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COMMUTATIVITY OF COONS AND TENSOR PRODUCT OPERATORS

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ABSTRACT. We show under what conditions Coons type surface approximation operators and tensor product approximation commute. An application is given for Bézier surfaces.

Definitions. We first define a *Coons patch*: Consider a surface patch $\mathbf{s}(u,v)$, which is a continuous map of the unit square into $\mathbf{R}^3$. We can define its (bilinearly blended) Coons approximation by

$$C\mathbf{s}(u,v) = (1-u)\mathbf{s}(0,v) + u\mathbf{s}(1,v) + (1-v)\mathbf{s}(u,0) + v\mathbf{s}(u,1) - (1-u,u)\begin{pmatrix}\mathbf{s}(0,0), \mathbf{s}(0,1)\\ \mathbf{s}(1,0), \mathbf{s}(1,1)\end{pmatrix}\begin{pmatrix}1-v\\ v\end{pmatrix}. \tag{1}$$

This Coons patch interpolates to all four boundary curves of $\mathbf{s}$; in fact, it only depends on data from the boundary curves. For more details, see [**1** or **6**].

Let us next define a *tensor product surface*: Let $\mathbf{x}(t)$ be a curve, i.e., a continuous map of the unit interval into $\mathbf{R}^3$. We can define an approximation to it by

$$A\mathbf{x}(t) = \sum_{i=0}^{m} \mathbf{x}_i A_i(t), \tag{2}$$

where $\mathbf{x}_i = \mathbf{x}(t_i)$ for $0 = t_0 \le t_1, \dots, \le t_m = 1$. The $A_i(t)$ are univariate functions; they determine the nature of the approximation scheme. A second such scheme might be of the form

$$B\mathbf{x}(t) = \sum_{j=0}^{n} \mathbf{x}_j B_j(t).$$

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The corresponding tensor product approximation $A \otimes B$ to a surface $\mathbf{s}(u,v)$ is given by

$$(A \otimes B)\mathbf{s}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{s}_{i,j} A_i(u) B_j(v), \tag{3}$$

where $\mathbf{s}_{i,j} = \mathbf{s}(u_i, v_j)$. For more details about tensor product approximation schemes, see [**4**].

We finally need the notion of *linear precision*: The operator A has linear precision if and only if for constants $\mathbf{a}$ and $\mathbf{b}$

$$A(\mathbf{a} + t\mathbf{b}) = \mathbf{a} + t\mathbf{b},$$

i.e., the operator A reproduces straight lines. Since linear precision implies constant precision, we have

$$\sum_{i=0}^{m} A_i(t) = 1. \tag{4}$$

The commutativity theorem. To a given surface $\mathbf{s}$, we can construct its Coons type approximation $C\mathbf{s}$ or, given two (not necessarily different) curve approximation schemes A and B, we can construct its tensor product approximation $(A \otimes B)\mathbf{s}$. We can then approximate the tensor product approximation by a Coons approximation: $C(A \otimes B)\mathbf{s}$ or vice versa: $(A \otimes B)C\mathbf{s}$. Under what conditions will those two approximations coincide? The following theorem provides the answer.

Theorem. *Tensor product and Coons surface approximations commute, i.e.,*

$$C(A \otimes B)\mathbf{s} = (A \otimes B)C\mathbf{s} \tag{5}$$

under the following conditions:

(6)
1. *Both A and B have linear precision,*
2. $A_i(0) = \delta_{i,0},\ A_i(1) = \delta_{i,m},\ B_j(0) = \delta_{j,0},\ B_j(1) = \delta_{j,n}$.

Remark. Equation (6) states that the curves $A\mathbf{x}, B\mathbf{x}$ agree with $\mathbf{x}$ at the endpoints of $\mathbf{x}$.

Proof. Let us first establish an auxiliary result. Provided that (6) holds, a boundary curve of a tensor product patch is the univariate approximation to the corresponding boundary curve of the original patch $\mathbf{s}$. To see this, we write

$$\begin{aligned}(A \otimes B)\mathbf{s}(0,v) &= \sum_i \sum_j \mathbf{s}_{i,j} A_i(0) B_j(v) \\ &= \sum_j \mathbf{s}_{0,j} B_j(v).\end{aligned} \tag{7}$$

The last follows because of (6). Analogous results hold for the three remaining boundary curves.

We can now write

$$\begin{aligned}&(A \otimes B)C\mathbf{s}(u,v) \\ &\quad= \sum_i \sum_j (1-u_i)\mathbf{s}_{0,j} A_i(u) B_j(v) \\ &\qquad+ \sum_i \sum_j u_i \mathbf{s}_{m,j} A_i(u) B_j(v) \\ &\qquad+ \sum_i \sum_j (1-v_j)\mathbf{s}_{i,0} A_i(u) B_j(v) \\ &\qquad+ \sum_i \sum_j v_j \mathbf{s}_{i,n} A_i(u) B_j(v) \\ &\qquad- \sum_i \sum_j (1-u_i, u_i) \begin{pmatrix} \mathbf{s}_{0,0}, \mathbf{s}_{0,n} \\ \mathbf{s}_{m,0}, \mathbf{s}_{m,n} \end{pmatrix} \begin{pmatrix} 1-v_j \\ v_j \end{pmatrix} A_i(u) B_j(v).\end{aligned}$$

The first term in this equation can be rewritten

$$\sum_i \sum_j (1-u_i)\mathbf{s}_{0,j} A_i(u) B_j(v) = \sum_j B_j(v)\mathbf{s}_{0,j} \sum_i (1-u_i) A_i(u).$$

By linear precision of A,

$$= (1-u) \sum_j B_j(v)\mathbf{s}_{0,j}.$$

Because of (7),

$$= (1-u)(A \otimes B)\mathbf{s}(0,v).$$

The remaining terms are treated analogously, and we get:

$$
\begin{aligned}
&(A \otimes B)C\mathbf{s}(u,v) \\
&\qquad = (1-u)(A \otimes B)\mathbf{s}(0,v) + u(A \otimes B)\mathbf{s}(1,v) \\
&\qquad\quad + (1-v)(A \otimes B)\mathbf{s}(u,0) + v(A \otimes B)\mathbf{s}(u,1) \\
&\qquad\quad - (1-u,u)\begin{pmatrix} (A \otimes B)\mathbf{s}(0,0), (A \otimes B)\mathbf{s}(0,1) \\ (A \otimes B)\mathbf{s}(1,0), (A \otimes B)\mathbf{s}(1,1) \end{pmatrix}\begin{pmatrix} 1-v \\ v \end{pmatrix} \\
&\qquad = C(A \otimes B)\mathbf{s}(u,v).
\end{aligned}
$$

□

Remark. I am grateful to a referee for realizing that the above theorem not only proves sufficiency but also shows the necessity of the above conditions.

Applications. Let us investigate a special case of the above theorem: Let A and B be Bernstein approximation operators, i.e., $A_i(t) = \binom{m}{i} t^i (1-t)^{m-i}$; more specifically, let

$$A\mathbf{x} = \sum_{i=0}^{m} \mathbf{x}\left(\frac{i}{m}\right) A_i(t).$$

If $\mathbf{s}$ is a piecewise bilinear surface with breakpoints $\mathbf{s}_{i,j}$, we say that $(A \otimes B)\mathbf{s}$ is the *Bézier patch* defined by the *control net* $\mathbf{s}$; for more details, see [**3**].

We now have an easy way to construct the Coons patch to four boundary Bézier curves (of compatible degrees): Suppose that each boundary curve is specified by its control polygon. We can then construct the Coons patch to the four boundary control polygons. (Note that in order to do this, we must assume that the polygons are evaluated at their vertices, corresponding to parameter values i/m or j/n, respectively.) This Coons patch will be piecewise bilinear and can thus be interpreted as a control net for a Bézier surface. By the above theorem, this Bézier surface is the Coons patch defined by the four boundary curves. This construction is much simipler than obtaining the control net from the original Coons approach (1).

If all four boundary curves are cubics, the patch generated by this method is bicubic and has as its *corner twists* the so-called Adini-twists (see [**2**, p. 34 or **3**, p. 83]). We thus have a direct method to determine Bézier points corresponding to Adini-twists.

If A and B denote B-spline approximations, the theorem is not in general valid; it is only applicable if the end knot multiplicities are chosen to be high enough so that (6) holds. Also, evaluation of the boundary polygons must now occur at the "Greville abscissae" (see [**5**, p. 11 or **6**, p. 122])

$$\xi_i = \left(\frac{1}{m}\right)(u_i + \cdots + u_{i+m-1}).$$

A generalization. One generalization of the standard Coons patch is the so-called *Gordon surface*, see [**3**]. Let F and G be two curve approximation schemes of the form (2), defined over knot sequences $0 = \hat{u}_0 \le \hat{u}_1, \dots, \le \hat{u}_M = 1$ and $0 = \hat{v}_0 \le \hat{v}_1 \dots \le \hat{v}_N = 1$, respectively. We can apply them to a surface $\mathbf{s}$ by means of a *Boolean sum*:

$$\begin{aligned}(F \oplus G)\mathbf{s}(u,v) = &\sum_{I=0}^{M} \mathbf{s}(\hat{u}_I, v) F_I(u) + \sum_{J=0}^{N} \mathbf{s}(u, \hat{v}_J) G_J(v) \\ &- \sum_{I=0}^{M}\sum_{J=0}^{N} \mathbf{s}(\hat{u}_I, \hat{v}_J) F_I(u) G_J(v).\end{aligned}$$

The functions F_I and G_J have to be chosen such that they are cardinal: $F_I(u_K) = \delta_{I,K}$ and $G_J(v_K) = \delta_{J,K}$. Hence, the Gordon surface interpolates to the isoparametric lines $u = u_I, v = v_J$ of $\mathbf{s}$, see also [**7**]. In the special case $M = N = 1$, $F_0(u) = 1 - u$, $F_1(u) = u$, $G_0(v) = 1 - v$, $G_1(v) = v$, one obtains the original Coons formula (1).

The result of the previous theorem can now be generalized under the assumption that the operators A and B possess a richer structure than do F and G, more precisely,

Theorem. *The Gordon and tensor product surface approximation schemes commute, i.e.,*

$$(A \otimes B)(F \otimes G)\mathbf{s} = (F \otimes G)(A \otimes B)\mathbf{s}$$

under the following conditions:

1. *The F_I (respectively, G_J) are in the precision set of A (respectively, B),*

2. $A_i(\hat{u}_I) = \delta(u_i, \hat{u}_I)$, $B_J(\hat{v}_J) = \delta(v_j, \hat{v}_J)$, *where δ denotes the Kronecker Delta.*

The proof proceeds exactly as for the preceding theorem.

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