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GEOMETRIC INEQUALITIES IN NORMED SPACES

S. BUSENBERG AND M. MARTELLI

1. Introduction. In a recent paper [**4**] of S. Busenberg, D. Fisher and M. Martelli, it was shown that the period T of any periodic solution of the first order system

$$\mathbf{x}' = f(\mathbf{x}) \tag{1.1}$$

where $f : \mathbf{E} \to \mathbf{E}$ is Lipschitz continuous with constant L and $\mathbf{E}$ is a normed space, satisfies the inequality

$$TL \geq 6. \tag{1.2}$$

This result refines an earlier estimate of A. Lasota and J. Yorke [**11**], who showed that $TL \geq 4$. Inequality (1.2) is optimal in the stated generality, since Busenberg, Fisher and Martelli [**5**] provided an example where $TL = 6$ in $L^1(Q)$ where Q is the unit square in $\mathbf{R}^2$. In general, the best lower bound for the product TL seems to depend strongly on the geometry of the underlying space. In fact, in the same paper, the three authors gave a simple proof of the better lower bound

$$TL \geq 2\pi \tag{1.3}$$

in spaces with the norm defined via an inner product. Inequality (1.3) was first proved by J. Yorke [**15**]. In [**3**] Busenberg and Martelli give an alternate proof of (1.3) which relies on corresponding inequalities for difference equations in Hilbert spaces. They also show in [**6**] that (1.3) is optimal in every Hilbert space of dimension larger than or equal to two.

Ideas and techniques developed to prove (1.2), (1.3), and to provide the example mentioned above, can be applied to a number of classical problems of geometrical nature to yield different proofs and/or new

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results which extend and sharpen the existing results. The purpose of this paper is to present these geometric applications.

In Section 2 we give the necessary background by restating without proof the fundamental inequalities on which (1.2) and (1.3) are based (see Propositions 2.1 and 2.3) and we obtain, in Hilbert and normed spaces, new geometric inequalities (Propositions 2.2 and 2.4) which, besides being interesting in their own right, also greatly simplify the proofs of certain results of Sections 3 and 4.

Section 3 is devoted to extensions of the classical result of Fenchel [**9**] on the total curvature of closed, smooth curves in $\mathbf{R}^3$. We give a simple proof of this result, based on Proposition 2.1. We extend Fenchel's Theorem to any finite or infinite dimensional Hilbert space $\mathbf{H}$, using again the same proposition and a simple geometric idea of Horn [**10**]. We also prove a theorem on closed curves contained in the unit sphere S of $\mathbf{H}$, from which Fenchel's result can be derived as a corollary.

The final section contains inequalities on sums of distances between unit vectors in normed spaces. Interesting and special cases of these inequalities were obtained previously by Chakerian and Klamkin [**7**]. Using Proposition 2.3, Proposition 2.4 and a result on the number of disjoint Hamiltonian cycles of a complete graph $\mathbf{G}$, we provide, via an elementary argument, a sharp lower bound (see (4.6)) for the total length of all the edges of $\mathbf{G}$. This last result improves an inequality concerning convex sets which was conjectured by B. Grünbaum and established by Andrew and Ghandehari [**1**].

2. Preliminary results in Hilbert and Banach spaces. Assume that $\mathbf{v}_1, \dots, \mathbf{v}_N$ are N vectors in a Hilbert space $\mathbf{H}$. Denote by θ_i the angle between $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$, with $\mathbf{v}_{N+1} = \mathbf{v}_1$.

Proposition 2.1. *Assume that the vectors* $\mathbf{v}_1, \dots, \mathbf{v}_N$ *are all different from zero and that*

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_N = \mathbf{0}. \tag{2.1}$$

Then

$$\theta_1 + \theta_2 + \cdots + \theta_N \geq 2\pi \tag{2.2}$$

with equality if and only if all vectors belong to the same plane and the path

$$T = \bigcup_{i=1}^{N} \{t\mathbf{v}_{i+1} + (1-t)\mathbf{v}_i\}, \qquad \mathbf{v}_{N+1} = \mathbf{v}_1,$$

$0 \le t \le 1$, *is the boundary of a convex polygon.*

Proof. See [**3**]. □

The above Proposition can be extended to the case when the vectors do not add up to the zero vector. To give this extension, we need the definition of the convex hull of a set.

Given any set A, $\mathbf{H} \supset A$, the convex hull of A, written $\text{co}(A)$, is the set of all convex finite linear combinations of elements of A (i.e., the coefficients must be positive and their sum must be 1). It is not a trivial matter to show that the convex hull of a compact set in a finite dimensional normed space is closed (see Valentine [**14**]). In infinite dimension the result is false as the following example shows.

Example. Let $\mathbf{H} = l^2$, the Hilbert space of square summable sequences of real numbers, and let A be the set

$$A = \{\mathbf{v}_1 = (1,0,0,\dots),\ \mathbf{v}_2 = (0,\frac{1}{2},0,0,\dots), \\ \mathbf{v}_3 = (0,0,\frac{1}{3},0,\dots),\dots(0,0,0,\dots)\}.$$

Then A is a compact sequence, and all points of $\text{co}(A)$ have all but finitely many coordinates different from zero. The point

$$\mathbf{w} = \sum_{1}^{\infty}{}_n \frac{1}{2^n}\mathbf{v}_n$$

is in the closure of $\text{co}(A)$ and all coordinates of $\mathbf{w}$ are different from zero.

Going back to the extension of Proposition 2.1, we assume that the vectors $\mathbf{v}_1, \dots, \mathbf{v}_N$ are of unit length and that $\text{dis}\,(\mathbf{0}, \text{co}(\mathbf{v}_1, \dots, \mathbf{v}_N)) =$

r. Then the induction argument used in [**3**] to establish (2.2) yields the following more general version of Proposition 2.1.

Proposition 2.1′. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_N$ *be unit vectors in* $\mathbf{H}$. *Assume that* $\operatorname{dis}(\mathbf{0}, \operatorname{co}(\mathbf{v}_1, \dots, \mathbf{v}_N)) = r$. *Denote by* θ_i *the angle between* $\mathbf{v}_i$ *and* $\mathbf{v}_{i+1}$, $i = 1, 2, \dots, N$, $\mathbf{v}_{N+1} = \mathbf{v}_1$. *Then*

$$\theta_1 + \theta_2 + \cdots + \theta_N \geq 4 \cos^{-1} r. \tag{2.3}$$

Proposition 2.2 below establishes an inequality on the lower bound of the maximum distance between any vector of $\mathbf{H}$ and a set of unit vectors.

Proposition 2.2. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_N$ *be* N *unit vectors in* $\mathbf{H}$ *and assume that* $\operatorname{dis}(\mathbf{0}, \operatorname{co}(\mathbf{v}_1, \dots, \mathbf{v}_N)) = r$. *Then for any vector* $\mathbf{w}$ *we have*

$$\max\{||\mathbf{w} - \mathbf{v}_i|| : i = 1, 2, \dots, N\} \geq 1 - r \tag{2.4}$$

Proof. There is a vector $\mathbf{v} \in \operatorname{co}(\mathbf{v}_1, \dots, \mathbf{v}_N)$ such that

$$||\mathbf{v}|| = r, \qquad \mathbf{v} = \sum_{1}^{N}{}_i \alpha_i \mathbf{v}_i, \qquad \alpha_i \geq 0, \qquad \sum_{1}^{N}{}_i \alpha_i = 1.$$

Assume that $\max\{||\mathbf{w} - \mathbf{v}_i|| : i = 1, 2, \dots, N\} < 1 - r$. Then

$$(1 - r)^2 > \sum_{1}^{N}{}_i \alpha_i ||\mathbf{v}_i - \mathbf{w}||^2 \geq 1 + ||\mathbf{w}||^2 - 2\langle \mathbf{v}, \mathbf{w} \rangle.$$

Since

$$1 + ||\mathbf{w}||^2 - 2\langle \mathbf{v}, \mathbf{w} \rangle \geq 1 + ||\mathbf{w}||^2 - 2r||\mathbf{w}|| \geq 1 - r^2$$

we reach a contradiction. $\square$

We now extend the previous results to normed spaces. Since we do not have an inner product compatible with the norm, we need to modify, in a suitable manner, the statement and the proof of Proposition 2.1 and Proposition 2.2.

Proposition 2.3. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_N$ *be* N *nonzero vectors in a normed space* $\mathbf{E}$*, such that*

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_N = \mathbf{0}. \tag{2.5}$$

Define

$$L = \min\{k : ||\mathbf{v}_{i+1} - \mathbf{v}_i|| \le k||\mathbf{v}_i||,\ i = 1, 2, \dots, N;\ \mathbf{v}_{N+1} = \mathbf{v}_1\}.$$

Then

$$L \ge \begin{cases} \frac{4}{N} & \text{if } N \text{ is even and} \\ \frac{4N}{N^2-1} & \text{if } N \text{ is odd.} \end{cases} \tag{2.6}$$

Proof. See [**5, 8**]. □

The following definition is needed below in Proposition 2.4 and in Section 4. Let $\mathbf{v}_1, \dots, \mathbf{v}_N$ be N vectors in $\mathbf{E}$. For every subset of indices $j_1, j_2, \dots, j_p$ of $1, 2, \dots, N$, consider the convex hull of the corresponding vectors. Denote this set by $C(j_1, j_2, \dots, j_p)$. Let $\mathbf{x}$ be any vector belonging to $C(j_1, j_2, \dots, j_p)$. Define

$$d(j_1, j_2, \dots, j_p) = \min_{\mathbf{x} \in C(j_1, j_2, \dots, j_p)} \max\{||\mathbf{x} - \mathbf{v}_{j_i}|| : i = 1, 2, \dots, p\}$$

and set

$$\delta = \max_{p \in (1, 2, \dots, N)} d(j_1, j_2, \dots, j_p). \tag{2.7}$$

Proposition 2.4. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_N$ *be* N *vectors in* $\mathbf{E}$. *Then for every vector* $\mathbf{w} \in E$ *we have*

$$\delta \ge \min\{||\mathbf{w} - \mathbf{v}_i|| : i = 1, 2, \dots, N\} - d(\mathbf{w}, \operatorname{co}(\mathbf{v}_1, \dots, \mathbf{v}_N)). \tag{2.8}$$

In particular, if $\mathbf{v}_1, \dots, \mathbf{v}_N$ *are unit vectors whose convex hull is at distance r from the origin, then*

$$\delta \geq 1 - r. \tag{2.9}$$

Proof. By the triangular inequality, it is enough to show that for every vector $\mathbf{v} \in \mathrm{co}(\mathbf{v}_1, \dots, \mathbf{v}_N)$ we have $\min\{||\mathbf{v}-\mathbf{v}_i|| : i = 1, 2, \dots, N\} \leq \delta$. Consider the N disks D_i, $i = 1, 2, \dots, N$ of radius δ, $D_i = \{\mathbf{v} \in \mathbf{E} : ||\mathbf{v} - \mathbf{v}_i|| \leq \delta\}$. The definition of δ implies that the intersection of all these disks and $\mathrm{co}(\mathbf{v}_1, \dots, \mathbf{v}_N)$ is not empty. Moreover, for every subset of indices $j_1, j_2, \dots, j_p$ of $1, 2, \dots, N$, the intersection of the corresponding disks and $C(j_1, j_2, \dots, j_p)$ is not empty. By an induction argument, we obtain that $\mathrm{co}(\mathbf{v}_1, \dots, \mathbf{v}_N)$ is contained in the union of the D_i's. Hence, each vector $\mathbf{v} \in \mathrm{co}(\mathbf{v}_1, \dots, \mathbf{v}_N)$ belongs to one of the disks D_i, $i = 1, 2, \dots, N$, and the result follows. □

3. Geometric applications in Hilbert spaces. In 1929, W. Fenchel [**9**] proved that the total curvature of a closed smooth curve in R^3 is never smaller than 2π and it is equal to 2π if and only if the curve is simple, plane and convex. This beautiful result was later obtained in a different manner by T. Levi Civita [**12**] and K. Borsuk [**2**]. A very elegant geometric proof was obtained by R. Horn [**10**], who reports that elementary proofs of it have appeared in the lecture notes of A. Morse, A.S. Besicovitch, and H. Flanders.

We now derive Fenchel's theorem from an elementary geometric result which is established via Proposition 2.1.

Recall first that a closed curve $\mathbf{x} : [a, b] \to \mathbf{R}^3$ is simple if $\mathbf{x} : [a, b) \to \mathbf{R}^3$ is injective, and observe that the total curvature of a smooth closed curve is the length of its tangent indicatrix when the curve is parameterized using the arc length as a parameter. Moreover, the curve is simple, plane and convex if and only if its tangent indicatrix is a (simple) great circle [**10**]. Therefore, Fenchel's theorem can be reformulated in the following manner.

Theorem 3.1. *Let S^2 be the unit sphere of $\mathbf{R}^3$ and $\mathbf{z} : [a, b] \to S^2$ be at least C^1. Assume that $\mathbf{z}[a] = \mathbf{z}[b]$ and*

$$\int_a^b \mathbf{z}(t)\, dt = \mathbf{0}. \tag{3.1}$$

Then

$$\int_a^b ||\mathbf{z}'(t)||\, dt \geq 2\pi$$

with equality if and only if the image Γ of $\mathbf{z}$ is a (simple) great circle.

As we mentioned previously, the convex hull of a compact set in a finite dimensional normed space is closed. Therefore, equality (3.1) implies that the convex hull of Γ, $\mathrm{co}(\Gamma)$, must contain the origin. In fact, if this were not the case, then we could separate the origin from $\mathrm{co}(\Gamma)$ with a plane α and by selecting in $\mathbf{R}^3$ a basis containing the unit vector $\mathbf{v}$ perpendicular to α in the direction of $\mathrm{co}(\Gamma)$, we would obtain that the integral of the $\mathbf{v}$ component of $\mathbf{z}(t)$ is not zero, contradicting (3.1). This observation leads us to the geometric result of which Fenchel's theorem is a special case.

Theorem 3.2. *Let S^2 be the unit sphere of $\mathbf{R}^3$ and let $\mathbf{z} : [a, b] \to S^2$ be simple, closed and at least C^1. Assume that $\mathbf{0} \in \mathrm{co}(\Gamma)$, where $\Gamma = \mathrm{Range}\, \mathbf{z}$. Then*

$$\int_a^b ||\mathbf{z}'(t)||\, dt \geq 2\pi \tag{3.3}$$

with equality if and only if Γ is a great circle.

Proof. Since $\mathbf{0} \in \mathrm{co}(\Gamma)$, we can assume without loss of generality that there are N vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_N$ on Γ, and N nonzero constants $a_1, a_2, \dots, a_N$, such that

$$\sum_1^N{}_i\, a_i \mathbf{w}_i = \mathbf{0}.$$

Let $\mathbf{v}_i = a_i \mathbf{w}_i$, $i = 1, 2, \dots, N$. Inequality (3.3) now follows from Proposition 2.1, since the measure in radians of each angle θ_i, $i =$

$1, 2, \ldots, N$, is smaller than or equal to the length of the arc of Γ joining $\mathbf{w}_i$ with $\mathbf{w}_{i+1}, \mathbf{w}_{N+1} = \mathbf{w}_1$.

If $\theta_1 + \theta_2 + \cdots + \theta_N = 2\pi$, then, by Proposition 2.1, the vectors $\mathbf{w}_i$, $i = 1, 2, \ldots, N$, belong to a plane. Assuming that equality holds in (3.3), we obtain that each θ_i is equal to the length of the arc of Γ joining $\mathbf{w}_i$ with $\mathbf{w}_{i+1}, \mathbf{w}_{N+1} = \mathbf{w}_1$. Since the curve Γ is simple and at least of class C^1, it must be a great circle. □

Theorem 3.3 below extends the above result and Fenchel's theorem to any Hilbert space $\mathbf{H}$, and it could be proved in essentially the same manner. However, we prefer to present here a different proof based on an elegant geometric idea of Horn [**10**] and Proposition 2.1.

Theorem 3.3. *Let S be the unit sphere of $\mathbf{H}$ and let $\mathbf{z} : [a, b] \to S$ be simple, closed and at least C^1 except possibly at finitely many points. Assume that $\mathbf{0} \in \mathrm{co}(\Gamma)$, where $\Gamma = \mathit{Range}\,\mathbf{z}$. Then*

$$L(\Gamma) = \int_a^b ||\mathbf{z}'(t)||\, dt \geq 2\pi. \tag{3.4}$$

Moreover, if equality holds in (3.4) *and*

$$\int_a^b \mathbf{z}(t)\, dt = \mathbf{0}, \tag{3.5}$$

then Γ is a great circle.

Proof. We shall first replace Γ with a curve which is symmetric with respect to the origin and has the same length as Γ. Let P and Q be two points of Γ which divide it in two arcs of equal length. If P and Q are symmetric with respect to the origin, then we form the new curve Δ by selecting one of the arcs in which Γ is divided by the two points and adding to it its symmetric image with respect to $\mathbf{0}$. Obviously, $L(\Delta) = L(\Gamma)$ and Δ is symmetric with respect to the origin. If P and Q are not symmetric with respect to the origin, choose the unit vector $\mathbf{w}$ which contains the middle point of the segment joining P with Q. By (3.5) one of the two arcs intersects the subspace orthogonal to $\mathbf{w}$. Let Φ be the union of this arc and its symmetric image with respect to $\mathbf{w}$. The closed curve Φ intersects the subspace orthogonal to $\mathbf{w}$ in at least two points symmetric with respect to $\mathbf{0}$, and which divide Γ in two arcs of equal length. From this point on, we proceed as above.

Therefore, to establish inequality (3.4) we can assume, without loss of generality, that Γ is symmetric with respect to the origin. The C^1 character of the parametrization $\mathbf{z}$ may be lost at a few points, but this is obviously of no concern.

We now select on Γ pairs of antipodal points P_i, P_i' so that the distance between each point P_i and its next P_{i+1} along Γ, is small. Denoting by $\mathbf{v}_i$ the unit vector ending at P_i and by θ_i the angle between $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$, we see that θ_i is smaller than the length of the arc of Γ joining P_i with P_{i+1}. Thus, the inequality (3.4) follows from Proposition 2.1. If

$$\theta_0 + \theta_1 + \cdots + \theta_{2N-1} = 2\pi,$$

then the points $P_0, P_1, \ldots, P_{2N-1}$ belong to the same plane and the curve Φ is a great circle. The geometric construction we have used requires now P and Q to be symmetric with respect to the origin and the selected arc of Γ to be a half circle. Therefore, using (3.5), we obtain that Γ is a great circle. □

Let us now examine the case when the distance between $\mathrm{co}(\Gamma)$ and the origin is $r > 0$. Using Proposition 2.1$'$ and the same argument as in the above proof, we establish the following generalization of Fenchel's result.

Theorem 3.4. *Let S be the unit sphere of $\mathbf{H}$ and $\mathbf{z} : [a, b] \dashrightarrow S$ be at least C^1. Assume that $\mathbf{z}[a] = \mathbf{z}[b]$ and $\mathrm{dis}\,(\mathbf{0}, \mathrm{co}(\Gamma)) = r$, where $\Gamma = \mathit{Range}\,\mathbf{z}$. Then*

$$L(\Gamma) = \int_a^b ||\mathbf{z}'(t)||\, dt \geq 4\cos^{-1} r. \tag{3.6}$$

We close this section with an extension of a result due to Chakerian-Klamkin [**7**], who proved it in the case when $r = 0$ and $\mathbf{H}$ is finite dimensional.

Theorem 3.5. *Let K be a closed curve of length $T(K)$ contained in the unit ball of a Hilbert space* $\mathbf{H}$. *If K intersects the unit sphere S in a set of points whose convex hull is at distance r from the origin, then*

$$T(K) \geq 4(1-r) \tag{3.7}$$

Proof. This result can be considered as a special case of Corollary 4.1, which is derived from Theorem 4.2 and which establishes inequality (3.7) in normed spaces. In this case, the proof depends on Proposition 2.4. Another proof, independent of Proposition 2.4, is provided here, using Theorem 4.1 below.

Let $\mathbf{g} : [0, T(K)] \to \mathbf{H}$ be a parametrization of K using the arc length as a parameter. Then the constant L of inequality (4.1) is equal to 1. Moreover, by Proposition 2.2, the constant M of (4.3) satisfies the inequality $M \geq 1 - r$. Hence, $T(K) \geq 4(1-r)$. □

4. Geometric applications in normed spaces. We start this section with an application of Proposition 2.3. The result provides a lower bound on the period T of periodic orbits in $\mathbf{E}$.

Theorem 4.1. *Let* $\mathbf{E}$ *be a normed space and let* $\mathbf{g} : \mathbf{R} \to \mathbf{E}$ *be continuous and periodic of period $T > 0$. Suppose there exists $\delta > 0$, $L \geq 0$, such that for $h \in [0, \delta)$*

$$||\mathbf{g}(t+h) - \mathbf{g}(t)|| \leq Lh. \tag{4.1}$$

Then

$$LT \geq 4M \tag{4.2}$$

where

$$M = \max_{t \in [0,T]} \left\{ \left\| \mathbf{g}(t) - \frac{1}{T} \int_0^T \mathbf{g}(s)\, ds \right\| \right\}. \tag{4.3}$$

Proof. Since the function

$$\mathbf{G}(t) = \mathbf{g}(t) - \frac{1}{T} \int_0^T \mathbf{g}(s)\, ds$$

satisfies all the assumptions of the theorem and has mean value zero, we may assume, without loss of generality, that

$$\int_0^T \mathbf{g}(s)\,ds = \mathbf{0}, \qquad M = \max_{t\in[0,T]} ||\mathbf{g}(t)|| = ||\mathbf{g}(0)||.$$

We now define $M\mathbf{f}(t) = \mathbf{g}(t)$. Then

$$\int_0^T \mathbf{f}(s)\,ds = \mathbf{0}, \qquad 1 = \max_{t\in[0,T]} ||\mathbf{f}(t)|| = ||\mathbf{f}(0)||$$

and we can apply to $\mathbf{f}$ Theorem 3.1 of [**3**] whose proof is based on Proposition 2.3 and which provides the desired estimate $LT \geq 4M$. □

We now establish the results which were announced at the end of the previous section.

Theorem 4.2. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_N$ *be* N *vectors of* $\mathbf{E}$ *and let* $\delta = \max\{d(j_1, j_2, \dots, j_p)\}$ *(see* (2.7)*). Denote by* P *the polygonal path* $P(1, 2, \dots, N) = [\mathbf{v}_1, \mathbf{v}_2] \cup \cdots \cup [\mathbf{v}_{N-1}, \mathbf{v}_N] \cup [\mathbf{v}_N, \mathbf{v}_1]$, *where* $[\mathbf{x}, \mathbf{y}]$ *is the line segment joining* $\mathbf{x}$ *with* $\mathbf{y}$. *Let* $T(P)$ *be the length of the path. Then*

$$T(P) \geq 4\delta. \tag{4.4}$$

Proof. Let $j_1, j_2, \dots, j_q$ be a subset of indices of $1, 2, \dots, N$ such that $\delta = d(j_1, j_2, \dots, j_q)$. Parametrize the polygonal path $P(j_1, j_2, \dots, j_q)$ using the arc length as a parameter. Then, using the notations of Theorem 4.1, we have $L = 1$ and $M \geq \delta$. By the triangular inequality and Theorem 4.1, we have $T(P) \geq T(P(j_1, j_2, \dots, j_q)) \geq 4\delta$. □

Corollary 4.1. *Let* K *be a closed curve of length* $T(K)$ *contained in the unit ball of a normed space* $\mathbf{E}$. *If* K *intersects the unit sphere* S *in a set of points whose convex hull is at distance* r *from the origin, then*

$$T(K) \geq 4(1 - r). \tag{4.5}$$

Proof. For every $\varepsilon > 0$, we may assume that there are N unit vectors $\mathbf{v}_1, \dots, \mathbf{v}_N$, whose end points belong to the curve K and whose convex hull is at distance at most $r + \varepsilon$ from the origin. By the triangle inequality, Theorem 4.2 and Proposition 2.4, we have $T(K) \geq 4(1 - r - \varepsilon)$. Since this is true for every ε, the result follows. $\square$

In 1973, D. Chakerian and M.S. Klamkin [**7**] proved that the length $T(K)$ of a closed curve K contained in the unit ball of the Euclidean space $\mathbf{R}^n$, and such that the origin is contained in the convex hull of the intersection of K with the unit sphere S of $\mathbf{R}^n$, satisfies the inequality $T(K) \geq 4$. Corollary 4.1 improves their result in two ways. It extends it to normed spaces, and it relaxes the constraint that the distance of the convex hull from the origin be zero.

Our next goal is to provide a lower bound on the sum of distances between finitely many vectors. The result is obtained using some ideas from graph theory, which will be presented first. Recall that a graph is *complete* when no edges can be added to the graph without repetitions. Therefore, the number of edges of a complete graph with n vertices is $(n-1)n/2$.

A *Hamiltonian* cycle is a cycle which passes through all vertices exactly once. We shall say that two Hamiltonian cycles are disjoint if they do not have any edges in common. It is obvious that a complete graph always has a Hamiltonian cycle. What is less obvious is how many disjoint Hamiltonian cycles there are in a complete graph.

It turns out that the answer to the above question becomes easier if we double all edges of the graph. More precisely, we have

Proposition 4.1. *Let* $\mathbf{G}$ *be a graph with* $n+1$ *vertices and assume that* $\mathbf{G}$ *has been obtained from a complete graph* $\mathbf{H}$ *by doubling all edges of* $\mathbf{H}$. *Then* $\mathbf{G}$ *has exactly* n *disjoint Hamiltonian cycles.*

Proof. Let $V_1, V_2, \dots, V_n, V_{n+1}$ be the $n+1$ vertices of $\mathbf{G}$. Arrange them counterclockwise on the unit circle with V_{n+1} at the center (see Figure 1). Consider the Hamiltonian cycle which starts at V_{n+1} and goes on to $V_1, V_2, V_n, V_3, V_{n-1}, \dots, V_{n+1}$ moving from V_1 one step counterclockwise, then two steps clockwise, followed by three steps

FIGURE 1. A Hamiltonian cycle of the graph **G**.

counterclockwise, four steps clockwise, etc., until all vertices have been reached, at which point we return to V_{n+1}. The theorem follows by considering all Hamiltonian cycles obtained in the manner described above, by selecting any one of the n vertices $V_1, V_2, \ldots, V_n$, of $\mathbf{G}$ as the vertex connected directly to V_{n+1}. $\square$

We are now ready to obtain a lower bound on the sum of distances of vectors. A weaker version of this result was announced without proof in [**13**]. It extends a similar inequality obtained by Chakerian and Klamkin [**7**] in the case when $r = 0$ and E is a Minkowski plane, and it contains Grünbaum's conjecture (see [**1**]) as a special case.

Theorem 4.3. *Let* $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N$ *be* N *vectors and let* $\delta = \max\{d(j_1, j_2, \dots, j_p) : p \in (1, 2, \dots, N)\}$. *Then*

$$\sum_{1 \le i < j}^{N} ||\mathbf{v}_i - \mathbf{v}_j|| \ge 2(N-1)\delta \tag{4.6}$$

Proof. Construct the complete graph $\mathbf{H}$ whose vertices are the end points of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N$ and notice that the left hand side of (4.6) is the sum of the lengths of all edges of $\mathbf{H}$. Let $\mathbf{G}$ be the graph obtained from $\mathbf{H}$ by doubling all edges. Then, according to Proposition 4.1, $\mathbf{G}$ has $N-1$ disjoint Hamiltonian cycles. Each cycle can be considered as a closed curve whose length, according to Theorem 4.2, is bounded below by 4δ. Therefore, the global length of all edges of $\mathbf{G}$ is at least $4(N-1)\delta$. Inequality (4.6) follows immediately since the edges of $\mathbf{H}$ are counted twice in $\mathbf{G}$.

Remark . After this paper was accepted for publication, the authors were made aware of the paper of A.D. Andrew and M.A. Ghandehari [**1**], in which the inequality

$$\sum_{i<j} ||\mathbf{v}_i - \mathbf{v}_j|| \ge 2(N-1)\min\{||\mathbf{v} - \mathbf{v}_i|| : i = 1, 2, \dots, N\} \tag{4.7}$$

is established, where $\mathbf{v} \in \text{co}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N)$. Inequality (4.7) resolves the following conjecture of B. Grünbaum: Given a set A of N points in a real normed space $\mathbf{E}$, and a point $\mathbf{v} \in \text{co}(A)$, the sum of all distances between pairs of points of A, is at least $2(N-1)$ times the minimum distance from $\mathbf{v}$ to the points of A. It is easily seen that (4.7) is a consequence of (4.6) since $\delta \ge \min\{||\mathbf{v} - \mathbf{v}_i|| : i = 1, 2, \dots, N\}$.

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HARVEY MUDD COLLEGE, MATHEMATICS DEPARTMENT, CLAREMONT, CA 91711

CALIFORNIA STATE UNIVERSITY, MATHEMATICS DEPARTMENT, FULLERTON, CA 92634