be a functional defined on $P(Y)$ as follows: $\chi(\varnothing) = 0$, and if A is a nonempty subset of Y, then $\chi(A) = \inf\{\varepsilon : A \text{ has a finite } \varepsilon - \text{dense subset}\}$. In the literature, χ has been called the *Hausdorff measure of noncompactness functional.* Basic facts about this functional and its relatives can be found in [**2, 13**]. We record the following easily verified facts as a lemma.

Lemma 2. *Let χ be the Hausdorff measure of noncompactness functional defined on $P(Y)$. Then*:

(a) $\chi(A) = \infty$ *if and only if A is unbounded*;

(b) $\chi(A) = 0$ *if and only if A is totally bounded*:

(c) *If $A \subset B$, then $\chi(A) \leq 2\chi(B)$*;

(d) *If A is totally bounded, then for each $\varepsilon > 0$, $\chi(S[A,\varepsilon]) \leq \varepsilon$*;

(e) $\chi(\operatorname{cl} A) = \chi(A)$.

Finally, we denote the positive integers by **N** in what follows.

2. Results.

Theorem 1. *Let* $\langle X, d_X \rangle$ *be a metric space. The following are equivalent*:

(1) *X is a locally compact space*;

(2) *for each complete metric space* $\langle Y, d_Y \rangle$, $\langle U(X,Y), h_\rho \rangle$ *is a closed subspace of* $\langle \mathrm{CL}\,(X \times Y), h_\rho \rangle$.

Proof. (2) $\Rightarrow$ (1). The proof in this direction uses some of the ideas of the proof of Theorem 7 of [**5**]. Suppose that X is not locally compact.

Let $x_0 \in X$ fail to have a local base of compact sets. Let $\delta_1 = 1$ and choose a countably infinite subset E_1 of $\{z : 0 < d_X(x_0, z) < \delta_1\}$ with no cluster point in X. There exists $\varepsilon_1 > 0$ such that for each z in E_1, $\varepsilon_1 < d_X(x_0, z)$. Next, let $\delta_2 = \varepsilon_1/2$ and let E_2 be a countably infinite subset of $\{z : 0 < d_X(x_0, z) < \delta_2\}$ with no cluster point in X. Choose $0 < \varepsilon_2 < \inf\{d_X(x_0, z) : z \in E_2\}$ and set $\delta_3 = \varepsilon_2/2$. Continuing, we can produce for each $n \in \mathbf{N}$ a countably infinite set E_n with no cluster point in X and numbers δ_n and ε_n such that (i) $\delta_n = \varepsilon_{n-1}/2$; (ii) $\varepsilon_n < \delta_n$; (iii) $E_n \subset \{z : \varepsilon_n < d_X(x_0, z) < \delta_n\}$.

For each positive integer n, let $\{x_i^n : i \in \mathbf{N}\}$ be an enumeration of E_n and let $\langle \lambda_i^n \rangle$ be a sequence of positive integers such that

(a) $\lambda_i^n < 1/i$ for each i;

(b) the family $\{\mathrm{cl}\, S[x_i^n, \lambda_i^n] : i \in \mathbf{N}\}$ is pairwise disjoint;

(c) $\mathrm{cl}\, S[x_i^n, \lambda_i^n] \subset \{z : \varepsilon_n < d_X(x_0, z) < \delta_n\}$ for each $i \in \mathbf{N}$.

Now let Y be an arbitrary noncompact complete metric space. Let $\langle y_n \rangle$ be a sequence in Y with distinct terms with no convergent subsequence. For each $n \in \mathbf{N}$, let g_n be a bijection from E_n to the set $\{y_m : m \geq n\}$. Define multifunctions F_n $(n = 1, 2, \dots)$ as follows:

$$F_n(x) = \begin{cases} \{y_1, g_n(x_i^n)\} & \text{if } x \in \mathrm{cl}\, S[x_i^n, \lambda_i^n] \quad i = 1, 2, \dots \\ \{y_1, y_m\} & \text{if } x \in \mathrm{cl}\, S[x_i^m, \lambda_i^m] \quad m < n \text{ and } i = 1, 2, \dots \\ \{y_1\} & \text{otherwise.} \end{cases}$$

Since a union of any subfamily of the family $\{\mathrm{cl}\, S[x_i^n, \lambda_i^n] : i \in \mathbf{N}\}$ is a closed subset of X for each $n \in \mathbf{N}$, it is easy to check that $F_n \in U(X,Y)$ for every $n \in \mathbf{N}$. Now define the multifunction F from X to Y as follows: $F(x) = \{y_1, y_m\}$ if $x \in \operatorname{cl} S[x_i^m, \lambda_i^m]$ for some m and i and $F(x) = \{y_1\}$ otherwise. Then F is a closed multifunction which is not upper semicontinuous at x_0 and the sequence $\langle G(F_n)\rangle$ converges in the Hausdorff metric to $G(F)$. Thus, $\langle U(X,Y), h_\rho\rangle$ is not a closed subspace of $\langle \mathrm{CL}\,(X \times Y), h_\rho\rangle$.

(1) $\Rightarrow$ (2). Let $\langle X, d_X\rangle$ be a locally compact metric space, and let $\langle Y, d_Y\rangle$ be a complete metric space. We show that $\langle U(X,Y), h_\rho\rangle$ is a closed subspace of $\langle \mathrm{CL}\,(X \times Y), h_\rho\rangle$. Let $\langle F_n\rangle$ be a sequence from $U(X,Y)$ such that the sequence $\langle G(F_n)\rangle$ converges in the Hausdorff metric to a closed subset R of $X \times Y$. We prove that the multifunction H_R induced by R belongs to $U(X,Y)$.

Put $A = \{x \in X : R(x) \neq \varnothing\}$. To see that A is dense in X, let V be a nonempty open set in X. Let $a \in V$; there exists $\delta > 0$ such that $S[a,\delta] \subset V$. There is a $k \in N$ such that $h_\rho(R, G(F_k)) < \delta$. Let $b \in F_k(a)$. There must exist $(x,y) \in R$ such that $\rho[(a,b),(x,y)] < \delta$, i.e., $x \in V$ and $R(x) \neq \varnothing$. We next show that $A = X$.

Suppose not. Let $x \in A^C$; there is $\delta > 0$ such that $\operatorname{cl} S[x,\delta]$ is compact. Put $B = \cup\{R(a) : a \in A \cap S[x,\delta/2]\}$. We show that $\chi(B) = 0$, where χ is the Hausdorff measure of noncompactness functional. Let $\varepsilon > 0$, and put $\alpha = \min\{\varepsilon/2, \delta/2\}$. There is a $j \in N$ such that $h_\rho(G(F_n), R) < \alpha$ for every $n \geq j$. Let $n \geq j$ be fixed. Then $B \subset S[F_n(S[x,\delta]),\alpha]$. By a well-known result of Berge [**9**, p. 110], $F_n(\operatorname{cl} S[x,\delta])$ is a compact subset of Y; so by (d) of Lemma 2, we have $\chi(S[F_n(\operatorname{cl} S[x,\delta]),\alpha]) \leq \alpha \leq \varepsilon/2$. The inclusion $B \subset S[F_n(\operatorname{cl} S[x,\delta]),\alpha]$ and (c) of Lemma 2 imply that $\chi(B) \leq \varepsilon$. Since $\chi(B) \leq \varepsilon$ for any $\varepsilon > 0$, we have $\chi(B) = 0$ so that $\chi(\operatorname{cl} B) = 0$. By (b) of Lemma 2, $\operatorname{cl} B$ is a totally bounded set, and the completeness of Y implies that $\operatorname{cl} B$ is compact.

By the density of A in X, there is a sequence $\langle x_n\rangle$ in $A \cap S[x,\delta/2]$ convergent to x. Let $\langle y_n\rangle$ be a sequence of points of Y such that $(x_n, y_n) \in \mathbf{R}$ for every $n \in \mathbf{N}$. Since $\langle y_n\rangle$ is a sequence of points of B and $\operatorname{cl} B$ is compact, there is a cluster point y_0 of the sequence $\langle y_n\rangle$. Then (x, y_0) is a cluster point of the sequence $\langle (x_n, y_n)\rangle$, i.e., $(x, y_0) \in \operatorname{cl} \mathbf{R} = \mathbf{R}$. This contradicts $x \notin A$, establishing the fact that A is all of X.

By a second result of Berge [**9**, p. 112], any closed multifunction with compact range is upper semicontinuous. Since upper semicontinuity is a local property, it suffices now to show that locally H_R has compact range. But this is obvious: choosing $S[x,\delta]$ with compact closure, the compact set $\operatorname{cl} B$ as defined above contains the closed set $H_R(\operatorname{cl} S[x,\delta/4]) = R \cap (\operatorname{cl} S[x,\delta/4] \times Y)$. Finally, since $R(x)$ is a closed subset of the compact set $\operatorname{cl} B$, $R(x)$ is a compact set. □

Theorem 1 fails if we consider the broader class of upper semicontinuous multifunctions, rather than the usco ones.

Example 1. We produce a sequence of upper semicontinuous multifunctions with nonempty closed values from the real line to itself that converges in Hausdorff distance to a multifunction with nonempty closed values that is not upper semicontinuous at the origin. Define F_n as follows:

$$F_n(x) = \begin{cases} [1/x^2, \infty) & \text{if } |x| \geq 1/n \\ [1/n^2, \infty) \cup \{0\} & \text{if } x = 0 \\ [1/n^2, \infty) & \text{otherwise.} \end{cases}$$

Evidently, $\langle F_n \rangle$ converges in Hausdorff distance to F defined by

$$F(x) = \begin{cases} [1/x^2, \infty) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Theorem 2. *Let $\langle X, d_X \rangle$ and $\langle Y, d_Y \rangle$ be complete metric spaces. Let X be a locally compact space. Then $\langle \cup(X,Y), h_\rho \rangle$ is complete.*

Proof. By Lemma 1, $\langle \operatorname{CL}(X \times Y), h_\rho \rangle$ is a complete metric space; so, by Theorem 1, $\langle U(X,Y), h_\rho \rangle$ as a closed subspace of a complete space is complete. □

It should be remarked that if X is a locally compact metrizable space and Y is completely metrizable (i.e., Y is a G_δ-subset of its completion), then since X admits a metric that is complete [**12**], $X \times Y$ admits a metric that makes $U(X,Y)$ completely metrizable with the induced Hausdorff metric.

3. Applications. Let $\langle X, d_X \rangle$ and $\langle Y, d_Y \rangle$ be metric spaces. Let $C(X,Y)$ be the space of all continuous functions from X to Y.

Let d_1 be the usual metric of uniform convergence on $C(X,Y)$, i.e., $d_1(f,g) = \sup\{d_Y(f(x),g(x)) : x \in X\}$. If we identify members of $C(X,Y)$ with their graphs, then h_ρ defines a metric on $C(X,Y)$, which we denote by d_2. It is easy to see that $d_2(f,g) \leq d_1(f,g)$. If X is compact, then d_1 and d_2 are equivalent [**14**]; more generally, if X is an *Atsuji space* [**1**]—a metric space on which each continuous function into a metric space is uniformly continuous—then d_1 and d_2 are also equivalent [**3**]. Thus, if X is an Atsuji space, then a subset Ω of $C(X,Y)$ is d_1-compact if and only if Ω is d_2-compact.

We have the following variant of the Arzela-Ascoli theorem.

Theorem 3. *Let $\langle X, d_X\rangle$ be a locally compact Atsuji space, and let $\langle Y, d_Y\rangle$ be a complete metric space. Let Ω be a subset of $C(X,Y)$. Then Ω is compact in the topology of uniform convergence if and only if*

(1) *Ω is d_1-closed and whenever $\langle f_n\rangle$ is a sequence in Ω convergent in the Hausdorff metric to a closed subset E of $X \times Y$, then E is the graph of a function, and*

(2) *Ω is d_2-totally bounded.*

Proof. Let Ω be d_1-compact. Then Ω is d_2-compact, which means that Ω is d_2-complete and d_2-totally bounded. Thus, (1) and (2) are satisfied. Conversely, suppose that (1) and (2) are satisfied. It is sufficient to prove that Ω is d_2-complete. Let $\langle f_n\rangle$ be d_2-Cauchy. Since Atsuji spaces are complete, $\langle \mathrm{CL}\,(X\times Y), h_\rho\rangle$ is complete, and there is a closed subset E of $X \times Y$ to which $\langle f_n\rangle$ is h_ρ-convergent. By condition (1), E is the graph of a function, and by Theorem 1, f is continuous. Since X is an Atsuji space, the sequence $\langle f_n\rangle$ d_1-converges to f and, thus, d_1-closedness of Ω implies that $f \in \Omega$. Thus, Ω is d_2-complete. □

Theorem 3 is proved for compact X in [**4**]; it also follows from Theorem 3 of [**6**]. If X is a locally comapct Atsuji space and Y is a complete metric space, then pointwise total boundedness and equicontinuity of Ω do not ensure compactness of Ω in the topology of uniform convergence.

Example 2. Put $X = [0,1/2] \cup \mathbf{N}$ with the usual metric, and let Y be the real line with the usual metric. For each $n \in \mathbf{N}$, define $f_n \in C(X,Y)$ by $f_n(n) = n$ and $f_n(x) = 0$ otherwise. Then $\Omega = \{f_n : n \in \mathbf{N}\}$ is pointwise totally bounded and equicontinuous, but Ω is noncompact.

It is very easy to see that $\langle C(X,Y), d_2 \rangle$ need not be complete. For example, for $n \geq 1$, let $f_n \in C([0,1], \mathbf{R})$ be the piecewise linear function whose graph connects the following points in succession:

$$(0,1), (1/n, 0) \quad \text{and} \quad (1,0).$$

Then $\langle f_n \rangle$ is a d_2-Cauchy sequence without a cluster point in $C(X,Y)$.

Some completeness criteria for closed subsets of $\langle C(X,Y), d_2 \rangle$ can be found in [**6**].

The last results give some sufficient conditions for the complete metrizability of $\langle C(X,Y), d_2 \rangle$. If $\langle X, d_X \rangle$ and $\langle Y, d_Y \rangle$ are complete metric spaces and X is locally compact, then by Theorem 2, $\langle U(X,Y), h_\rho \rangle$ is complete. Thus, by the theorem of Alexandroff, complete metrizability of $\langle C(X,Y), d_2 \rangle$ will be stated if we show that $C(X,Y)$ is a G_δ-subset of $\langle U(X,Y), h_\rho \rangle$.

We will need the following definition:

Definition 1. [**8**] A metric space $\langle X, d \rangle$ is called boundedly Atsuji provided each closed and bounded subset of X is Atsuji.

The following result [**8**] will be further useful:

Lemma 3. *Let $\langle X, d \rangle$ be a metric space. The following are equivalent*:

(1) *X is boundedly Atsuji*;

(2) *Whenever B is a closed and bounded subset of X and $\{V_i : i \in I\}$ is a collection of open subsets of X with $B \subset \cup\{V_i : i \in I\}$, then there exists $\delta > 0$ such that each subset of X of diameter less than δ which meets B lies entirely within some V_i.*

Theorem 4. *let $\langle X, d_X \rangle$ be a boundedly Atsuji space and $\langle Y, d_Y \rangle$ be* *a metric space. Then $C(X,Y)$ is a G_δ-subset of $\langle U(X,Y),h_\rho\rangle$.*

Proof. Clearly $C(X,Y)=\{F\in U(X,Y):\forall x\in X, F(x)$ is a singleton$\}$. We use some ideas of the proof of Lemma 3.3 of [**7**]. Let H_k $(k\in\mathbf{N})$ be the following sets of relations

$$H_k=\{F\in U(X,Y):\forall x\in X\,\mathrm{diam}\,F(x)\le 1/k\}.$$

Clearly $C(X,Y)=\cap\{H_k:k\in\mathbf{N}\}$. It is very easy to see that $X=\cup\{B_n:n\in\mathbf{N}\}$ where each B_n is a closed bounded set, i.e., B_n is Atsuji for every $n\in N$.

Let $k\in\mathbf{N}$. For every $j\in N$ put $H_k^j=\{F\in U(X,Y):\forall x\in B_j\,\mathrm{diam}\,F(x)\le 1/k\}$. Clearly $H_k=\cap\{H_k^j:j\in\mathbf{N}\}$. We show that H_k^j is a G_δ-subset of $\langle U(X,Y),h_\rho\rangle$ for every $j\in\mathbf{N}$.

Let $j\in\mathbf{N}$. Let $\varepsilon>0$ $(0<\varepsilon<1)$. First we prove that there is an open set G_ε in $\langle U(X,Y),h_\rho\rangle$ such that $H_k^j\subset G_\varepsilon\subset\{F\in U(X,Y):\forall x\in B_j\,\mathrm{diam}\,F(x)\le 1/k+\varepsilon\}$. Let $F\in H_k^j$. The upper semicontinuity of F implies that for every $x\in B_j$ there is a neighborhood 0_x of x such that $F(z)\subset S[F(x),\varepsilon/4]$ for every $z\in 0_x$ (1).

Since X is a boundedly Atsuji space and B_j is a closed and bounded set, by Lemma 3 there is a real $\alpha>0$ such that the family $\{S[x,\alpha]:x\in B_j\}$ is a refinement of $\{0_x:x\in B_j\}$. Put $\eta=\min\{\varepsilon/4,\alpha\}$ and $0_F^\varepsilon=\{R\in U(X,Y):h_\rho(G(R),G(F))<\eta\}$. Then 0_F^ε is contained in the set $\{R\in U(X,Y):\forall x\in B_j,\mathrm{diam}\,R(x)\le 1/k+\varepsilon\}$.

Let $R\in 0_F^\varepsilon$. Let $x\in B_j$ and y_1,y_2 are two different points from $R(x)$ (if they exist). There are points (x_1,z_1) and $(x_2,z_2)\in G(F)$ such that $\rho[(x,y_1),(x_1,z_1)]<\eta$ and also $\rho[(x,y_2),(x_2,z_2)]<\eta$. Clearly $S[x,\alpha]$ contains points x_1 an dx_2. There is a $u\in B_j$ such that $S[x,\alpha]\subset 0_u$. By (1) $\{z_1,z_2\}\subset S[F(u),\varepsilon/4]$, i.e., there are $v_1,v_2\in F(u)$ such that $d_Y(z_1,v_1)\le\varepsilon/4$ and $d_Y(z_2,v_2)\le\varepsilon/4$, i.e., $d_Y(z_1,z_2)\le 1/k+\varepsilon/2$. Thus we have $d_Y(y_1,y_2)\le d_Y(y_1,z_1)+d_Y(z_1,z_2)+d_Y(z_2,y_2)\le 1/k+\varepsilon$, i.e., $\mathrm{diam}\,R(x)\le 1/k+\varepsilon$ for every $x\in B_j$. Put $G_\varepsilon=\cup\{0_F^\varepsilon:F\in H_k^j\}$. It is very easy to see that

$$H_k^j\subset\cap\{G_{1/n}:n\in\mathbf{N}\}\subset\cap\{\{F\in U(X,Y):\\ \forall x\in B_j,\mathrm{diam}\,F(x)\le 1/k+1/n\}:n\in\mathbf{N}\}\subset H_k^j.$$

Thus $H_k^j = \cap\{G_{1/n} : n \in \mathbf{N}\}$, i.e., H_k is a G_δ-subset of $\langle U(X,Y), h_\rho\rangle$. Since $k \in N$ was arbitrary, $C(X,Y)$ is a G_δ-subset of $\langle U(X,Y), h_\rho\rangle$.

Theorem 5. *Let $\langle X, d_X\rangle$ be a locally compact boundedly Atsuji space, and let $\langle Y, d_Y\rangle$ be a complete metric space. Then $\langle C(X,Y), d_2\rangle$ is completely metrizable.*

Proof. Since boundedly Atsuji spaces are complete [**8**], by Theorem 2 $\langle U(X,Y), h_\rho\rangle$ is complete. By Theorem 4, $C(X,Y)$ is a G_δ-subset of $\langle U(X,Y), h_\rho\rangle$. By the well-known theorem of Alexandroff, $\langle C(X,Y), d_2\rangle$ is completely metrizable. □

Another sufficient condition for the complete metrizability of $\langle C(X,Y), d_2\rangle$ can be found in [**7**].

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