ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 3, Summer 1992

TOPOLOGICAL TYPES OF SEVEN CLASSES OF ISOLATED SINGULARITIES WITH $\mathbf{C}^*$-ACTION

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1. Introduction. In 1982, Mather and the second author [**11**] proved that two germs of complex analytic hypersurfaces of the same dimension with isolated singularities are biholomorphically equivalent if and only if their moduli algebras are isomorphic. It is a natural question to ask for a necessary and sufficient condition for two germs of complex analytic hypersurfaces with isolated singularities (V_1, p_1) and (V_2, p_2) of the same dimension to have the same topological type. We say that (V_1, p_1) and (V_2, p_2) in $\mathbf{C}^{n+1}$ have the same topological type if $(\mathbf{C}^{n+1}, V_1, p_1)$ is homeomorphic to $(\mathbf{C}^{n+1} V_2, p_2)$. Even for $n = 1$, the case is not trivial. It took more than 40 years to get a complete solution. In 1928, Brauner [**2**] proved that the topological type of plane irreducible curve singularity is determined by its Puiseux pairs. In 1932, Burau [**3**] discovered (and also independently by Zariski [**28**]) that for plane irreducible curves the Puiseux exponents are invariant of topological type. Finally, Lejeune [**9**] and Zariski [**26**] proved that the topological type of plane curve singularity is determined by the topological type of all its irreducible components and all the pairs of intersection multiplicity of those components. This together with the theorem of J. Reeve [**17**], which asserts that the intersection multiplicity of two plane curves is the same as the linking number of the corresponding knots, gives a complete answer to our question for $n = 1$.

A polynomial $h(z_0, \dots, z_n)$ is weighted homogeneous of type $(w_0, \dots, w_n)$, where $(w_0, \dots, w_n)$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} \dots z_n^{i_n}$ for which $(i_0/w_0) + \dots + (i_n/w_n) = 1$. $(w_0, \dots w_n)$ is called the weights of h.

Orlik and Wagreich [**15**] and Arnold [**1**] showed that if $h(z_0, z_1, z_2)$ is a weighted homogeneous polynomial in $\mathbf{C}^3$ and $V = \{h(z) = o\}$ has an isolated singularity at origin, then V can be deformed into one of the following seven classes below while keeping the link K_V differentiably

Received by the editors on August 7, 1989.
Research of the second author partly supported by NSF Grant No. DMS 8411477.

constant.

Class 1. $V(a_0, a_1, a_2; 1) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\}$

Class 2. $V(a_0, a_1, a_2; 2) = \{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} = 0\}$, $a_1 > 1$

Class 3. $V(a_0, a_1, a_2; 3) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2} = 0\}$, $a_1 > 1$, $a_2 > 1$

Class 4. $V(a_0, a_1, a_2; 4) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$, $a_0 > 1$

Class 5. $V(a_0, a_1, a_2; 5) = \{z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$

Class 6. $V(a_0, a_1, a_2; 6) = \{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\}$, $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2$

Class 7. $V(a_0, a_1, a_2; 7) = \{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\}$, $(a_0 - 1)(a_1 b_2 + a_2 b_1) = a_2(a_0 a_1 - 1)$

The purpose of this paper is to give precise conditions when two classes of singularities above have the same topological type. These conditions are stated but without proof in [**21**, Theorem 4.1] which were used to prove the Zariski multiplicity conjecture for quasi-homogeneous surface singularities in [**21**]. In Orlik's paper, "Weighted homogeneous polynomials and fundamental groups" (Topology **9** (1970) 267–272) he asserted that, for weighted homogeneous polynomials f and g, if the links K_f and K_g are not lens spaces and $\pi_1(K_f)$ is isomorphic to $\pi_1(K_g)$, then f and g have the same weights. If this assertion were true, then it would not make the results of our paper easier to obtain. However, the above assertion of Orlik is not true. For example, $x^2 + y^7 + z^{14}$ and $x^3 + y^4 + z^{12}$ have homeomorphic links which are not lens spaces; however, they have distinct weights. Since there is a desire for detailed proofs of those results stated in [**2**, Theorem 4.1], we decided to publish them for the convenience of readers. The original proofs of these results depend only on the deep theorems of Neumann [**14**] and Varchenko [**20**]; however, the proofs are very long. Here we shall, in addition, use a result of Yoshinaga [**24**] and the result of Milnor and Orlik [**13**] to obtain simpler proofs of these results.

In Section 2 we recall the definitions of zeta function and characteristic polynomials of isolated hypersurface singularity. Varchenko's result is used to compute the zeta function of the above seven classes weighted homogeneous singularities explicitly. We also recall the result of Orlik and Wagreich [**15**] on description of resolution in terms of weights for weighted homogeneous singularities.

After the completion of this manuscript, we received a paper by O. Saeki ("Topological invariance of weights for weighted homogeneous isolated singularities in $\mathbf{C}^3$," Proc. Amer. Math. Soc. **103** (1988), 905–908). In his paper, O. Saeki stated that the topological type of a quasi-homogeneous isolated surface singularity in $\mathbf{C}^3$ is determined by its weights. However, the proof presented in his paper is incomplete. He uses the following unknown fact that the knot type of (S^5, K) is determined by $S^5 - K$. On the other hand, it is well known that, at least for nonalgebraic knot (S^5, K), the complement $S^5 - K$ does not determine the knot type of (S^5, K).

Notation and convention. In the rest of this paper we shall assume that the weights of the weighted homogeneous polynomial are greater than or equal to two without loss of generality by [**18**] (notice that the weights we adopt here are reciprocal of those used by Saito). If a_1 and a_2 are integers and a_1 divides a_2, we shall write a_1/a_2.

2. Zeta function and resolution for weighted homogeneous singularity. We shall first recall a deep result due to Varchenko [**20**]. Let $f : (\mathbf{C}^{n+1}, 0) \mapsto (\mathbf{C}, 0)$ be an analytic function. Let ε, δ be positive numbers $0 < \delta \ll \varepsilon \ll 1$. We define

$$T = \{t \in \mathbf{C} : |t| < \delta\}, \qquad B = \{z \in \mathbf{C}^{n+1} : |z_0|^2 + \cdots + |z_{n+1}| < \varepsilon\},$$
$$X = B \cap f^{-1}(T), \qquad X(t) = B \cap f^{-1}(t).$$

Milnor proved that $f : X \backslash X(0) \mapsto T \backslash \{0\}$ is a locally trivial smooth fiber bundle. If f has an isolated singularity at the origin, the fiber $X(t)$ of this fiber bundle has the homotopy type of a bouquet of n-spheres. The generator of $\pi_1(T\backslash\{0\})$ (represented by a counter-clockwise oriented circle around the origin), induces the monodromy automorphism $h : H^*(X(t), \mathbf{C}) \mapsto H^*(X(t), \mathbf{C})$.

Definition. The zeta-function of the monodromy at the origin is the function

$$\zeta_f(z) = \Pi_{q \geq 0} \{\det [Id - zh; H^q(X(t), \mathbf{C})]\}^{(-1)^q}. \tag{2.1}$$

If f has an isolated singularity at the origin, then $H^q(X(t), \mathbf{C}) = 0$ for $q \neq 0, n$. In this case the characteristic polynomial $\Delta_f(z)$ of the monodromy $h : H^n(X(t), \mathbf{C}) \mapsto H^n(X(t), \mathbf{C})$ is expressed by means of $\zeta_f(z)$ according to the formula

$$\Delta_f(z) = z\left[\left(\frac{z}{z-1}\right)\zeta_f\left(\frac{1}{z}\right)\right]^{(-1)^n} \tag{2.2}$$

where $\mu = \dim H^n(X(0), \mathbf{C})$ is the Milnor number.

Let $\mathbf{N} \subset \mathbf{R}_+$ be the set of all nonnegative integers and of all nonnegative real numbers. Let $f = \sum a_k x^k$, $a_k \in \mathbf{C}$, $k \in \mathbf{N}^{n+1}$, be an element in $\mathbf{C}\{x_0, \ldots, x_n\}$ and $\operatorname{supp} f$ be the set $\{k \in \mathbf{N}^{n+1} : a_k \neq 0\}$. We denote by $\Gamma_+(f)$ the convex hull of the set $\cup_{k \in \operatorname{supp} f}(k + \mathbf{R}_+^{n+1})$ in $\mathbf{R}_+^{n+1}$. The polyhedron $\Gamma(f)$ which is the union of all compact facets of $\Gamma_+(f)$ will be called Newton's diagram of the power series f. The polynomial $\sum_{k\in\Gamma(f)} a_k x^k$ will be called the main part of the power series f. Let γ be a closed facet of $\Gamma(f)$. Let us denote the polynomial $\sum_{k\in\gamma} a_k x^k$ by f_γ. The main part of the power series f will be called nondegenerate if for any closed facet $\gamma \in \Gamma(f)$ the polynomials $(x_0(\partial f_\gamma/\partial x_0)), \ldots, (x_n(\partial f_\gamma/x_n))$ have no common zero in $\{x \in \mathbf{C}^{n+1} : x_0 \ldots x_n \neq 0\}$.

We shall define the notions of zeta-function ζ_Γ associated with the Newton's diagram $\Gamma(f)$. Let

$$\zeta_\Gamma(z) = \Pi_{i=1}^{n+1}(\zeta^l(z))^{(-1)^{l-1}} \tag{2.3}$$

where ζ^l is a polynomial defined as below. ζ^l is defined by the $(l-1)$ dimensional facets of the intersections of $\Gamma(f)$ with all possible l-dimensional coordinate planes.

Let L be an l-dimensional affine subspace of $\mathbf{R}^{n+1}$ such that $L \cap \mathbf{Z}^{n+1}$ is l-dimensional lattice. By definition, let the l-dimensional volume of the cube (spanned by any basis of $L \cap \mathbf{Z}^k$) be equal to one.

Now we shall define ζ^l. Let $I \subseteq \{0, 1, \ldots, n\}$ and $|I| = l$, where $|I|$ is the number of the elements of I. Let us consider the pair $L_I, L_I \cap \Gamma(f)$, where $L_I = \{k \in \mathbf{R}^{n+1} : k_i = 0, \forall i \neq I\}$. Let $\Gamma_1(I), \ldots, \Gamma_{j(I)}(I)$ be all $(l-1)$-dimensional facets of $L_I \cap \Gamma(f)$ and $L_1, \ldots, L_{j(I)}$ be the $(l-1)$-dimensional affine subspaces, containing them respectively. Let $\sum_{i\in I} a_i^j k_i = m_j(I)$ be the equation of L_j in L_I where $a_i^j, m_j(I) \in \mathbf{N}$ and the greatest common divisor of the numbers a_i^j, $i \in I$, is equal to one. The numbers $a_i^j, m_j(I)$ are defined by these conditions uniquely. The numbers $m_j(I)$ will take part in the definition of ζ^l. Another definition of $m_j(I)$ is the following. Consider the quotient of the lattice $\mathbf{Z}^{n+1} \cap L_I$ by the subgroup generated by vectors of $\mathbf{Z}^k \cap L_j$. This is a cyclic group of order $m_j(I)$. Let $V(\Gamma_j(I))$ be the $l-1$-dimensional volume of $\Gamma_j(I)$ in L_j. Let

$$\zeta^l(z) = \Pi_{I,|I|=l} \Pi_{j=1}^{j(I)} (1 - z^{m_j(I)})^{(l-1)! V(\Gamma_j(I))}. \tag{2.4}$$

It was observed by Varchenko [**20**] that $m_j(I)(l-1)!V(\Gamma_j(I))$ is equal to $l!$ multiplied by the l-dimensional volume of the cone over $\Gamma_j(I)$ with vertex at origin. According to this remark $\deg \zeta^l$ contains the following geometric sense. Let $\Gamma_-(f)$ be the cone over $\Gamma(f)$ with vertex at the origin. Then $\deg \zeta^l$ is the sum of l-dimensional volumes of the intersections of $\Gamma_-(f)$ with all possible l-dimensional coordinate planes, multiplied by $l!$ The following theorem plays an important role in the proof of our theorem.

Theorem 2.1. (Varchenko) *Let f belong to the square of the maximal ideal of $\mathbf{C}\{x_0, \dots, x_n\}$, and let the main part of the power series f be nondegenerate. Then the zeta-function of the monodromy of f at the origin is equal to the zeta-function ζ_Γ of the Newton diagram Γ of f.*

Let a, b, and c be integers. We shall denote (a, b, c) to be the greatest common divisor of a, b, and c.

Proposition 2.2. (i) *Class 1. The function $f(z_0, z_1, z_2) = z_0^{a_0} + z_1^{a_1} + z_2^{a_2}$ is nondegenerate. Its zeta function is given by*

$$\zeta_I(z) = (1 - z^{a_0})(1 - z^{a_1})(1 - z^{a_2}) \left(1 - z^{\frac{a_0 a_1}{(a_0, a_1)}}\right)^{-(a_0, a_1)}$$

$$\left(1 - z^{\frac{a_1 a_2}{(a_1, a_2)}}\right)^{-(a_1, a_2)} \left(1 - z^{\frac{a_0 a_2}{(a_0, a_2)}}\right)^{-(a_0, a_2)} \left(1 - z^{\frac{a_0 a_1 a_2}{(a_0 a_1, a_1 a_2, a_0 a_2)}}\right)^{(a_0 a_1, a_1 a_2, a_0 a_2)}$$

(ii) *Class 2. The function $f(z_0, z_1, z_2) = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2}$ is nondegenerate. Its zeta function is*

$$\zeta_{II}(z) = (1 - z^{a_0})(1 - z^{\alpha_1}) \left(1 - z^{\frac{a_0 a_1}{(a_0, a_1)}}\right)^{-(a_0, a_1)}$$

$$\left(1 - z^{\frac{a_1 a_2}{(a_1 - 1, a_2)}}\right)^{-(a_1 - 1, a_2)} \left(1 - z^{\frac{a_0 a_1 a_2}{(a_0 a_2, a_1 a_2, a_0(a_1 - 1))}}\right)^{(a_0 a_2, a_1 a_2, a_0(a_1 - 1))}$$

(iii) *Class* 3. *The function* $f(z_0, z_1, z_2) = z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2}$ *is nondegenerate. Its zeta function is*

$$\zeta_{III}(z) = (1 - z^{a_0}) \left(1 - z^{\frac{a_1 a_2 - 1}{(a_1 - 1, a_2 - 1)}}\right)^{-(a_1 - 1, a_2 - 1)}$$
$$\left(1 - z^{\frac{a_0 (a_1 a_2 - 1)}{(a_1 a_2 - 1, a_0 (a_2 - 1), a_0 (a_1 - 1))}}\right)^{(a_1 a2 - 1, a_0 (a_2 - 1), a_0 (a_1 - 1))}$$

(iv) *Class* 4. *The function* $f(z_0, z_1, z_2) = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2}$ *is nondegenerate. Its zeta function is*

$$\zeta_{IV}(z) = (1 - z^{a_0}) \left(1 - z^{\frac{a_0 a_1}{(a_1, a_0 - 1)}}\right)^{-(a_1, a_0 - 1)}$$
$$\left(1 - z^{\frac{a_0 a_1 a_2}{(a_1 a_2, a_0 a_1 - a_0 + 1, a_2 (a_0 - 1))}}\right)^{(a_1 a_2, a_0 a_1 - a_0 + 1, a_2 (a_0 - 1))}$$

(v) *Class* 5. *The function* $f(z_0, z_1, z_2) = z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2}$ *is nondegenerate. Its zeta function is*

$$\zeta_V(z)$$
$$= \left(1 - z^{\frac{a_0 a_1 a_2 + 1}{(a_1 a_2 - a_2 + 1, a_0 a_2 - a_0 + 1, a_0 a_1 - a_1 + 1)}}\right)^{(a_1 a_2 - a_2 + 1, a_0 a_2 - a_0 + 1, a_0 a_1 - a_1 + 1)}$$

(vi) *Class* 6. *The function* $f(z_0, z_1, z_2) = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$, *where* $(a_0 - 1)(a_2 b_1 + a_1 b_2) = a_0 a_1 a_2$, *is nondegenerate for suitable chosen* b_1 *and* b_2. *Its zeta function is*

$$\zeta_{VI}(z) = (1 - z^{a_0}) \left(1 - z^{\frac{a_0 a_1}{(a_1, a_0 - 1)}}\right)^{-(a_1, a_0 - 1)} \left(1 - z^{\frac{a_0 a_2}{(a_2, a_0 - 1)}}\right)^{-(a_2, a_0 - 1)}$$
$$\left(1 - z^{\frac{a_0 a_1 a_2}{(a_1 a_2, a_2 (a_0 - 1), a_1 (a_0 - 1))}}\right)^{(a_1 a_2, a_2 (a_0 - 1), a_1 (a_0 - 1)) \frac{a_0}{(a_0 - 1)}}$$

(vii) *Class* 7. *The function* $f(z_0, z_1, z_2) = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$, *where* $(a_0 - 1)(a_2 b_1 + a_1 b_2) = (a_2 (a_0 a_1 - 1)$ *is nondegenerate* *for suitable chosen b_1 and b_2. Its zeta function is*

$$\zeta_{VII}(z) = \left(1 - z^{\frac{a_0a_1-1}{(a_1-1,a_0-1)}}\right)^{-(a_1-1,a_0-1)}$$
$$\left(1 - z^{\frac{a_2(a_0a_1-1)}{(a_2(a_1-1),a_2(a_0-1),a_1(a_0-1))}}\right)^{(a_2(a_1-1),a_2(a_0-1),a_1(a_0-1))\frac{a_0}{a_0-1}}$$

Proof. Easy exercise. □

Let f be a weighted homogeneous polynomial in $\mathbf{C}[z_0, z_1, z_2]$. Let (w_0, w_1, w_2) be weights of f. Let $w_i = u_i/v_i$ be the reduced fraction of w_i, i.e., u_i and v_i are integers with $(u_i, v_i) = 1$. Define $d = \langle u_0, u_1, u_2\rangle$ the least common multiple of u_0, u_1 and u_2; $c = (u_0, u_1, u_2)$ the greatest common divisor of u_0, u_1, u_2; $q_i = d/w_i$; $c_0 = (u_1, u_2)/c$; $c_1 = (u_0, u_2)/c$; $c_2 = (u_0, u_1)/c$.

Finally, we define γ_0, γ_1 and γ_2 by $u_0 = cc_1c_2\gamma_0$, $u_1 = cc_0c_2\gamma_1$, $u_2 = cc_0c_1\gamma_2$.

The link $K_f = f^{-1}(0)\cap S^5$, where S^5 is a sphere with center at origin, is a Seifert fibered 3-manifold. Orlik and Wagreich [**15**] have calculated the Seifert invariants of K_f,

$$\{-b; g; n_1(\alpha_1, \beta_1), n_2(\alpha_2, \beta_2), n_3(\alpha_3, \beta_3), n_4(\alpha_4, \beta_4)\},$$

which are given as follows.

TABLE 1.

	α_0	n_0	α_1	n_1	α_2	n_2	α_3	n_3
I	γ_0	cc_0	γ_1	cc_1	γ_2	cc_2		0
II	γ_0	$(cc_0-1)/v_2$	$v_2\gamma_0$	1	γ_2	c		0
III	γ_0	$(cc_0-v_1-v_2)/v_1v_2$	$v_2\gamma_0$	1	$v_1\gamma_0$	1		0
IV	γ_2	$(c-1)/v_1$	v_2	1	$v_1\gamma_2$	1		0
V	v_0	1	v_1	1	v_2	1		0
VI	γ_1	$(c-1)/v_1$	γ_2	$(c-1)/v_2$	$v_1\gamma_2$	1	$v_2\gamma_1$	1
VII	γ_2	$(c-v_0-v_1)/v_0v_1$	v_2	1	$v_1\gamma_2$	1	$v_0\gamma_2$	1

TABLE 2.

	w_0	w_1	w_2
I	a_0	a_1	a_2
II	a_0	a_1	$\frac{a_1a_2}{a_1-1}$
III	a_0	$\frac{a_1a_2-1}{a_2-1}$	$\frac{a_1a_2-1}{a_1-1}$
IV	a_0	$\frac{a_0a_1}{a_0-1}$	$\frac{a_0a_1a_2}{a_0a_1-a_0+1}$
V	$\frac{a_0a_1a_2+1}{a_1a_2-a_2+1}$	$\frac{a_0a_1a_2+1}{a_0a_2-a_0+1}$	$\frac{a_0a_1a_2+1}{a_0a_1-a_1+1}$
VI	a_0	$\frac{a_0a_1}{a_0-1}$	$\frac{a_0a_2}{a_0-1}$
VII	$\frac{a_0a_1-1}{a_1-1}$	$\frac{a_0a_1-1}{a_0-1}$	$\frac{a_2(a_0a_1-1)}{a_1(a_0-1)}$

TABLE 2. (Continued)

	u_0	u_1	u_2
I	a_0	a_1	a_2
II	a_0	a_1	$\frac{a_1a_2}{(a_2,a_1-1)}$
III	a_0	$\frac{a_1a_2-1}{(a_1-1,a_2-1)}$	$\frac{a_1a_2-1}{(a_1-1,a_2-1}$
IV	a_0	$\frac{a_0a_1}{(a_0-1,a_1)}$	$\frac{a_0a_1a_2}{(a_1a_2,a_0a_1-a_0+1)}$
V	$\frac{a_0a_1a_2+1}{k}$	$\frac{a_0a_1a_2+1}{k}$	$\frac{a_0a_1a_2+1}{k}$
VI	a_0	$\frac{a_0a_1}{(a_0-1,a_1)}$	$\frac{a_0a_2}{(a_0-1,a_2)}$
VII	$\frac{a_0a_1-1}{(a_0-1,a_1-1)}$	$\frac{a_0a_1-1}{(a_0-1,a_1-1)}$	$\frac{a_2(a_0a_1-1)}{(a_2(a_0a_1-1),a_1(a_0-1))}$

where $k = (a_1a_2 - a_2 + 1, a_0a_2 - a_0 + 1, a_0a_1 - a_1 + 1)$
$= (a_0a_1a_2 + 1, a_1a_2 - a_2 + 1)$

TABLE 2 (Continued).

	v_0	v_1	v_2
I	1	1	1
II	1	1	$\frac{a_1}{(a_2,a_1-1)}$
III	1	$\frac{a_2-1}{(a_1-1,a_2-1)}$	$\frac{a_1-1}{(a_1-1,a_2-1)}$
IV	1	$\frac{a_0-1}{(a_0-1,a_1)}$	$\frac{a_0a_1-a_0+1}{(a_1a_2,a_0a_1-a_0+1)}$
V	$\frac{a_1a_2-a_2+1}{k}$	$\frac{a_0a_2-a_0+1}{k}$	$\frac{a_0a_1-a_1+1}{k}$
VI	1	$\frac{a_0-1}{(a_0-1,a_1)}$	$\frac{a_0-1}{(a_0-1,a_2)}$
VII	$\frac{a_1-1}{(a_0-1,a_1-1)}$	$\frac{a_0-1}{(a_0-1,a_1-1)}$	$\frac{a_1(a_0-1)}{(a_1(a_0-1),a_2(a_0a_1-1))}$

By the theorem of Saito [**18**], we may assume from now on that $w_i \geq 2$ for $i = 0, \dots, n$. We have, in particular, $u_i \geq 2$ for $i = 0, \dots, n$. The following theorem which is a consequence of Milnor and Orlik [**13**] is due to Yoshinaga [**24**].

Theorem 2.3. *Let $f(x_0, \dots, x_n)$ (respectively, $g(x_0, \dots, x_n)$) be a weighted homogeneous polynomial with weights $(u_0/v_0, \dots, u_n/v_n)$ (respectively, $u'_0/v'_0, \dots, u'_n/v'_n)$) where u_i/v_i (respectively u'_i/v'_i) is the reduced fraction of w_i (respectively w'_i). Assume that f (respectively g) has an isolated singularity at origin. Then $\Delta_f(z) = \Delta_g(z)$ if and only if the following two conditions are satisfied.*

(1) $\{2, u_0, \dots, u_n\} = \{2, u'_0, \dots, u'_n\}$

(2) *For any $u \in \{2, u_0, \dots, u_n\}$,*

$$\prod_{u_i=u}\left(1-\frac{u_i}{v_i}\right) = \prod_{u'_j=u}\left(1-\frac{u'_j}{v'_j}\right)$$

where the product over an empty set is assumed to be one.

3. Classification of the topological types of seven classes of singularities.

Lemma 3.1. *Let $f(z_0, z_1, z_2)$ and $g(z_0, z_1, z_2)$ be weighted homogeneous polynomials with weights $(u_0/v_0, u_1/v_1, u_2/v_2)$ and $(u'_0/v'_0, u'_1/v'_1, u'_2/v'_2)$, respectively, having isolated singularities at origin.*

(i) *If* $u'_{i_1} = u'_{i_2} = 2$ *where* $\{i_0, i_1, i_2\} = \{0, 1, 2\}$*, then*

$$\{(z_0, z_1, z_2) \in \mathbf{C}^3 : g(z_0, z_1, z_2) = 0\}$$

is biholomorphically equivalent to

$$\{(z_0, z_1, z_2) \in \mathbf{C}^3 : z_0^2 + z_1^2 + z_2^{k+1} = 0\}$$

with $k \geq 2$*.*

(ii) *Suppose that* f *and* g *are one of the seven types. If* $\Delta_f(z) = \Delta_g(z)$ *and* $u_0 = u_1 = u_2 = u'_{i_0} > u'_{i_1} = u'_{i_2} = 2$ *where* $\{i_0, i_1, i_2\} = \{0, 1, 2\}$*, then* $f(z_0, z_1, z_2)$ *is of type* V *and* $\pi_1(K_f)$ *is infinite while* $\pi_1(K_g)$ *is finite.*

Proof. (i) Observe that $w'_{i_1} = u'_{i_1}/v'_{i_1} = 2/v'_{i_1} \leq 2$ and similarly $w'_{i_2} \leq 2$. By our convention, we conclude that $w'_{i_1} = w'_{i_2} = 2$. Therefore (i) follows from a proposition of Saito [**18**].

(ii) From the proof of (i) above, we see that $v'_{i_0} = v'_{i_1} = v'_{i_2} = 1$. By Proposition 2.3, we have

$$1 - u'_{i_0} = (1 - u_0/v_0)(1 - u_1/v_1)(1 - u_2/v_2).$$

If $v_0 = 1$, then $1 = ((1 - u_1)/v_1)((1 - u_2)/v_2)$. Since $w_1 \geq 2$ and $w_2 \geq 2$, we have $1 - u_1/v_1 \leq -1$ and $1 - u_2/v_2 \leq -1$. It follows that $u_2/v_2 = 2 = u_1/v_1$. As u_1/v_1 and u_2/v_2 are reduced forms, we conclude that $u_1 = u_2 = 2$ and $v_1 = v_2 = 1$. This contradicts the assumption that $u_1 = u_2 > 2$. Hence, v_0 cannot be one. Similarly, v_1 and v_2 cannot be one. This implies that $f(z_0, z_1, z_2)$ is either of type V or of type VII.

(a) Suppose that f is of type V, i.e.,

$$f = z_0^{a_0} z_1 + z^{a_1} z_2 + z_0 z_2^{a_2}.$$

Recall that $g = z_0^2 + z_1^2 + z_2^{k+1}$, $k \geq 2$ and $u_0 = u_1 = 2$, $u_2 = k + 1$. By Proposition 2.2,

$$\zeta_f(z) = \left(1 - z^{\frac{a_0 a_1 a_2 + 1}{l}}\right)^l$$
$$\zeta_g(z) = (1 - z^{k+1})$$

where $l = (a_1a_2 - a_2 + 1, a_0a_2 - a_0 + 1, a_0a_1 - a_1 + 1)$. Since $\zeta_f(z) = \zeta_g(z)$ by assumption, we have $l = 1$ and $(a_0a_1a_2 + 1)/l = k+1 \geq 3$. Note that $\pi_1(K_g)$ is finite while $\pi_i(K_f)$ is finite only if $1/w_0 + 1/w_1 + 1/w_2 \geq 1$. We conclude that

$$\frac{(a_0a_1 + a_0a_2 + a_1a_2) - (a_0 + a_1 + a_2) + 3}{a_0a_1a_2 + 1} \geq 1$$

which is equivalent to

$$a_0a_1a_2 + 1 \leq (a_0a_1 + a_0a_2 + a_1a_2) - (a_0 + a_1 + a_2) + 3,$$

i.e., $(a_0 - 1)(a_1 - 1)(a_2 - 1) \leq 1$. Thus $1 \leq a_i \leq 2$ for $i = 0, 1, 2$. Since $a_0a_1a_2 + 1 \geq 3$, we have only the following subcases.

(α) $a_0 = a_1 = 1$, $a_2 = 2$, i.e., $f = z_0z_1 + z_1z_2 + z_0z_2^2$. In this case, we see that $w_0 = 3 = w_2$, $w_1 = 3/2$, which contradicts our assumption that $w_i \geq 2$ for all $i = 0, 1, 2$.

(β) $a_0 = 1$, $a_1 = a_2 = 2$, i.e., $f = z_0z_1 + z_1^2z_2 + z_0z_2^2$. In this case, we have $w_0 = 5/3$, $w_1 = 5/2$, $w_2 = 5$, which contradicts our assumption that $w_i \geq 2$ for all $i = 0, 1, 2$.

(γ) $a_0 = a_1 = a_2 = 2$. In this case

$$l = (a_1a_2 - a_2 + 1, a_0a_2 - a_0 + 1, a_0a_1 - a_1 + 1) = 3 > 1,$$

which again gives a contradiction.

(b) Suppose that f is of type VII, i.e., $f = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$. Recall that $g = z_0^2 + z_1^2 + z_2^{k+1}$. By Proposition 2.2,

$$\zeta_f(z) = \left(1 - z^{\frac{a_0a_1-1}{(a_0-1,a_1-1)}}\right)^{\left(\frac{a_0a_2}{a_0-1}-1\right)(a_1-1,a_0-1)}$$
$$\zeta_g(z) = (1 - z^{k+1}).$$

To have $\zeta_f(z) = \zeta_g(z)$, we need

$$\frac{a_0a_1 - 1}{(a_0 - 1, a_1 - 1)} = k + 1 \quad \text{and} \quad \left(\frac{a_0a_2}{a_0 - 1} - 1\right)(a_1 - 1, a_0 - 1) = 1.$$

Thus, $(a_0-1, a_1-1)=1$, $a_0a_1-1=k+1$ and $((a_0a_2)/(a_0-1))-1=1$. It follows that $a_0=2$, $a_2=1$ and $2a_1=k+2$. From the equation $(b_1+a_1b_2)/(2a_1-1)=1$, we infer that $b_1=a_1-1$ and $b_2=1$. Hence, $f=z_0^2z_1+z_0z_1^{a_1}+z_0z_2+z_1^{a_1-1}z_2=(z_0+z_1^{a_1-1})(z_0z_1+z_2)$ which do not have isolated singularity at origin. So f cannot be of type VII. □

Lemma 3.2. *Let $f(z_0,z_1,z_2)$ and $g(z_0,z_1,z_2)$ be weighted homogeneous polynomials having isolated singularities at origin with weights (w_0,w_1,w_2) and (w_0',w_1',w_2'), respectively. Suppose that $\Delta_f(z)=\Delta_g(z)$ and $\pi_1(K_f)=\pi_1(K_g)$. If $u_1=u_2$, then $u_1=u_2=u_{i_1}'=u_{i_2}'$ for some $i_1\neq i_2$ with $0\le i_1,i_2\le 2$, and $u_0=u_{i_0}'$ where $\{0,1,2\}=\{i_0,i_1,i_2\}$.*

Proof. Suppose first that u_0',u_1',u_2' are pairwise distinct. From Theorem 2.3, we see that $\{2,u_0,u_1\}=\{2,u_0',u_1',u_2'\}$. This implies that one of the $u_i'=2$, say $u_0'=2$. We have $\{2,u_0,u_1\}=\{2,u_1',u_2'\}$ with $2,u_1',u_2'$ are pairwise distinct. Without loss of generality, we may assume $u_0=u_1'$, $u_1=u_2'$. By Theorem 2.3 (ii) with $u=2$, we see that $1-u_0'/v_0'=1$, i.e., $u_0'/v_0'=0$ which contradicts the hypothesis $w_0'\ge 2$. So u_0',u_1',u_2' cannot be pairwise distinct. Without loss of generality we shall assume that $u_1'=u_2'$.

Suppose on the contrary that $u_1=u_2\neq u_1'=u_2'$. Since

$$\{2,u_0,u_1,u_2\}=\{2,u_0',u_1',u_2'\},$$

we have either (α) $u_1'=u_2'=u_0$ or (β) $u_1'=u_2'=2\neq u_0$.

In case (β), we have $\{2,u_0,u_1\}=\{2,u_0'\}$. Note that $u_0\neq 2$ and $u_1\neq 2=u_1'=u_2'$. Thus we must have

$$u_0=u_1=u_2=u_0'\neq u_1'=u_2'=2.$$

By Lemma 3.1, we have $\pi_1(K_f)\neq\pi_1(K_g)$, which contradicts our assumption.

In case (α), we have either $u_1'=u_2'=u_0'$ or $u_1'=u_2'\neq u_0'$. In the first case, we have $u_0'=u_1'=u_2'=u_0\neq u_1=u_2$. It follows that $u_1=u_2=2$, which again leads to a contradiction by Lemma 3.1 because the assumption $\pi_1(K_f)=\pi_1(K_g)$. In the second case, we have

$u'_0 \neq u'_1 = u'_2 = u_0$. Hence $u_0 \neq u_1 = u_2$. By Theorem 2.3 (ii) with $u = u_0$, we have

$$(1 - u'_1/v'_1)(1 - u'_2/v'_2) = 1 - u_0/v_0.$$

Since $w'_1 \geq 2$, $w'_2 \geq 2$ and $w_0 \geq 2$, the left hand side is positive while the right hand side is negative. This gives a contradiction.

Now we have shown $u_1 = u_2 = u'_{i_1} = u'_{i_2}$. By Theorem 2.3 (i), we have

$$\{2, u_0, u_1\} = \{2, u'_{i_0}, u'_{i_1}\}.$$

Hence, one of the following subcases has to be satisfied.

(1) If $u_0 = u_1$, then either $u'_{i_0} = u'_{i_1} = u_0 = u_1$ or $2 = u'_{i_0} \neq u'_{i_1} = u_0 = u_1$. In the latter case, we have $u_0 = u_1 = u_2 = u'_{i_1} = u'_{i_2} \neq 2 = u'_{i_0}$. This contradicts Theorem 2.3 (ii) with $u = 2$.

(2) If $u_0 \neq u_1$ and $u_0 = 2$, then either $u'_{i_0} = 2 = u_0$ or $u'_{i_0} = u_1 = u_{i_1}$. In the latter case, we have $u'_{i_0} = u'_{i_1} = u'_{i_2} = u_1 = u_2 \neq u_0 = 2$. This contradicts Theorem 2.3 (ii) with $u = 2$.

(3) If $u_0 \neq u_1$ and $u_1 = 2$, then either $u'_{i_0} = u_0$ or $u'_{i_0} = 2 = u_1$. In the latter case, we have $u'_{i_0} = 2 = u_1 = u_2 = u'_{i_1} = u'_{i_2}$. It follows that $u_0 = 2 = u'_{i_0}$ as well.

(4) If $u_0 \neq u_1$, $u_0 \neq 2$ and $u_1 \neq 2$, then we have $u'_{i_0} = u_0$ immediately.

Thus we conclude that $u_0 = u'_{i_0}$. □

Lemma 3.3. *Let $f(z_0, z_1, z_2)$ and $g(z_0, z_1, z_2)$ be weighted homogeneous polynomials having isolated singularity at origin with weights (w_0, w_1, w_2) and (w'_0, w'_1, w'_2), respectively. Suppose that $\Delta_f(z) = \Delta_g(z)$ and $\pi_1(K_f) = \pi_1(K_g)$. If u_1/u_2 and $u_1 < u_2$, then for some indices i_1, i_2 with $i_1 \neq i_2$, $0 \leq i_1$, $i_2 \leq 2$, we have u'_{i_1}/u'_{i_2}, $u_1 = u'_{i_1}$, $u_2 = u'_{i_2}$ and $u_0 = u'_{i_0}$ where $\{0, 1, 2\} = \{i_0, i_1, i_2\}$.*

Proof. Let $u_2 = ku_1$ where k is an integer bigger than one. By Proposition 2.3 (i), we have $\{2, u_0, u_1, ku_1\} = \{2, u'_0, u'_1, u'_2\}$. Because the hypothesis of Lemma 3.1 is not satisfied, there are the following three subcases.

(1) u_0, u_1 and ku_1 are pairwise distinct. By Lemma 3.2, u'_0, u'_1 and u'_2 must be pairwise distinct. Thus it is clear that we have $u_0 = u'_0$, $u_1 = u'_1$, $u_2 = u'_2$ after reindexing u'_0, u'_1, u'_2.

(2) $u_0 = u_1$. By Lemma 3.2, we may assume that $u_2 \neq u_0 = u_1 = u'_0 = u'_1$. Thus $\{2, u_1, u_2 = ku_1\} = \{2, u'_0, u'_2\}$. Observe that $u_2 = ku_1 > 2$ since $u_1 \geq 2$ and $k > 1$. Therefore, $u'_2 = ku_1$ as required.

(3) $u_0 = u_2 = ku_1 > u_1$. By Lemma 4.2, we may assume that $u_0 = u_2 = u'_0 = u'_2$. Thus $\{2, u_1, u_2 = ku_1\} = \{2, u'_1, u'_2\}$. If $u_1 \neq 2$, then $2, u_1, u_2$ are distinct. Hence, $u'_1 = u_1$ as required. If $u_1 = 2$, then we have $\{2, u_2 = 2k\} = \{2, u'_1, u'_2 = 2k\}$. u'_1 cannot be $u_2 = 2k$, otherwise $u'_1 = u'_2 = u'_0 = u_2 = u_0 > u_1$ which is exactly the hypothesis in Lemma 3.1. Hence, $u'_1 = 2 = u_1$ as required. □

Lemma 3.4. *Let $f(z_0, z_1, z_2)$ and $g(z_0, z_1, z_2)$ be weighted homogeneous polynomials in seven types. Let $\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$ (respectively $\alpha_0, \alpha_1, \alpha_2, \alpha_3$) be the orders of stabilizer subgroups of the action of $SO(2)$ at the link K_f (respectively K_g). If $\pi_1(K_f) = \pi_1(K_g)$, then we have $\{1, \bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3\} = \{1, \alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. Moreover, for any $\alpha \in \{1, \alpha_0, \alpha_1, \alpha_2, \alpha_3\} \backslash \{1\}$, we have*

$$\sum_{\alpha_i = \alpha} n_i = \sum_{\bar{\alpha}_j = \alpha} \bar{n}_j,$$

where n_i (respectively $\bar{n}_j$) is the number of orbits whose stabilizer subgroup has order α_i (respectively $\bar{\alpha}_j$).

Proof. By Neumann's result [**14**], the minimal good resolution graph of weighted homogeneous two-dimensional hypersurface singularities are determined by the fundamental group of their links. In [**15**], the weighted dual graph of minimal good resolution of $(f^{-1}(0,0)$ (respectively $(g^{-1}(0), 0))$ is star-shaped. The self-intersection numbers of all the vertices in a branch of the weighted dual graph determine α_i/β_i where $(\alpha_i, \beta_i) = 1$ and β_i determine the representation $\mathbf{Z}_{\alpha_i} \mapsto SO(2)$. Thus, if $\pi_1(K_f) = \pi_1(K_g)$, then

$$(*) \qquad \{\alpha_i/\beta_i : \alpha_i > 1\} = \{\bar{\alpha}_j/\bar{\beta}_i : \bar{\alpha}_j > 1\}$$

From $(*)$, we infer that $\{1, \alpha_0, \alpha_1, \alpha_2, \alpha_3\} = \{1, \bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3\}$. Note that n_i is the number of the branches in the weighted dual graph such that the continuous fraction formed by the self-intersection numbers is exactly equal to α_i/β_i with $\alpha_i > 1$. The last statement of the lemma follows easily. □

Lemma 3.5. *Let $f(z_0, z_1, z_2)$ and $g(z_0, z_1, z_2)$ be weighted homogeneous polynomials with weights $(u_0/v_0, u_1/v_1, u_2/v_2)$ and $(u'_0/v'_0, u'_1/v'_1, u'_2/v'_2)$, respectively, having isolated singularities at origin. Suppose that f and g are one of seven types. If $\Delta_f(z) = \Delta_g(z)$ and $\pi_1(K_f) = \pi_1(K_g)$, then*

$$\min\{u_0, u_1, u_2\} = \min\{u'_0, u'_1, u'_2\}$$

and

$$\max\{u_0, u_1, u_2\} = \max\{u'_0, u'_1, u'_2\}.$$

Proof. Let us assume without loss of generality that

$$u_0 = \min\{u_0, u_1, u_2\} \quad \text{and} \quad u'_0 = \min\{u'_0, u'_1, u'_2\}.$$

If $u_0 \neq u'_0$, then we have either $u_0 > 2$, $u'_0 = 2$ or $u_0 = 2$, $u'_0 > 2$, since $\{2, u_0, u_1, u_2\} = \{2, u'_0, u'_1, u'_2\}$ by (i) of Theorem 2.3. By symmetry, we only need to show that the case $u_0 > 2$, $u'_0 = 2$ cannot occur. Suppose on the contrary that $u_0 > 2$, $u'_0 = 2$. Then by (ii) of Theorem 2.3, we have

$$(*) \qquad 1 = \prod_{u_i=2}\left(1 - \frac{u_i}{v_i}\right) = \prod_{u'_j=2}\left(1 - \frac{u'_j}{v'_j}\right).$$

Since $w'_j = u'_j/v'_j \geq 2$, i.e., $1 - w'_j \leq -1$, we conclude that $1 - w'_j = -1$ for $u'_j = 2$. Moreover from $(*)$, we know that there exists exactly one j which is different from zero such that $u'_j = 2$. For the sake of argument, let us assume that $u'_1 = 2$. Then we have $u'_0 = u'_1 = 2$ and $u'_2 > 2$. As $\min\{u_0, u_1, u_2\} = u_0 > 2$, we infer that $u_0 = u_1 = u_2 = u'_2 > u'_0 = u'_1 = 2$. However, we know that $\pi_1(K_f)$ is infinite while $\pi_1(K_g)$ is finite by Lemma 3.1. This contradicts our hypothesis that $\pi_1(K_f) = \pi_1(K_g)$. We conclude that $u_0 = u'_0$. It is clear that $\max\{u_0, u_1, u_2\} = \max\{u'_0, u'_1, u'_2\}$ by Theorem 2.3. □

Lemma 3.6. *Let $f(z_0, z_1, z_2)$ and $g(z_0, z_1, z_2)$ be weighted homogeneous polynomials with weights $(u_0/v_0, u_1/v_1, u_2/v_2)$ and $(u'_0/v'_0, u'_1/v'_1,$*

u'_2/v'_2), respectively, having isolated singularity at origin, where $u_0 \leq u_1 \leq u_2$ and $u'_0 \leq u'_1 \leq u'_2$. Suppose that f and g are one of the seven types. If $\Delta_f(z) = \Delta_g(z)$ and $\pi_1(K_f) = \pi_1(K_g)$, then $u_i = u'_i$ for $0 \leq i \leq 2$.

Proof. By Lemma 3.5, we know that $u_0 = u'_0$ and $u_2 = u'_2$. We only need to consider the following three subcases.

Case 1. $u_0 = u_1$. If $u'_1 = u'_0$, then $u'_1 = u'_0 = u_0 = u_1$. If $u'_1 \neq u'_0$, then, by Lemma 3.2, we have $u'_1 = u'_2 = u_1 = u_0$.

Case 2. $u_1 = u_2$. If $u'_1 = u'_2$, then $u'_1 = u'_2 = u_2 = u_1$. If $u'_1 = u'_2$, then $u_1 = u_2 = u'_0 = u'_1$ by Lemma 3.2.

Case 3. $u_0 < u_1 < u_2$. It is clear that $u'_1 = u_1$ by (i) of Proposition 2.3. □

Now let us recall the following well-known fact. If $(V, 0)$ and $(W, 0)$ are germs of isolated hypersurface singularities in $\mathbf{C}^{n+1}$, having the same topological type, then $\pi_1(K_V) = \pi_1(K_W)$ and $\Delta_V(z) = \Delta_W(z)$.

Theorem 3.7. *Suppose that $f = z_0^{a'_0} + z_1^{a'_1} + z_2^{a'_2}$ is of type I and $g = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2}$ is of type II. Then $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ have the same topological type if and only if*

$$\left\{ \begin{array}{l} f = z_0^{a_0} + z_1^{a_1} + z_2^{\frac{a_1 a_2}{a_1 - 1}} \\ g = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} \end{array} \right\}$$

with $(a_1 - 1)|a_2$.

Proof. Since $u_1 = a_1$ and $u_2 = a_1 a_2/(a_1 - 1, a_2)$, we see that u_1/u_2. As $\pi_1(K_f) = \pi_1(K_g)$, the hypothesis of Lemma 3.1 cannot be satisfied. So by Lemma 3.2 and Lemma 3.3, we may assume a'_1/a'_2, $a_1 = a'_1$ and $a'_2 = a_1 a_2/(a_1 - 1, a_2)$. By Theorem 2.3 (i), we have $\{2, a'_0, a'_1, a'_2\} = \{2, a_0, a'_1, a'_2\}$. If $a'_0 \neq a_0$, then either (α) $a'_0 = 2$ or (β) $a'_0 = a'_1$, or (γ) $a'_0 = a'_2$. In case (α), we have $\{2, a'_1, a'_2\} = \{2, a_0, a'_1, a'_2\}$ and hence either $a_0 = a'_1 = a_1$ or $a_0 = a'_2 = u_2$. In both cases, we have

$a_1' = a_2'$ by Lemma 3.2. Thus,

$$a_0 = a_1 = \frac{a_1 a_2}{(a_1 - 1, a_2)} = a_1' = a_2' \neq a_0' = 2.$$

This contradicts Theorem 2.3 (ii) with $u = 2$. In case (β), we have $a_0' = a_1' = a_1 = a_2'$ by Lemma 3.2, since $a_0' \neq a_0$. Thus, $a_0' = a_1' = a_2' = a_1 = a_1 a_2/(a_1 - 1, a_2) \neq a_0$. This contradicts Theorem 2.3 (ii) with $u = a_0'$. In case (γ), a similar argument as case (β) will give a contradiction. Therefore, we see that $a_0' \neq a_0$ is impossible, i.e., we must have $a_0' = a_0$.

By Theorem 2.3 (ii), we have

$$(1 - w_0')(1 - w_1')(1 - w_2') = (1 - w_0)(1 - w_1)(1 - w_2)$$

which implies

$$(1 - a_0')(1 - a_1')(1 - a_2') = (1 - a_0')(1 - a_1')\left(1 - \frac{a_2'}{v_2}\right)$$

we conclude that $v_2 = 1$. Recall that $v_2 = a_1 - 1/(a_2, a_1 - 1)$. We have $(a_1 - 1)/a_2$. □

Theorem 3.8. *Suppose that* $f = z_0^{a_0'} + z_1^{a_1'} + z_2^{a_2'}$ *is of type I and* $g = z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2}$ *is of type III. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$\left\{ \begin{array}{l} f = z_0^{a_0} + z_1^{a_1+1} + z_2^{a_1+1} \\ g = z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_1} \end{array} \right\}.$$

Proof. Since $u_1 = u_2 = a_1 a_2 - 1/(a_1 - 1, a_2 - 1)$, we may assume that $a_1' = a_2' = u_1 = u_2$. It follows that $\gamma_1' = \gamma_2' = 1$. Since the minimal good resolution graphs for f and g are the same, by Lemma 3.4 and Table 1, we have

$$\{1, \gamma_0', \gamma_1' = 1, \gamma_2' = 1\} = \{1, \gamma_0, \gamma_0 v_2, \gamma_0 v_1\}.$$

There are two subcases.

Case 1. $\gamma_0 > 1$. It follows that $v_1 = v_2 = 1$ and $\gamma_0 = \gamma_0' > 1$. As $v_1 = a_2 - 1/(a_1 - 1, a_2 - 1)$ and $v_2 = a_1 - 1/(a_1 - 1, a_2 - 1)$, we see that $a_1 = a_2$ and $u_1 = u_2 = a_1 + 1 = a_1' = a_2'$. By Proposition 2.3 (ii), we have

$$(1 - w_0')(1 - w_1')(1 - w_2') = (1 - w_0)(1 - w_1)(1 - w_2)$$

which implies that

$$(1 - a_0')(1 - a_1')(1 - a_1') = (1 - a_0)(1 - a_1')(1 - a_1'),$$

i.e., $a_0' = a_0$. So f and g are in the form as required.

Case 2. $\gamma_0 = 1$. In this case, we have $\{\gamma_0', 1\} = \{1, v_1, v_2\}$. Then we have either (α) $v_1 = 1$, $v_2 > 1$ or (β) $v_2 = 1$, $v_1 > 1$ or (γ) $v_1 = v_2$. In the case (α), we have $v_2 = \gamma_0' > 1$. Hence, $n_0 = n_1'$ which implies $(a_1', a_2') = c'c_0' = n_0 = n_1' = 1$ (cf. Table 1). It follows that $a_1' = a_2' = 1$, which is impossible. In case (β), the same argument above will give a contradiction. In case (γ), since $v_1 = a_2 - 1/(a_1 - 1, a_2 - 1)$ and $v_2 = a_1 - 1/(a_1 - 1, a_2 - 1)$, we have $a_1 = a_2$ and hence $v_1 = v_2 = 1$. The result follows from the same argument as Case 1. □

Theorem 3.9. *Suppose that* $f = z_0^{a_0'} + z_1^{a_1'} + z_2^{a_2'}$ *is of type I and* $g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2}$ *is of type IV. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$\left\{ \begin{array}{l} f = z_0^{k+1} + z_1^{m(k+1)} + z_2^{tm(k+1)} \\ g = z_0^{k+1} + z_0 z_1^{mk} + z_1 z_2^{t(mk+m-1)} \end{array} \right\}.$$

Proof. By Table 2, we have u_0/u_1 and u_1/u_2. Therefore, we may assume that $a_0'/a_1', a_1'/a_2'$, $a_0 = u_0 = a_0'$, $u_1 = a_1'$ and $u_2 = a_2'$. By Theorem 2.3 (ii), we have

$$(1 - w_0')(1 - w_1')(1 - w_2') = (1 - w_0)(1 - w_1)(1 - w_2)$$

which implies

$$(1 - a_0')(1 - a_1')(1 - a_2') = (1 - a_0')\left(1 - \frac{a_1'}{v_1}\right)\left(1 - \frac{a_2'}{v_2}\right).$$

Clearly, this is possible only if $v_1 = v_2 = 1$, i.e., $a_0 - 1 = (a_0 - 1, a_1)$ and $a_0a_1 - a_0 + 1 = (a_1a_2, a_0a_1 - a_0 + 1)$. We infer from Table 2 that $a'_1 = u_1 = a_0a_1/(a_0 - 1, a_1) = a_0a_1/a_0 - 1$ and $a'_2 = u_2 = a_0a_1a_2/(a_1a_2, a_0a_1 - a_0 + 1) = a_0a_1a_2/(a_0a_1 - a_0 + 1)$. Let $a'_0 = k+1$, $a'_1 = m(k+1)$ and $a'_2 = tm(k+1)$. It follows that $a_0 = k+1$, $a_1 = mk$ and $a_2 = t(mk + m - 1)$. □

Theorem 3.10. *Suppose that* $f = z_0^{a'_0} + z_1^{a'_1} + z_2^{a'_2}$ *is of type I and* $g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2}$ *is of type V. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$\left\{ \begin{array}{l} f = z_0^{k+1} + z_1^{k+1} + z_2^{k+1} \\ g = z_0^k z_1 + z_1^k z_2 + z_0 z_2^k \end{array} \right\}.$$

Proof. By Table 2, we have $u_0 = u_1 = u_2$. It follows from Lemma 3.2 and Theorem 2.3 (ii) that $a'_0 = a'_1 = a'_2$. Since $\gamma'_0 = \gamma'_1 = \gamma'_2 = 1$, we see, by Lemma 3.4 and Table 1, that $v_0 = v_1 = v_2 = 1$. It follows that $a_1a_2 - a_2 = a_0a_2 - a_0 = a_0a_1 - a_1$ and, hence, $a_0 = a_1 = a_2 = k$. By Proposition 2.2, we have

$$\zeta_f(z) = (1 - z^{a'_0})^{a'^2_0 - 3a'_0 + 3}$$
$$\zeta_g(z) = (1 - z^{k+1})^{k^2 - k + 1}.$$

In order to have $\zeta_f(z) = \zeta_g(z)$, we must have $a'_0 = k+1$. □

Theorem 3.11. *Suppose that* $f = z_0^{a'_0} + z_1^{a'_1} + z_2^{a'_2}$ *is of type I and* $g = z_0^{a_0} + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$ *is of type VI. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$\left\{ \begin{array}{l} f = z_0^{k+1} + z_1^{m(k+1)} + z_2^{t(k+1)} \\ g = z_0^{k+1} + z_0 z_1^{mk} + z_0 z^{tk} 2 + z_1^{b_1} z_2^{b_2} \end{array} \right\}.$$

Proof. By Table 2, we see that u_0/u_1 and u_0/u_2. It follows that we may assume without loss of generality that a'_0/a'_1, a'_0/a'_2, $a'_0 = u_0$, $a'_1 = u_1$, $a'_2 = u_2$. By Theorem 2.3 (ii), we have

$$(1 - w_0)(1 - w_1)(1 - w_2) = (1 - w'_0)(1 - w'_1)(1 - w'_2)$$

which implies

$$\left(1-\frac{a'_0}{v_0}\right)\left(1-\frac{a'_1}{v_1}\right)\left(1-\frac{a'_2}{v_2}\right)=(1-a'_0)(1-a'_1)(1-a'_2).$$

It follows that $v_0 = v_1 = v_2 = 1$. By Table 2, we infer that $(a_0-1)/a_1$ and $(a_0-1)/a_2$. Therefore, $a'_0 = a_0$, $a'_1 = a_0a_1/(a_0-1)$ and $a'_2 = a_0a_2/(a_0-1)$ by Table 2 again. Set $a_0 = k+1$, $a_1 = mk$, $a_2 = tk$, then we have $a'_0 = k+1$, $a'_1 = m(k+1)$, $a'_2 = t(k+1)$. So f and g are in the form as required. □

Theorem 3.12. *Suppose that* $f = z_0^{a'_0} + z_1^{a'_1} + z_2^{a'_2}$ *is of type I and* $g = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$ *is of type VII. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{k+1} + z_2^{m(k+1)} \\ g = z_0^k z_1 + z_0 z_1^k + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}.$$

Proof. By Table 2, $u_0 = u_1 = a_0a_1 - 1/(a_0-1, a_1-1)$. We may assume, without loss of generality, that $a'_0 = a'_1 = u_0 = u_1$ and $a'_2 = u_2$. By Theorem 2.3 (ii), we have

$$(1-w'_0)(1-w'_1)(1-w'_2) = (1-w_0)(1-w_1)(1-w_2)$$

which implies that

$$(1-a'_0)(1-a'_1)(1-a'_2) = \left(1-\frac{a'_0}{v_0}\right)\left(1-\frac{a'_1}{v_1}\right)\left(1-\frac{a'_2}{v_2}\right).$$

It follows that $v_0 = v_1 = v_2 = 1$. The fact that $v_0 = v_1$ implies $a_0 = a_1$ by Table 2. Since $v_2 = 1$, it is easy to see that a_1/a_2 by Table 2 again. If we set $a_0 = a_1 = k$, then $a_2 = mk$, $a'_0 = a'_1 + k + 1$ and $a'_2 = m(k+1)$ (see Table 2). Hence, f and g are in the form as required. □

Theorem 3.13. *Suppose that* $f = z_0^{a'_0} + z_1^{a'_1} + z_2^{a'_2}z_1$ *is of type II and* $g = z_0^{a_0} + z_1^{a_1}z_2 + z_1z_2^{a_2}$ *is of type III. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

(1) $$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{k+1} + z_1 z_2^{mk} \\ g = z_0^{mk+1} + z_1^k z_2 + z_1 z_2^k \end{array}\right\}$$

or

$$(2)\qquad \left\{\begin{array}{l} f = z_0^{mk+1} + z_1^{mk+1} + z_1 z_2^k \\ g = z_0^{mk+1} + z_1^{m(k-1)+1} z_2 + z_1 z_2^k \end{array}\right\}$$

or

$$(3)\qquad \left\{\begin{array}{l} f = z_0^{a_0} + z_1^{mk+m+1} + z_1 z_2^{k+1} \\ g = z_0^{a_0} + z_1^{mk+1} z_2 + z_1 z_2^{k+1} \end{array}\right\}$$

or

$$(4)\qquad \left\{\begin{array}{l} f = z_0^{mk+m+1} + z_1^{mn+1} + z_1 z_2^{\frac{(mk+m+1)n}{mn+1}} \\ g = z_0^{mn+1} + z_1^{mk+1} z_2 + z_1 z_2^{k+1} \end{array}\right\}$$

with $(mn+1)/(m(k+1)+1)$.

Proof. By Table 2, we have $u_1 = u_2$. It follows by Lemma 3.2 that we have the following subcases.

(i) $u_0' = u_1' = u_1 = u_2$ and $u_2' = u_0$. This implies $\gamma_0' = 1$. By Theorem 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_0' = 1, \alpha_1' = v_2'\gamma_0' = v_2', \alpha_2' = \gamma_2'\} \\ = \{1, \alpha_0 = \gamma_0, \alpha_1 = v_2 g_0, \alpha_2 = v_1\gamma_0\}.$$

Case (α). If $\gamma_0 > 1$, then we have $v_1 = v_2 = 1$ because $(v_2', \gamma_2') = 1$. It follows from Table 2 that $a_1 = a_2$. Recalling that $u_0' = u_1' = u_1 = u_2$, we infer that $a_0' = a_1' + a_1 + 1 = a_2 + 1$ by Table 2. By Theorem 2.3 (ii), we have $1 - w_0 = 1 - w_2'$, i.e., $a_0 = a_1' a_2'/(a_1' - 1)$. Thus, $(a_1' - 1)/a_2'$. Set $a_0' = a_1' = k+1$. Then $a_1 = a_2 = k$, $a_2' = mk$ and $a_0 = m(k+1)$. So we are in case (1) asserted in the theorem.

Case (β). If $\gamma_0 = 1$, then $\gamma_2' = 1$ since $u_0 = u_2'$ and $u_0' = u_1' = u_1 = u_2$. This implies that u_2'/u_1'. From Table 2, we see that u_1'/u_2'. Thus, we have $u_2' = u_1' = u_0' = u_2 = u_1 = u_0$. By Table 2, we have

$$(*)\qquad a_0' = a_1' = \frac{a_1' a_2'}{(a_2', a_1' - 1)} = a_0 = \frac{a_1 a_2 - 1}{(a_1 - 1, a_2 - 1)}$$

By Lemma 3.4, we have $\{1, v_2'\} = \{1, v_2, v_1\}$. It follows that if neither v_2 nor v_1 is one, then $v_1 = v_2 = v_2'$. Let $\alpha = v_1$. Then $\sum_{\alpha_i=\alpha} n_i = 2$ while $\sum_{\alpha_i'=\alpha} n_i' = 1$ by Table 1. This contradicts the second part of the conclusion of Lemma 3.4. Therefore, we conclude that either $v_1 = 1$ or $v_2 = 1$. We shall first assume that $v_1 = 1$. Thus $v_2 = v_2'$ and $w_2 = w_2'$. From Table 2, we have $a_1'a_2'/(a_1' - 1) = (a_1a_2 - 1)/(a_1 - 1)$. Observe that $(*)$ implies $a_2'/(a_1' - 1)$. If we set $a_2' = k$, then we have $a_0' = a_1' = mk + 1 = a_0$. From Table 1, the fact that $v_2 = v_2'$ implies $(a_1' - 1)/a_2' = (a_1 - 1)/(a_1 - 1, a_2 - 1)$, i.e., $m = (a_1 - 1)/(a_1 - 1, a_2 - 1)$. However, $(a_2 - 1)/(a_1 - 1)$ because $v_1 = 1$. Thus $(a_1 - 1)/(a_2 - 1) = m$. Set $a_2 = t+1$. Then $a_1 = mt+1$ and $(a_1a_2 - 1)/(a_2 - 1) = mt+m+1 = a_0 = mk + 1$. Hence, $t = k - 1$, i.e., $a_2 = k$ and $a_1 = m(k - 1) + 1$. So we are in case 2 as asserted in the theorem.

(ii) $u_1' = u_2' = u_1 = u_2$. By Lemma 3.2, we have $u_0' = u_0$, i.e., $a_0' = a_0$. From Table 2, we have $a_2'/(a_1' - 1)$. Since $u_1' = u_2'$, we have $\gamma_2' = 1$. By Theorem 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_0', \alpha_1' = v_2'\gamma_0', \alpha_2' = \gamma_2' = 1\} = \{1, \gamma_0, v_2\gamma_0, v_1\gamma_0\}.$$

Note that from Table 2, we have $(v_1, v_2) = 1$. It follows that either $v_1 = 1$ or $v_2 = 1$.

We first assume that $v_1 = 1$, i.e., $(a_2 - 1)/(a_1 - 1)$. In this case, we have $w_1' = w_1$. Since $w_0' = a_0' = a_0 = w_0$, we infer that $w_2' = w_2$ by Theorem 2.3 (ii). If we set $a_2 = k + 1$, then $a_1 = mk + 1$, $a_1' = w_1' = w_1 = (a_1a_2 - 1)/(a_2 - 1) = mk + m + 1$. On the other hand, $a_1'a_2'/(a_1' - 1) = w_2' = w_2 = (a_1a_2 - 1)/(a_1 - 1)$ implies that $a_2' = k + 1$. Hence, we are in case (3) of the theorem.

We next assume that $v_2 = 1$, i.e., $(a_1 - 1)/(a_2 - 1)$. By the similar argument as above, we have $w_2' = w_2$ and $w_1' = w_1$. If we set $a_1 = k + 1$, then $a_2 = mk + 1$, $a_1' = w_1' = w_1 = (a_1a_2 - 1)/(a_2 - 1) = (mk + m + 1)/m = k + 1 + (1/m)$. This implies that $m = 1$ and hence $a_1 = k + 1 = a_2$, $a_1' = k + 2$. On the other hand, $a_1'a_2'/(a_1' - 1) = w_2' = w_2 = (a_1a_2 - 1)/(a_1 - 1)$ implies that $a_2' = k+1$. So we are in case (3) again.

(iii) $u_0' = u_2' = u_1 = u_2$. By Lemma 3.2, we have $u_1' = u_0$. From Table 1, we know that $u_1'/u_2' = u_0'$. As $u_0' = u_2'$, we have $\gamma_0' = \gamma_2' = 1$.

(α) If $\gamma_0 = 1$, then Lemma 3.4 implies

$$\{1, \alpha'_0 = 1, \alpha'_1 = v'_2, \alpha'_2 = 1\} = \{1, \alpha_0 = 1, \alpha_1 = v_2, \alpha_2 = v_1\}.$$

By the second part of Lemma 3.4, we have either $v_2 = 1$ or $v_1 = 1$.

Let us first assume that $v_1 = 1$, i.e., $(a_2 - 1)/(a_1 - 1)$. The fact that $u'_1 = u_0$ and $u_1 = u_2 = u'_0 = u'_2$ imply $a_0 = a'_1$, a_1/a_0 and $a'_0 = a'_1 a'_2/(a'_2, a'_1 - 1) = (a_1 a_2 - 1)/(a_2 - 1)$. Set $a_2 = k + 1$. Then $a_1 = mk + 1$ and $a'_0 = mk + m + 1$. By Table 2 and Theorem 2.3 (ii), we have

$$\begin{aligned}(1 - u'_0)(1 - u'_1)\left(1 - \frac{u'_2}{v'_2}\right) &= (1 - w'_0)(1 - w'_1)(1 - w'_2) \\ &= (1 - w_0)(1 - w_1)(1 - w_2) \\ &= (1 = u'_1)(1 - u'_0)\left(1 - \frac{u'_2}{v_2}\right).\end{aligned}$$

This implies that $v'_2 = v_2$ and hence $(a'_1 - 1)/(a'_2, a'_1 - 1) = (a_1 - 1)/(a_2 - 1) = m$. Set $(a'_2, a'_1 - 1) = n$. Then $a'_1 = mn + 1$ and $a'_2 = (mk + m + 1)n/(mn + 1)$. Hence, we are in case (4).

(β) If $\gamma_0 > 1$, then $v_1 = v_2 = 1$, since by Lemma 3.4

$$\{1, \alpha'_0 = 1, \alpha'_1 = v'_2, \alpha'_2 = 1\} = \{1, \alpha_0 = \gamma_0, \alpha_1 = v_2\gamma_0, \alpha_2 = v_1\gamma_0\}.$$

Taking $\alpha = \alpha_0 = \alpha_1 = \alpha_2$, we see from Table 1 that $\sum_{\alpha_i = \alpha} n_i > 1$ while $\sum_{\alpha'_i = \alpha} n'_i = 1$. This contradicts the second part of Lemma 3.4. $\square$

Theorem 3.14. *Suppose that* $f = z_0^{a'_0} + z_1^{a'_1} + z_1 z_2^{a'_2}$ *is of type II and* $g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2}$ *is of type IV. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if either*

$$\text{(1)} \qquad \left\{\begin{array}{l} f = z_0^{a_0} + z_1^{\frac{a_0 a_1}{a_0 - 1}} + z_1 z_2^{a_2} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$$

with $(a_0 - 1)|a_1$*; or*

$$\text{(2)} \qquad \left\{\begin{array}{l} f = z_0^{a_0} + z_1^{\frac{a_0 a_1 a_2}{a_0 a_1 - a_0 + 1}} + z_1 z_2^{\frac{a_0 a_1 a_2 - a_0 a_1 + a_0 - 1}{a_2(a_0 - 1)}} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$$

with $(a_0a_1 - a_0 + 1)/a_1a_2$ *and* $a_2(a_0 - 1)/(a_0a_1a_2 - a_0a_1 + a_0 - 1)$ *or*

$$\left\{\begin{array}{l} f = z_0^{\frac{a_0a_1a_2}{a_0a_1-a_0+1}} + z_1^{a_0} + z_1z_2^{a_1} \\ g = z_0^{a_0} + z_0z_1^{a_1} + z_1z_2^{a_2} \end{array}\right\} \tag{3}$$

with $(a_0a_1 - a_0 + 1)/a_1a_2$ *or*

$$\left\{\begin{array}{l} f = z_0^{\frac{a_0a_1}{a_0-1}} + z_1^{a_0} + z_1z_2^{\frac{a_1a_2(a_0-1)}{a_0a_1-a_0+1}} \\ g = z_0^{a_0} + z_0z_1^{a_1} + z_1z_2^{a_2} \end{array}\right\} \tag{4}$$

with $(a_0 - 1)/a_1$ *and* $(a_0a_1 - a_0 + 1)/a_1a_2(a_0 - 1)$.

Proof. From Table 2, we have u'_1/u'_2 and $u_0/u_1/u_2$. From Lemma 4.3, we have the following subcases.

(i) $u'_0/u'_1/u'_2$ with $u'_i = u_i$, $i = 0, 1, 2$.

(ii) $u'_1/u'_2/u'_0$ with $u'_1 = u_0$, $u'_2 = u_1$, $u'_0 = u_2$.

(iii) $u'_1/u'_0/u'_2$ with $u'_1 = u_0$, $u'_0 = u_1$, $u'_2 = u_2$.

Case (i). Since $u'_0/u'_1/u'_2$, we have $\gamma'_0 = 1$. We also have $\gamma'_i = \gamma_i$ as $u_i = u'_i$. Note $(v'_2, \gamma'_2) = 1$. We see that $(v'_2\gamma'_0, \gamma'_2) = 1$.

(a) $\gamma_2 > 1$. In this case, we have

$$\begin{aligned} \{1, \alpha'_0 = \gamma'_0 = 1, \alpha'_1 = v'_2\gamma'_0 = v'_2, \alpha'_2 = \gamma'_2\} \\ = \{1, \alpha_0 = \gamma_2, \alpha_1 = v_2, \alpha_2 = v_1\gamma_2\}. \end{aligned}$$

Since $\gamma'_2 = \gamma_2 > 1$, $(\alpha'_1, a'_2) = 1$ and $(\alpha_0, \alpha_2) = \gamma_2 > 1$, it follows that $v_1 = 1$ (i.e., $(a_0 - 1)/a_1$) and $v_2 = v'_2$. Since $u'_2 = u_2$, we have $w_2 = w'_2$, i.e., $a'_1a'_2/(a'_1 - 1) = a_1a_1a_2/(a_0a_1 - a_0 + 1)$. By Table 2, $u_0 = u'_0$, $u_1 = u'_1$ imply $a_0 = a'_0$, $a'_1 = a_0a_1/(a_0 - 1)$. This implies $a'_2 = a_2$. Thus, we are in case (1) of the theorem.

(b) $\gamma_2 = 1$. In this case, we have

$$\{1, \alpha'_0 = 1, \alpha'_1 = v'_2, \alpha'_2 = \gamma'_2 = \gamma_2 = 1\} = \{1, \alpha_0 = 1, \alpha_1 = v_2, \alpha_2 = v_1\}.$$

We claim that either $v_1 = 1$ or $v_2 = 1$. Suppose on the contrary that both v_1 and v_2 are not equal to one. Then necessarily $v'_2 = v_1 = v_2 > 1$.

Let α be $v_2' = v_1 = v_2$. Then $\sum_{\alpha_i=\alpha} n_i = 2$ while $\sum_{\alpha_i'=\alpha} n_i' = 1$ by Table 1. This contradicts the second part of Lemma 3.4. We conclude that we have either $v_1 = 1$ or $v_2 = 1$. Using this fact that $u_0'/u_1'/u_2'$ and $\gamma_0' = 1$, we see easily that $u_1' = u_2' = u_2 = u_1$.

If $v_1 = 1$, then $w_1' = w_1$ since $v_1' = 1$. Note also that $w_0' = w_0$ as $v_0' = v_0 = 1$. By Theorem 2.3 (ii), we have

$$(1 - w_0')(1 - w_1')(1 - w_2') = (1 - w_0)(1 - w_1)(1 - w_2).$$

It follows that $w_2 = w_2'$. The rest of the argument is the same as case (a). So we are in case (1) of the theorem.

If $v_2 = 1$ and $v_1 = v_2' > 1$, then we have $w_2' = w_1$ and $w_1' = w_2$ since $v_1' = 1$. It follows from Theorem 2.3 (ii) that $w_0' = w_0$. Thus, we have

$$\frac{a_1'a_2'}{a_1' - 1} = \frac{a_0a_1}{a_0 - 1}, \qquad a_1' = \frac{a_0a_1a_2}{a_0a_1 - a_0 + 1}, \qquad a_0' = a_0$$

with $(a_0a_1 - a_0 + 1)/a_1a_2$. It follows that $a_2' = (a_0a_1a_2 - a_0a_1 + a_0 - 1)/a_2(a_0 - 1)$ and we are in case (2) of the proposition.

Case (ii). Since $u_1'/u_2'/u_0'$, we have $\gamma_2' = 1$ and a_1'/a_0'. As $u_1' = u_0$, $u_2' = u_1$ and $u_0' = u_2$, we have $c = c'$, $c_0 = c_1'$ and $c_2 = c_2'$ and hence $\gamma_2 = \gamma_0'$.

(a) $\gamma_2 > 1$. In this case we have $\gamma_0' > 1$. By Lemma 3.4, we have

$$\{1, \alpha_0' = \gamma_0', \alpha_1' = v_2'\gamma_0', \alpha_2' = \gamma_2' = 1\} = \{1, \alpha_0 = \gamma_2, \alpha_1 = v_2, \alpha_2 = v_1\gamma_2\}.$$

Since $(v_2, \gamma_2) = 1$, we infer that $v_2 = 1$, $v_1 = v_2'$ and $\gamma_0' = \gamma_2 > 1$. It follows that $w_0' = w_2$ and $w_2' = w_1$. By Theorem 2.3 (ii), we have that $w_1' = w_0$. Thus, we have $a_1' = a_0$, $a_0' = a_0a_1a_2/(a_0a_1 - a_0 + 1)$ and $a_0a_1/(a_0 - 1) = a_1'a_2'/(a_1' - 1)$ by Table 2. It is easy to see that $a_2' = a_1$. Therefore, we are in case (3) of the theorem.

(b) $\gamma_2 = 1$. In this case we have $\gamma_0' = 1$. Since $u_0/u_1/u_2$, we have $\gamma_1 = 1$ and $cc_2 = (u_0, u_1) = u_0 = (u_0, u_2) = cc_1$. It follows that $u_1 = u_2 = u_0' = u_2'$. By Lemma 3.4, we have

$$\{1, \alpha_0' = 1, \alpha_1' = v_2', \alpha_2' = 1\} = \{1, \alpha_0 = 1, \alpha_1 = v_2, \alpha_2 = v_1\}.$$

We claim that either $v_1 = 1$ or $v_2 = 1$. Suppose on the contrary that both v_1 and v_2 are not equal to one. Then necessarily $v_2' = v_1 = v_2$. Let α be $v_2' = v_1 = v_2$. Then $\sum_{\alpha_i=\alpha} n_i = 2$ while $\sum_{\alpha_i'=\alpha} n_i' = 1$ by Table 1. This contradicts the second part of Lemma 4.4. We conclude that either $v_1 = 1$ or $v_2 = 1$. If $v_2 = 1$, then $v_1 = v_2'$. The rest of the argument is the same as in case (a) above. Therefore, we are in case (3) of the theorem.

If $v_2 > 1$ and $v_1 = 1$, then $v_2' = v_2 > 1$. Since $v_1 = 1$, we have $(a_0 - 1)/a_1$. It follows that $w_1 = w_0'$ and $w_2 = w_2'$. By Theorem 2.3 (ii), we also have $w_1' = w_0$. By Table 2, we infer that $a_0 = a_1'$, $a_0' = a_0a_1/(a_0-1)$ with $(a_0-1)/a_1$ and $a_1'a_2'/(a_1'-1) = a_0a_1a_2/(a_0a_1 - a_0 + 1)$. This implies that $a_2' = a_1a_2(a_0 - 1)/(a_0a_1 - a_0 + 1)$. Hence, we are in case (4) of the proposition.

Case (iii). Since $u_1'/u_0'/u_2'$, $u_1' = u_0$, $u_0' = u_1$ and $u_2' = u_2$, we have $\gamma_0' = 1$ and $\gamma_2 = \gamma_2'$.

(a) $\gamma_2 = \gamma_2' = 1$. In this case, since $c'c_0' = (u_1', u_2') = u_1' = (u_1', u_0') = c'c_2'$, we have $u_0' = u_2' = u_1 = u_2$. The rest of the argument is the same as case (ii) (b).

(b) $\gamma_2 = \gamma_2' > 1$. In this case, since u_0'/u_2', we have $u_2 = u_2' > u_0' = u_1$. From (ii) of Theorem 2.3, we have $1 - w_2 = 1 - w_2'$ and $(1 - w_0)(1 - w_1) = (1 - w_0')(1 - w_1')$. It follows that $v_2 = v_2'$ and $v_1 = 1$ as $v_0 = 1 = v_0' = v_1'$ by Table 2. Hence we have $w_0 = w_1'$, $w_1 = w_0'$ and $w_2 = w_2'$, i.e., $a_1' = a_0$, $a_0' = a_0a_1/(a_0 - 1)$ with $(a_0-1)/a_1$, and $a_1'a_2'/(a_1'-1) = a_0a_1a_2/(a_0a_1-a_0+1)$. It follows that $a_2' = a_1a_2(a_0 - 1)/(a_0a_1 - a_0 + 1)$. We are in case (4) of the theorem. □

Theorem 3.15. *Suppose that* $f = z_0^{a_0'} + z_1^{a_1'} + z_1z_2^{a_2'}$ *is of type II and* $g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2}$ *is of type V. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$\left\{\begin{array}{l} f = z_0^{kl+1} + z_1^{kl+1} + z_1z_2^{l} \\ g = z_0^{kl}z_1 + z_1^{k(l-1)+1}z_2 + z_0z_2^{l} \end{array}\right\}.$$

Proof. By Table 2, we know that $u_0 = u_1 = u_2$. Thus, we have

$u_0 = u_1 = u_2 = u'_0 = u'_1 = u'_2$ by Lemma 3.2. By Table 2, this implies that $a'_0 = a'_1$ and $a'_2/(a'_1 - 1)$. It follows that $\gamma'_0 = 1 = \gamma'_2$ and hence

$$\{1, \alpha'_0 = \gamma'_0 = 1, \alpha'_1 = v'_2\gamma'_0 = v'_2, \alpha'_2 = \gamma'_2 = 1\} \\ = \{1, \alpha_0 = v_0, \alpha_1 = v_1, \alpha_2 = v_2\}$$

by Table 1, Table 2 and Lemma 3.4. Since $n'_1 = 1 = n_0 = n_1 = n_2$, we may assume without loss of generality that $v_0 = v_1 = 1$ and $v_2 = v'_2$. The fact that $v_0 = v_1 = 1$ implies

$$a_1a_2 - a_2 + 1 = (a_0a_1a_2 + 1, a_1a_2 - a_2 + 1) = a_0a_2 - a_0 + 1,$$

i.e.,

$$(*) \qquad a_2(a_1 - 1) = a_0(a_2 - 1)$$

$(*)$ implies a_2/a_0. Set $a_0 = ka_2$ where k is a positive integer. Then $(*)$ implies $a_1 = 1 + k(a_2 - 1)$. Hence, $a'_0 = u'_0 = u_0 = (a_0a_1a_2 + 1)/(a_1a_2 - a_2 + 1) = ka_2 + 1$. By Proposition 2.2, we have

$$\zeta_f(z) = (1 - z^{\alpha'_0})^{2-\alpha'_0-a'_2+a'_0a'_2} = (1 - z^{a'_0})^{(a'_0-1)(a'_2-1)+1}$$
$$\zeta_g(z) = \left(1 - \frac{a_0a_1a_2 + 1}{z^{a_0a_2-a_0+1}}\right)^{a_0a_2-a_0+1}.$$

Since $\Delta_f(z) = \Delta_g(z)$, we have

$$(a'_0 - 1)(a'_2 - 1) + 1 = a_0(a_2 - 1) + 1$$

which implies $a_2 = a'_2$ because $a'_0 - 1 = ka_2 = a_0$. Let us denote $l = a_2 = a'_2$. Then $a'_1 = a'_0 = kl + 1$, $a_0 = kl$, $a_1 = k(l - 1) + 1$. Then f and g are in the form as required in the theorem. □

Theorem 3.16. *Suppose that $f = z_0^{a'_0} + z_1^{a'_1} + z_1z_2^{a'_2}$ is of type II and $g = z_0^{a_0} + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$ is of type IV. Then $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ have the same topological type if and only if*

$$(1) \qquad \left\{\begin{array}{l} f = z_0^{k+1} + z_1^{m(k+1)} + z_1z_2^{t(mk+m-1)} \\ g = z_0^{k+1} + z_0z_1^{mk} + z_0z_2^{tmk} + z_1^{b_1}z_2^{b_2} \end{array}\right\}$$

or

$$(2)\qquad \left\{\begin{aligned} f &= z_0^{k+1} + z_1^{m(k+1)} + z_1 z_2^{t(mk+m-1)} \\ g &= z_0^{k+1} + z_0 z_1^{tmk} + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{aligned}\right\}$$

or

$$(3)\qquad \left\{\begin{aligned} f &= z_0^{m(k+1)} + z_1^{k+1} + z_1 z_2^{a} \\ g &= z_0^{k+1} + z_0 z_1^{mk} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{aligned}\right\}$$

or

$$(4)\qquad \left\{\begin{aligned} f &= z_0^{m(k+1)} + z_1^{k+1} + z_1 z_2^{a} \\ g &= z_0^{k+1} + z_0 z_1^{a} + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{aligned}\right\}.$$

Proof. By Table 2, we have u_0/u_1, u_0/u_2 and u_1'/u_2'. By Lemma 3.2 and Lemma 3.3, it is easy to see that either (i) u_0'/u_1' or (ii) u_1'/u_0'.

Case (i). In this case, we have $u_0'/u_1'/u_2'$. By Lemma 3.3, we see that $u_0 = u_0'$, i.e., $a_0 = a_0'$. Moreover, we have either u_1/u_2 or u_2/u_1. We shall assume without loss of generality that u_1/u_2. This implies that $u_1' = u_1$ and $u_2' = u_2$.

(a) If $u_1' < u_2'$, then $u_1 = u_1' < u_2 = u_2'$. From Theorem 2.3 (ii), we see that $1-w_2 = 1-w_2'$. As $u_0 = u_0'$ and $v_0 = 1 = v_0'$, we have $w_0 = w_0'$. It follows from (ii) of Theorem 2.3 that $w_1 = w_1'$. Thus, we have $v_1 = v_1'$ since $u_1 = u_1'$. This implies $(a_0-1)/a_1$ and $a_1' = a_0a_1/(a_0-1)$. By Table 2, $w_2' = w_2$ implies $a_1'a_2'/(a_1'-1) = a_0a_2/(a_0-1)$, i.e.,

$$a_2' = \frac{a_0a_2}{a_0-1}\left(1-\frac{1}{a_1'}\right) = \frac{a_2(a_0a_1-a_0+1)}{a_1(a_0-1)}.$$

Observe that among the three integers a_1, a_0-1 and $a_0a_1-a_0+1$, any one of them can be expressed as an integral combination of the other two. We have $(a_0a_1-a_0+1, a_0-1) = (a_1, a_0-1) = (a_1, a_0a_1-a_0+1) = a_0-1$. It follows that a_1/a_2 as a_2' is an integer. Set $a_0 = k+1$. Then there exist integers m and t such that $a_1 = mk$ and $a_2 = tmk$. Thus, $a_0' = a_0 = k+1$, $a_1' = m(k+1)$ and $a_2' = t(mk+m-1)$. We are in the case (1) of the theorem.

(b) If $u_1' = u_2'$, then $u_1' = u_2' = u_1 = u_2$ and hence $\gamma_0' = \gamma_2' = \gamma_1 = \gamma_2 = 1$. It follows from Table 1 and Lemma 3.4 that

$$\{1, \alpha_0' = 1, \alpha_1' = v_2', \alpha_2' = 1\} = \{1, \alpha_0 = 1, \alpha_1 = 1, \alpha_2 = v_1, \alpha_3 = v_2\}.$$

Thus we have either $v_1 = 1$ or $v_2 = 1$. If $v_1 = 1$, then $v_1 = v_1' = 1$ and hence $w_1 = w_1'$. Since $w_0 = u_0 = u_0' = w_0'$, we conclude that $w_2 = w_2'$ by (ii) of Theorem 2.3. The rest of the argument is the same as in (a) above.

If $v_2 = 1$, then $(a_0 - 1)/a_2$ and $v_1 = v_2'$. It follows that $w_1 = w_2'$ and $w_1' = w_2$. A similar argument as in case (a) shows that we are in case (2) of the theorem.

Case (ii). In this case we have u_1'/u_0', u_1'/u_2', u_0/u_1 and u_0/u_2. By Lemma 4.5, we have $u_0 = u_1'$, i.e., $a_0 = a_1'$. Since $v_0 = v_1'$ by Table 2, we have $w_0 = w_1'$. We may assume that u_1' is not divisible by u_0'. Hence, we have $u_0' > u_1'$, i.e., $a_0' > a_1'$. By Lemma 3.6, we have either $u_0 = u_1'$, $u_1 = u_0'$, $u_2 = u_2'$ or $u_0 = u_1'$, $u_1 = u_2'$, $u_2 = u_0'$.

Suppose first that $u_0 = u_1'$, $u_1 = u_0'$, $u_2 = u_2'$. From $u_1 = u_0'$ and Table 2, we have $a_0' = a_0 a_1/(a_0 - 1, a_1)$. There are two further subcases.

Case (a). If $u_1 = u_0' \neq u_2' = u_2$, then by (ii) of Proposition 2.3 with $u = u_1 = u_0'$, we have $w_1 = w_0'$ and hence $v_1 = 1$, i.e., $(a_0 - 1)/a_1$, since $v_0' = 1$ by Table 2. As $w_0 = w_1'$ and $w_1 = w_0'$, we conclude that $w_2 = w_2'$ by (ii) of Theorem 2.3. Since $u_2 = u_2'$, it follows that $(a_1' - 1)/(a_2', a_1' - 1) = v_2' = v_2 = (a_0 - 1)/(a_2, a_0 - 1)$. Noting that $a_0 = a_1'$, we have $(a_2', a_1' - 1) = (a_2, a_0 - 1)$. From $a_1' a_2'/(a_2', a_1' - 1) = u_2' = u_2 = a_0 a_2/(a_2, a_0 - 1)$, we have $a_2 = a_2'$. Thus we conclude that $a_0' = a_0 a_1/(a_0 - 1)$ with $(a_0 - 1)/a_1$, $a_1' = a_0$ and $a_2' = a_2$. Set $a_0 = k + 1$, $a_2 = a = a_2'$. Then $a_1 = mk$, $a_0' = m(k + 1)$, $a_1' = k + 1$. So we are in case (3) of the theorem.

Case (b). If $u_1 = u_0' = u_2' = u_2$, then $\gamma_0' = \gamma_2' = 1 = \gamma_1 = \gamma_2$. From Table 1 and Lemma 3.4, we have

$$\begin{aligned}&\{1, \alpha_0' = \gamma_0' = 1, \alpha_1' = v_2'\gamma_0' = v_2', \alpha_2' = \gamma_2' = 1\}\\ &\quad = \{1, \alpha_0 = \gamma_1 = 1, \alpha_1 = \gamma_2 = 1, \alpha_2 = v_1\gamma_2 = v_1, \alpha_3 = v_2\gamma_1 = v_2\}.\end{aligned}$$

Since $n_2 = n_3 = 1$ and $n_1' = 1$ by Table 1, we have either $v_1 = 1$, $v_2 = v_2'$ or $v_2 = 1$, $v_1 = v_2'$. In the former case, we have $w_1 = w_0'$ and

$w_2 = w_2'$. The same argument as in Case (a) above shows that we are in case (3) of the proposition. In the latter case, we have $w_0' = w_2$ and $w_1 = w_2'$. The same argument as in Case (a) above shows that we are in case (4) of the proposition.

We next suppose that $u_0 = u_1'$, $u_1 = u_2'$, $u_2 = u_0'$. A similar argument as above shows that we are either in case (3) or case (4) of the theorem. □

Theorem 3.17. *Suppose that* $f = z_0^{a_0'} + z_1^{a_1'} + z_1 z_2^{a_2'}$ *is of type II and* $g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$ *is of type VII. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$\text{(1)}\qquad \left\{\begin{array}{l} f = z_0^{a_0+1} + z_1^{a_0+1} + z_1 z_2^{a_2} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_0} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

or

$$\text{(2)}\qquad \left\{\begin{array}{l} f = z_0^{n(mk+m+1)} + z_1^{mk+m+1} + z_1 z_2^{k+1} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{n(mk+1)} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

or

$$\text{(3)}\qquad \left\{\begin{array}{l} f = z_0^{n(mk+m+1)} + z_1^{mk+m+1} + z_1 z_2^{k+1} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{k+1} + z_0 z_2^{nm(k+1)} + z_1^{b_1} z_2^{b_2} \end{array}\right\}.$$

Proof. By Table 2, we have $u_0 = u_1$, u_1/u_2 and u_1'/u_2'. There are four subcases to be considered.

Case (a). $u_1' < u_2'$. By Lemma 3.2 and Lemma 3.3, we have $u_0 = u_0' = u_1 = u_1' < u_2' = u_2$. Then from (ii) of Theorem 2.3, we have $(1 - u_0')(1 - u_1') = (1 - (u_0'/v_0))(1 - (u_1'/v_1))$ and $w_2 = w_2'$ as $v_0' = v_1' = 1$ by Table 1. It follows that $v_0 = v_1 = 1$ which implies that $a_0 = a_1$ and hence $u_0 = u_1 = a_0 + 1 = u_0' = u_1'$ by Table 2. On the other hand, $w_2' = w_2$ implies $a_1' a_2'/(a_1' - 1) = (a_2(a_0 a_1 - 1))/(a_1(a_0 - 1))$ and hence $a_2' = a_2$. We are in case (1) of the theorem.

Case (b). $u_1' = u_2' > u_0'$. This case cannot occur by Lemma 3.2 and Lemma 3.3.

Case (c). $u_1' = u_2' < u_0'$. By Lemma 3.2 and Lemma 3.3, we have $u_1' = u_2'/u_0'$ and $u_0 = u_1 = u_1' = u_2'$, $u_2 = u_0'$. By (ii) of Theorem 2.3 with $u = u_2 = u_0'$, we have $w_2 = w_0'$ and hence $v_2 = 1$ since $v_0' = 1$. It follows that w_2 is an integer. On the other hand, since $(a_1, a_0a_1 - 1) = 1$, we have a_1/a_2. As $u_1' = u_2' = u_0 = u_1$, we have $\gamma_2' = 1$. By Lemma 3.4, we have

$$\{1, \alpha_0' = \gamma_0', \alpha_1' = v_2'\gamma_0', \alpha_2' = \gamma_2' = 1\}$$
$$= \{1, \alpha_0 = \gamma_2, \alpha_1 = v_2 = 1, \alpha_2 = v_1\gamma_2, \alpha_3 = v_0\gamma_2\}.$$

Since $(v_0, v_1) = 1$, we see that we have either $v_0 = 1$ or $v_1 = 1$.

If $v_1 = 1$, then $(a_0 - 1)/(a_1 - 1)$ and $w_1 = w_1'$ since $v_1' = 1$. As $w_1 = w_1'$ and $w_2 = w_0'$, we have $w_0 = w_2'$ by (ii) of Proposition 2.3. Set $a_0 = k + 1$. Then we have $a_1 = mk + 1$ and $a_2 = n(mk + 1)$. From the fact that $w_1 = w_1'$, $w_0' = w_2$, $w_0 = w_2'$ and Table 2, we infer that $a_0' = n(mk + m + 1)$, $a_1' = mk + m + 1$, and $a_2' = k + 1$. So we are in case (2) of the theorem.

If $v_0 = 1$, then $(a_1 - 1)/(a_0 - 1)$ and $w_0 = w_1'$ since $v_1' = 1$. As $w_0 = w_1'$ and $w_2 = w_0'$, we have $w_1 = w_2'$ by (ii) of Theorem 2.3. Set $a_1 = k + 1$. Then we have $a_0 = mk + 1$ and $a_2 = nm(k + 1)$. From the fact that $w_0 = w_1'$, $w_1 = w_2'$, $w_2 = w_0'$ and Table 2, we infer that $a_0' = n(mk + m + 1)$, $a_1' = mk + m + 1$ and $a_2' = k + 1$. So we are in case (3) of the theorem.

Case (d). $u_0' = u_1' = u_2' = u_0 = u_1 = u_2$. In this case we have $\gamma_0' = \gamma_2' = \gamma_2 = 1$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_0' = 1, \alpha_1' = v_2'\gamma_0' = v_2', \alpha_2' = \gamma_2' = 1\}$$
$$= \{1, \alpha_0 = \gamma_2 = 1, \alpha_1 = v_2, \alpha_2 = v_2\gamma_2 = v_1, \alpha_3 = v_0\gamma_2 = v_0\}.$$

By the second part of Lemma 3.4 and Table 1, we conclude that two of v_0, v_1 and v_2 must be one since $n_1' = 1 = n_1 = n_2 = n_3$. If $v_2 = 1$, then a similar argument as in case (c) above will lead to case (2) or case (3) of the theorem. If $v_2 > 1$, then $v_0 = v_1 = 1$. This implies that $a_0 = a_1$, $w_0 = w_1 = w_0' = w_1'$. By (ii) of Theorem 2.3, we have $w_2 = w_2'$. The same argument as in case (a) above will lead to case (1) of the theorem. □

Theorem 3.18. *Suppose that* $f = z_0^{a_0'} + z_1^{a_1'}z_2 + z_1z_2^{a_2'}$ *is of type III and* $g = z_0^{a_0} + z_0z_1^{a_1} + z_1z_2^{a_2}$ *is of type IV. Then* $(f^{-1}(0), 0)$ *and*

$(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$(1)\qquad \left\{ \begin{array}{l} f = z_0^{n(mk+1)} + z_1^{mk-m+1} z_2 + z_1 z_2^k \\ g = z_0^{mk+1} + z_0 z_1^k + z_1 z_2^{n(mk-m+1)} \end{array} \right\}$$

or

$$(2)\qquad \left\{ \begin{array}{l} f = z_0^{n(mk+1)} + z_1^k z_2 + z_1 z_2^{mk-m+1} \\ g = z_0^{mk+1} + z_0 z_1^k + z_1 z_2^{n(mk-m+1)} \end{array} \right\}$$

or

$$(3)\qquad \left\{ \begin{array}{l} f = z_0^{a_0} + z_1^{\frac{a_0 a_1 a_2 - a_0 a_1 + a_0 - 1}{a_2(a_0-1)}} z_2 + z_1 z_2^{a_2} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array} \right\}$$

or

$$(4)\qquad \left\{ \begin{array}{l} f = z_0^{a_0} + z_1^{a_2} z_2 + z_1 z_2^{\frac{a_0 a_1 a_2 - a_0 a_1 + a_0 - 1}{a_2(a_0-1)}} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array} \right\}.$$

Proof. By Table 2, we have $u_1' = u_2'$ and $u_0/u_1/u_2$. By Lemma 3.2 and Lemma 3.3, we have two subcases.

Case (i). $u_1 = u_0 = u_1' = u_2'/u_0' = u_2$ and $u_0' > u_1' = u_2'$. In this case we have $1 - w_0' = 1 - w_2$ by (ii) of Proposition 2.3 with $u = u_0' = u_2$. It follows that $v_2 = v_0' = 1$ which implies that $(a_0 a_1 - a_0 + 1)/a_1 a_2$. From $u_0 = u_1 = u_1' = u_2'$, and $u_0' = u_2$, we have

$$(*)\qquad a_0 = \frac{a_0 a_1}{(a_1, a_0 - 1)} = \frac{a_1' a_2' - 1}{(a_1' - 1, a_2' - 1)}.$$

$(*)$ implies that $a_1/(a_0 - 1)$. Set $a_1 = k$. Then we have $a_0 = mk + 1$, $a_2 = n(mk - m + 1)$ and $a_0' = n(mk + 1)$. Note that $\gamma_0' = \gamma_2 > 1$, by our assumption, $u_0 = u_1 = u_1' = u_2'$, $v_0' = v_2$ and $u_2' > u_1'$. From Lemma 4.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_0', \alpha_1' = v_2' \gamma_0', \alpha_2' = v_1' \gamma_0'\} \\ = \{1, \alpha_0 = \gamma_2, \alpha_1 = v_2 = 1, \alpha_2 = v_1 \gamma_2\}.$$

Thus, one of v_1' or v_2' must be one.

If $v_1' = 1$, then we see that $w_1' = w_0$. Notice that $w_0' = w_2$. By Theorem 2.3 (ii), we have

$$(1 - w_0)(1 - w_1)(1 - w_2) = (1 - w_0')(1 - w_1')(1 - w_2').$$

Thus, we have $w_0' = w_2$, $w_1' = w_0$, $w_2' = w_1$. From $w_1' = w_0$ and $w_2' = w_1$, we have $(a_1'a_2' - 1)/(a_1' - 1) = a_0a_1/(a_0 - 1)$ and $a_1'a_2'/(a_2' - 1) = a_0$. From these we see that $a_1' = (a_0a_1 - a_0 + 1)/a_1 = mk - m + 1$ and $a_2' = a_1 = k$. So we are in case (1) of the theorem.

If $v_2' = 1$, then $w_0' = w_2$, $w_1' = w_1$, $w_2' = w_0$. A similar argument as above leads to case (2) of the proposition.

Case (ii). $u_0 = u_0'/u_1 = u_2 = u_1' = u_2'$. In this case we have $a_0' = a_0$, $\gamma_0' = 1 = \gamma_2$ and $a_0a_1/(a_1, a_0 - 1) = (a_1'a_2' - 1)/(a_1' - 1, a_2' - 1)$. From Lemma 3.4 and Table 1, we have

$$\begin{aligned}\{1, \alpha_0 = \gamma_2 = 1, \alpha_1 = v_2, \alpha_2 = v_1\gamma_2 = v_1\} \\ = \{1, \alpha_0' = \gamma_0' = 1, \alpha_1' = v_2'\gamma_0' = v_2', \alpha_2' = v_1'\gamma_0' = v_1'\}.\end{aligned}$$

So we have either $v_1' = v_1$, $v_2' = v_2$ or $v_1' = v_2$, $v_2' = v_1$.

If $v_1' = v_1$, $v_2' = v_2$, we have $w_1' = w_1$, $w_2' = w_2$. Thus,

$$\frac{a_1'a_2' - 1}{a_2' - 1} = \frac{a_0a_1}{a_0 - 1}, \qquad \frac{a_1'a_2' - 1}{a_1' - 1} = \frac{a_0a_1a2}{a_0a_1 - a_0 + 1}.$$

From this, we have

$$a_1' = \frac{a_0a_1a_2 - a_0a_1 + a_0 - 1}{a_2(a_0 - 1)}, \qquad a_2' = a_2.$$

So we are in case (3) of the theorem.

If $v_1' = v_2$, $v_2' = v_1$, we have $w_1' = w_2$, $w_2' = w_1$. Thus,

$$\frac{a_1'a_2' - 1}{a_1' - 1} = \frac{a_0a_1}{a_0 - 1}, \qquad \frac{a_1'a_2' - 1}{a_2' - 1} = \frac{a_0a_1a_2}{a_0a_1 - a_0 + 1}.$$

This implies that

$$a_1' = a_2, \qquad a_2' = \frac{a_0a_1a_2 - a_0a_1 + a_0 - 1}{a_2(a_0 - 1)}.$$

So we are in case (4) of the theorem. □

Theorem 3.19. *Suppose that* $f = z_0^{a'_0} + z_1^{a'_1}z_2 + z_1z_2^{a'_2}$ *is of type III and* $g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2}$ *is of type V. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if either*

$$(1) \qquad \left\{\begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_1a_2-a_2+1}} + z_2z_1^{\frac{a_0(a_1a_2-a_2+1)}{a_0a_1-a_1+1}} + z_1z_2^{a_1} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array}\right\}$$

or

$$(2) \qquad \left\{\begin{array}{l} f = z^{\frac{a_0a_1a_2+1}{a_1a_2-a_2+1}} + z_1^{a_1}z_2 + z_1z_2^{\frac{a_0(a_1a_2-a_2+1)}{a_0a_1-a_1+1}} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array}\right\}$$

or

$$(3) \qquad \left\{\begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_0a_2-a_0+1}} + z_2z_1^{\frac{a_1(a_0a_2-a_0+1)}{a_1a_2-a_2+1}} + z_1z_2^{a_2} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array}\right\}$$

or

$$(4) \qquad \left\{\begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_0a_2-a_0+1}} + z_1^{a_2}z_2 + z_1z_2^{\frac{a_1(a_0a_2-a_0+1)}{a_1a_2-a_2+1}} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array}\right\}$$

or

$$(5) \qquad \left\{\begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_0a_1-a_1+1}} + z_1^{a_0}z_2 + z_1z_2^{\frac{a_2(a_0a_1-a_1+1)}{a_0a_2-a_0+1}} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array}\right\}$$

or

$$(6) \qquad \left\{\begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_0a_1-a_1+1}} + z_2z_1^{\frac{a_2(a_0a_1-a_1+1)}{a_0a_2-a_0+1}} + z_1z_2^{a_0} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array}\right\}.$$

Proof. By Table 2, we have $u_0 = u_1 = u_2$. So by Lemma 3.6, we have $u'_0 = u'_1 = u'_2 = u_0 = u_1 = u_2$. Thus, $\gamma'_0 = 1$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha'_0 = 1, \alpha'_1 = v'_2, \alpha'_2 = v'_1\} = \{1, \alpha_0 = v_0, \alpha_1 = v_1, \alpha_2 = v_2\}.$$

There are six subcases.

Case (1). $v_0 = 1$, $v_1 = v_2'$, $v_2 = v_1'$. In this case we have $w_0' = w_0$, $w_1' = w_2$, $w_2' = w_1$ and $(a_1a_2 - a_2 + 1)/(a_0a_1a_2 + 1)$, i.e., $a_0' = (a_0a_1a_2 + 1)/(a_1a_2 - a_2 + 1)$, and

$$\frac{a_1'a_2' - 1}{a_2' - 1} = \frac{a_0a_1a_2 + 1}{a_0a_1 - a_1 + 1}, \qquad \frac{a_1'a_2' - 1}{a_1' - 1} = \frac{a_0a_1a_2 + 1}{a_0a_2 - a_0 + 1}.$$

This implies that

$$a_1' = \frac{a_0a_1a_2 - a_0a_2 + a_0}{a_0a_1 - a_0 + 1} = \frac{a_0(a_1a_2 - a_2 + 1)}{a_0a_1 - a_1 + 1}, \qquad a_2' = a_1.$$

So we are in case (1) of the theorem.

Case (2). $v_0 = 1$, $v_1 = v_1'$, $v_2 = v_2'$. In this case we have $w_0' = w_0$, $w_1' = w_1$, $w_2' = w_2$. A similar argument leads to

$$a_0' = \frac{a_0a_1a_2 + 1}{a_1a_2 - a_2 + 1}, \qquad a_1' = a_1, \qquad a_2' = \frac{a_0(a_1a_2 - a_2 + 1)}{a_0a_1 - a_1 + 1}.$$

So we are in case (2) of the theorem.

Case (3). $v_1 = 1$, $v_0 = v_1'$, $v_2 = v_2'$. In this case we have $w_0' = w_1$, $w_1' = w_0$, $w_2' = w_2$, and $(a_0a_2 - a_0 + 1)/(a_0a_1a_2 + 1)$, i.e.,

$$a_0' = \frac{a_0a_1a_2 + 1}{a_0a_2 - a_0 + 1}, \qquad \frac{a_1'a_2' - 1}{a_2' - 1} = \frac{a_0a_1a_2 + 1}{a_1a_2 - a_2 + 1},$$
$$\frac{a_1'a_2' - 1}{a_1' - 1} = \frac{a_0a_1a_2 + 1}{a_0a_1 - a_1 + 1}.$$

This implies

$$a_1' = \frac{a_1(a_0a_2 - a_0 + 1)}{a_1a_2 - a_2 + 1}, \qquad a_2' = a_2.$$

So we are in case (3).

Case (4). $v_1 = 1$, $v_0 = v_2'$, $v_2 = v_1'$. In this case we have $w_0' = w_1$, $w_1' = w_2$, $w_2' = w_0$ and $(a_0a_2 - a_0 + 1)/(a_0a_1a_2 + 1)$. A similar argument leads to $a_1' = a_2$, $a_2' = a_1(a_0a_2 - a_0 + 1)/(a_1a_2 - a_2 + 1)$. So we are in case (4).

Case (5). $v_2 = 1$, $v_0 = v_1'$, $v_1 = v_2'$, i.e., $a_0' = (a_0a_1a_2 + 1)/(a_0a_1 - a_1 + 1)$. In this case we have $a_0' = (a_0a_1a_2 + 1)/(a_0a_1 - a_1 + 1)$

$$\frac{a_1'a_2' - 1}{a_2' - 1} = \frac{a_0a_1a_2 + 1}{a_1a_2 - a_2 + 1}, \qquad \frac{a_1'a_2' - 1}{a' - 1} = \frac{a_0a_1a_2 + 1}{a_0a_2 - a_0 + 1}.$$

This implies

$$a_1' = a_0, \qquad a_2' = \frac{a_2(a_0a_1 - a_1 + 1)}{a_0a_2 - a_0 + 1}.$$

So we are in case (5).

Case (6). $v_2 = 1$, $v_0 = v_2'$, $v_1 = v_1'$. In this case we have $w_0' = w_2$, $w_1' = w_1$, $w_2' = w_0$. A similar argument leads to

$$a_0' = \frac{a_0a_1a_2 + 1}{a_0a_1 - a_1 + 1}, \qquad a_1' = \frac{a_2(a_0a_1 - a_1 + 1)}{a_0a_2 - a_0 + 1}, \qquad a_2' = a_0.$$

So we are in case (6). □

Theorem 3.20. *Suppose that* $f = z_0^{a_0'} + z_1^{a_1'}z_2 + z_1z_2^{a_2'}$ *is of type III and* $g + z_0^{a_0} + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$ *is of type VI. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if either*

$$(1) \qquad \left\{ \begin{matrix} f = z_0^{k+1} + z_1^{mk+m-1}z_2 + z_1z_2^{mk+m-1} \\ g = z_0^{k+1} + z_0z_1^{mk} + z_0z_2^{mk} + z_1^{b_1}z_2^{b_2} \end{matrix} \right\}$$

or

$$(2) \qquad \left\{ \begin{matrix} f = z_0^{m(k+1)} + z_1^{k}z_2 + z_1z_2^{k} \\ g = z_0^{m(k+1)} + z_0z_1^{k} + z_0z_2^{mk} + z_1^{b_1}z_2^{b_2} \end{matrix} \right\}.$$

Proof. From Table 2, we have $u_1' = u_2'$, u_0/u_1 and u_0/u_2. By Lemma 3.2, we have the following subcases.

Case (1). $u'_1 = u'_2 = u_1 = u_2$ and $u_0 = u'_0$. Since $u'_0 = u_0/u_1 = u_2 = u'_1 = u'_2$, we have $\gamma'_0 = 1 = \gamma_1 = \gamma_2$ and $w'_0 = w_0$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha'_0 = 1, \alpha'_1 = v'_2, \alpha'_2 = v'_1\} = \{1, \alpha_0 = 1, \alpha_1 = 1, \alpha_2 = v_1, \alpha_3 = v_2\}.$$

Therefore, we have either $v'_1 = v_1$, $v'_2 = v_2$ or $v'_1 = v_2$, $v'_2 = v_1$.

If $v'_1 = v_2$, $v'_2 = v_1$, then $w_1 = w'_2$, $w_2 = w'_1$. It follows that

$$(*) \qquad a'_1 = \frac{a_2(a_0a_1 - a_0 + 1)}{a_1(a_0 - 1)}$$

$$(**) \qquad a'_2 = \frac{a_1(a_0a_2 - a_0 + 1)}{a_2(a_0 - 1)}.$$

Since $(a_1, a_0 - 1) = (a_1, a_0a_1 - a_0 + 1) = (a_0a_1 - a_0 + 1, a_0 - 1)$, from $(*)$ we see that a_1/a_2, $(a_0 - 1)/a_2$. Similarly, from $(**)$, we see that a_2/a_1, $(a_0 - 1)/a_1$. Thus, we have $a_1 = a_2$. Set $a_0 = k + 1$. Then $a_1 = a_2 = mk$, $a'_0 = k + 1$ and $a'_1 = a'_2 = mk + m - 1$. So we are in case (1) of the theorem.

If $v'_1 = v_1$, $v'_2 = v_2$, then $w_1 = w'_1$, $w_2 = w'_2$. A similar argument shows that we are in case (1) again.

Case (2). $u'_1 = u'_2 = u_0 = u_1$, $u'_0 = u_2$. In this case, we have $u'_1 = u'_2 = u_0 = u_1/u'_0 = u_2$. We may assume that $u_2 > u_0 = u_1$. By (ii) of Theorem 2.3, we have $w_2 = w'_0$, and hence $v_2 = v'_0 = 1$. From Table 2, we have $w'_0 = a'_0 = a_0a_2/(a_0-1) = w_2$ and $a_1/(a_0-1)/a_2$, since $u_0 = u_1$ and $v_2 = 1$. As $u_0 = u'_1 = u'_2$, we have $a_0 = (a'_1a'_2 - 1)/(a'_1 - 1, a'_2 - 1)$. Since $u'_0 = u_2 > u_0 = u_1 = u'_1 = u'_2$, we infer that $\gamma_1 = 1$ and $\gamma'_0 = \gamma_2 > 1$. By Lemma 3.4 and Table 1, we have

$$\begin{aligned}\{1, \alpha'_0 = \gamma'_0, \alpha'_1 = v'_2\gamma'_0, \alpha_2 = v'_1\gamma'_0\}\\ = \{1, \alpha_0 = 1, \alpha_1 = \gamma_2, \alpha_2 = v_1\gamma_2, \alpha_3 = v_2\gamma_1 = 1\}.\end{aligned}$$

Since $\alpha'_0, \alpha'_1, \alpha'_2, \alpha_1, \alpha_2$ are bigger than one, hence $\{\alpha'_0, \alpha'_1, \alpha'_2\} = \{\alpha_1, \alpha_2\}$. We have three subcases.

(a) $\alpha'_0 = \alpha'_1$. In this case we have $v'_2 = 1$, i.e., $(a'_1 - 1)/(a'_2 - 1)$. From Table 1, we have $u'_0 = (c'c'_0 - v'_1 - v'_2)/v'_1v'_2 = a'_1 - 1$, $n'_1 = n'_2 = 1$, $n_1 = (c-1)/v_2 = c - 1 = a_0 - 1$, $n_2 = 1$. By the second part of Lemma

3.4, we have $n'_0 + n'_1 + n'_2 = n_1 + n_2$, which implies $a'_1 = a_0 - 1$. Note $a_0 = u_0 = u'_1 = (a'_1 a'_2 - 1)/(a'_1 - 1)$. It follows that $a'^2_1 - 1 = a'_1 a'_2 - 1$, i.e., $a'_1 = a'_2 = a_0 - 1$. From Table 2, we have $v'_1 = v'_2 = 1$. By (ii) of Theorem 2.3, we have $(1 - w_0)(1 - w_1) = (1 - w'_1)(1 - w'_2)$. As $u_0 = u_1 = u'_1 = u'_2$, $v_0 = v'_1 = v'_2 = 1$, we infer that $v_1 = 1$, i.e., $(a_0 - 1)/a_1$. Since we have shown $a_1/(a_0 - 1)$, we conclude that $a_1 = a_0 - 1$. Set $a_0 = k + 1$. Then $a_1 = a'_1 = a'_2 = k$, $a_2 = mk$ and $a'_0 = m(k + 1)$. So we are in case (2) of the proposition.

(b) $\alpha'_0 = \alpha'_2$. The same argument as in case (a) shows that we are in case (2) again.

(c) $\alpha'_1 = \alpha'_2$. In this case we have $v'_1 = v'_2$ which implies that $a'_1 = a'_2$, and hence $v'_1 = v'_2 = 1$ from Table 2. It follows that $\alpha'_0 = \alpha'_1 = \alpha'_2$. So we are in case (2) again.

Case (3). $u'_0 = u'_2 = u_0 = u_2$ and $u_1 = u'_0$. The same argument as in case (2) will lead to case (2) of the theorem. □

Theorem 3.21. *Suppose that* $f = z_0^{a'_0} + z_1^{a'_1} z_2 + z_1 z_2^{a'_2}$ *is of type III and* $g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$ *is of type VII. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$(1) \qquad \left\{ \begin{array}{l} f = z_0^{t(mnk+m+n)} + z_1^{nk+1} z_2 + z_1 z_2^{mk+1} \\ g = z_0^{nk+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{n(mk+1)} + z_1^{b_1} z_2^{b_2} \end{array} \right\}$$

with $(m, n) = 1$, *or*

$$(2) \qquad \left\{ \begin{array}{l} f = z_0^{t(mnk+m+n)} + z_1^{nk+1} z_2 + z_1 z_2^{nk+1} \\ g = z_0^{nk+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{tn(mk+1)} + z_1^{b_1} z_2^{b_2} \end{array} \right\}$$

with $(m, n) = 1$, *or*

$$(3) \qquad \left\{ \begin{array}{l} f = z_0^{k+2} + z_2 z_1^{k+2-\frac{k+1}{\alpha}} + z_1 z_2^{a} \\ g = z_0^{k+1} z_1 + z_0 z_1^{k+1} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array} \right\}$$

with $a/(k + 1)$, *or*

$$(4) \qquad \left\{ \begin{array}{l} f = z_0^{k+2} + z_1 z_2^{k+2-\frac{k+1}{a}} + z_1^{a} z_2 \\ g = z_0^{k+1} z_1 + z_0 z_1^{k+1} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array} \right\}$$

with $a/(k+1)$, *or*

$$(5)\qquad \left\{\begin{array}{l} f = z_0^{mk+m+1} + z_1^{s(tm+1)+1} z_2 + z_1 z_2^{sm+1} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{sm+1} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

with $k = s(tm+1)+t$, *or*

$$(6)\qquad \left\{\begin{array}{l} f = z_0^{mk+m+1} + z_2^{s(tm+1)+1} z_1 + z_2 z_1^{sm+1} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{sm+1} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

with $k = s(tm+1)+t$.

Proof. By Table 2, we have $u_0 = u_1/u_2$ and $u_1' = u_2'$. Then, by Lemma 3.2 and Lemma 3.3, we have the following two subcases.

(i). $u_0 = u_1 = u_1' = u_2'/u_2 = u_0'$, and $u_2 = u_0' > u_0 = u_1 = u_1' = u_2'$. By (ii) of Theorem 2.3, we have $w_2 = w_0'$ and hence $v_2 = 1$ since $v_0' = 1$. This implies $a_1(a_0-1)/a_2(a_0a_1-1)$ and hence $(a_1(a_0-1)/(a_1-1, a_0-1))/a_2$. From Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_0', \alpha_1' = v_2'\gamma_0', \alpha_2' = v_1'\gamma_0'\} \\ = \{1, \alpha_0 = \gamma_2, \alpha_1 = v_2 = 1, \alpha_2 = v_1\gamma_2, \alpha_3 = v_0\gamma_2\}.$$

From $u_0' = u_2$, we have $\gamma_0' = \gamma_2$. By the second part of Lemma 3.4, one sees that either $v_1' = v_0$, $v_2' = v_1$ or $v_1' = v_1$, $v_2' = v_0$.

If $v_1' = v_0$, $v_2' = v_1$, we have $w_0' = w_2$, $w_1' = w_0$, $w_2' = w_1$, i.e., $a_0' = a_2(a_0a_1-1)/a_1(a_0-1), (a_1'a_2'-1)/(a_2'-1) = (a_0a_1-1)/(a_1-1)$ and $(a_1'a_2'-1)/(a_1'-1) = (a_0a_1-1)/(a_0-1)$. This implies that $a_1' = a_0$, $a_2' = a_1$. Set $(a_0-1, a_1-1) = (a_1'-1, a_2'-1) = k$. Then $a_1' = a_0 = nk+1$, $a_2' = a_1 = mk+1$ with $(m,n) = 1$, and $a_2 = tn(mk+1)$, $a_0' = t(mnk+m+n)$. So we are in case (1) of the proposition.

If $v_1' = v_1$, $v_2' = v_0$, we have $w_0' = w_2$, $w_1' = w_1$, $w_2' = w_0$. A similar argument leads to case (2) of the theorem.

(ii). $u_0 = u_1 = u_1' = u_2' = u_2 = u_0'$. It follows that $\gamma_0' = \gamma_2 = 1$ and $a_2/a_1((a_0-1)/(a_0-1, a_1-1))$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_0' = 1, \alpha_1' = v_2'\gamma_0' = v_2', \alpha_2' = v_1'\gamma_0' = v_1'\} \\ = \{1, \alpha_0 = \gamma_2 = 1, \alpha_1 = v_2, \alpha_2 = v_1\gamma_2 = v_1, \alpha_3 = v_0\gamma_2 = v_0\}.$$

Note that $n'_1 = n'_2 = n_1 = n_2 = n_3 = 1$. If v_0, v_1 and v_2 were all bigger than one, by the second part of Lemma 3.4, we would have $n'_1 + n'_2 \geq n_1 + n_2 + n_3 = 3$. It is a contradiction. So one of the v_0, v_1, v_2 must be one.

(α) $v_0 = 1$. Then we have $(a_1 - 1)/(a_0 - 1)$. By Lemma 3.4 and Table 1, we have $\{1, v'_1, v'_2\} = \{1, v_1, v_2\}$, hence either $v'_1 = v_1$, $v'_2 = v_2$ or $v'_1 = v_2$, $v'_2 = v_1$.

If $v'_1 = v_1$, $v'_2 = v_2$, we have $w'_0 = w_0$, $w'_1 = w_1$, $w'_2 = w_2$, i.e.,

$$a'_0 = \frac{a_0a_1 - 1}{a_1 - 1}, \frac{a'_1a'_2 - 1}{a'_2 - 1} = \frac{a_0a_1 - 1}{a_0 - 1}, \frac{a'_1a'_2 - 1}{a'_1 - 1} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}.$$

Set $a_1 = k + 1$, $a_2 = a$. Then $a_0 = mk + 1$, $a'_0 = mk + m + 1$, $a'_1 = (a(mk + m + 1) - m(k + 1))/ma$, $a'_2 = a(mk + 1)/m(k + 1)$. To have $a'_2 \in \mathbf{Z}$, we need m/a. Let $a = sm$, then $a'_1 = (s(mk + m + 1) - (k+1))/ms$. From the fact that $a_2/(a_1(a_0 - 1)/(a_0 - 1, a_1 - 1))$, we get $s/(k + 1)$. Set $k = us - 1$, then $a'_1 = (mk + m + 1 - u)/m \in \mathbf{Z}$. This implies that $m/(u-1)$. Set $u = km+1$, then $k = us-1 = s(hm+1)-1$. To have $a'_2 \in \mathbf{Z}$, we need $a'_2 = sm(ms(hm+1) - m+1)/ms(hm+1) = ms - (m - 1)/(hm + 1) \in \mathbf{Z}$. Thus, $(hm + 1)/(m - 1)$. Note $h \geq 0$, $m > 0$. We have two possible cases: $h = 0$ or $h > 0$. When $h = 0$, we have $u = 1$, hence $k = s - 1$. So $a_2 = m(k + 1)$, $a'_1 = k + 1$ and $a'_2 = mk + 1$. Then $a'_0 = (a_0a_1 - 1)/(a_1 - 1) = mk + m + 1$. It leads to case (2) of the theorem where $t = n = 1$. When $h > 0$, we have $hm + 1 > m - 1 \geq 0$. Thus to have $(hm + 1)/(m - 1)$, it forces $m = 1$. Then $a_0 = a_1 = a'_0 - 1 = k + 1$, $a'_2 = a_2 = a$, $a'_1 = k + 2 - ((k + 1)/a)$ with $a/(k + 1)$. This leads to case (3) of the theorem.

If $v'_1 = v_2$ and $v'_2 = v_1$, we have $w'_0 = w_0$, $w'_1 = w_2$, $w'_2 = w_1$. A similar argument leads to case (4) or a special case of case (1) of the theorem.

(β) $v_1 = 1$. Then $(a_0 - 1)/(a_1 - 1)$. By Lemma 3.4 and Table 1, we have $\{1, v'_1, v'_2\} = \{1, v_0, v_2\}$, hence either $v'_1 = v_0$, $v'_2 = v_2$ or $v'_1 = v_2$, $v'_2 = v_0$.

If $v'_1 = v_0$, $v'_2 = v_2$, we have $w'_0 = w_1$, $w'_1 = w_0$, $w'_2 = w_2$, i.e.,

$$a'_0 = \frac{a_0a_1 - 1}{a_0 - 1}, \qquad \frac{a'_1a'_2 - 1}{a'_2 - 1} = \frac{a_0a_1 - 1}{a_1 - 1}, \qquad \frac{a'_1a'_2 - 1}{a'_1 - 1} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}.$$

This implies $a_2' = a_2$, $a_1' = a_2((a_0a_1 - 1) - a_1(a_0 - 1))/a_2(a_1 - 1)$. Set $a_0 = k+1$, $a_1 = mk+1$ and $a_2 = a$. Then $a_0' = mk+m+1$, $a_1' = (a(mk+m+1) - (mk+1))/ma$ and $a_2' = a$. To have $a_1' \in \mathbf{Z}$, we need $m/a - 1$. Set $a = sm+1$. We have

$$a_1' = \frac{sm(mk+m+1)+m}{m(sm+1)} = \frac{s(mk+m+1)+1}{sm+1} = k+1+\frac{s-k}{sm+1}.$$

Now we need $(sm+1)/(s-k)$. Set $-t(sm+1) = s-k$, i.e., $k = (tm+1)s+t$. One sees that $t \geq 0$ since $s \geq 0$, $k > 0$, $m > 0$. We have two possible cases: $t = 0$ or $t > 0$. If $t = 0$, we have $s = k$. Thus, $a_1' = k+1$, $a_2' = a_2 = km+1$. Hence, $w_2 = w_0' = mk+m+1 = w_1 = w_2'$. It leads to case (2) of the theorem. If $t > 0$, then $a_1' = (tm+1)s+1$. It leads to case (5) of the theorem.

If $v_1' = v_2$, $v_2' = v_0$, we have $w_0' = w_1$, $w_1' = w_2$, $w_2' = w_0$. A similar argument leads to case (1) or case (6) of the theorem.

(γ) $v_2 = 1$. By Lemma 3.4 and Table 1, we have $\{1, v_1', v_2'\} = \{1, v_0, v_1\}$. Thus, we have either $w_0' = w_2$, $w_1' = w_0$, $w_2' = w_1$ or $w_0' = w_2$, $w_1' = w_1$, $w_2' = w_0$. The same argument as in (i) leads to case (1) or case (2) of the proposition. $\square$

Theorem 3.22. *Suppose that* $f = z_0^{a_0'} + z_0 z_1^{a_1'} + z_1 z_2^{a_2'}$ *is of type IV and* $g = z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2}$ *is of type V. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$\text{(1)} \qquad \left\{ \begin{array}{l} f = z_0^{ms+m+1} + z_0 z_1^{s+1} + z_1 z_2^{sm+1} \\ g = z_0^{ms+1} z_1 + z_1^{s+1} z_2 + z_0 z_2^{m(s+1)} \end{array} \right\}$$

or

$$\text{(2)} \qquad \left\{ \begin{array}{l} f = z_0^{st+s+t+2} + z_0 z_1^{st+s+t+1} + z_1 z_2^{s+1} \\ g = z_0^{st+s+t+1} z_1 + z_1^{ts+s+1} z_2 + z_0 z_2^{s+1} \end{array} \right\}$$

$t > 0$, *or*

$$\text{(3)} \qquad \left\{ \begin{array}{l} f = z_0^{mk+1} + z_0 z_1^{k} + z_1 z_2^{a} \\ g = z_0^{a_0} z_1 + z_1^{a} z_2 + z_0 z_2^{k} \end{array} \right\}$$

where $a_0 = (mk+1) + ((mk-m+1)/a)$, *with* $a/(mk-m+1)$, *or*

$$\left\{\begin{array}{l} f = z_0^{(ms+m+1)} + z_0 z_1^{s+1} + z_1 z_2^{sm+1} \\ g = z_0^{s+1} z_1 + z_1^{m(s+1)} z_2 + z_0 z_2^{sm+1} \end{array}\right\} \tag{4}$$

or

$$\left\{\begin{array}{l} f = z_0^{st+s+t+2} + z_0 z_1^{st+s+t+1} + z_1 z_2^{s+1} \\ g = z_0^{st+s+1} z_1 + z_1^{s+1} z_2 + z_0 z_2^{st+s+t+1} \end{array}\right\} \tag{5}$$

or

$$\left\{\begin{array}{l} f = z_0^{mk+1} + z_0 z_1^{k} + z_1 z_2^{a} \\ g = z_0^{a} z_1 + z_1^{k} z_2 + z_0 z_2^{a_2} \end{array}\right\} \tag{6}$$

where $a_2 = (mk+1) + ((mk-m+1)/a)$ *with* $a/(mk-m+1)$, *or*

$$\left\{\begin{array}{l} f = z_0^{sm+m+1} + z_0 z_1^{s+1} + z_1 z_2^{sm+1} \\ g = z_0^{m(s+1)} z_1 + z_1^{sm+1} z_2 + z_0 z_2^{s+1} \end{array}\right\} \tag{7}$$

or

$$\left\{\begin{array}{l} f = z_0^{st+s+t+2} + z_0 z_1^{st+s+t+1} z_2 + z_0 z_2^{s+1} \\ g = z_0^{s+1} z_1 + z_1^{st+s+t+1} z_2 + z_0 z_2^{st+s+1} \end{array}\right\} \tag{8}$$

or

$$\left\{\begin{array}{l} f = z_0^{mk+1} + z_0 z_1^{k} + z_1 z_2^{a} \\ g = z_0^{k} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a} \end{array}\right\} \tag{9}$$

where $a_1 = (mk+1) + ((mk-m+1)/a)$ *with* $a/(mk-m+1)$.

Proof. From Table 2, we have $u_0 = u_1 = u_2$. By Lemma 3.5, we have $u_0 = u_1 = u_2 = u_0' = u_1' = u_2'$. Thus, we have $\gamma_2' = 1$. From Table 1 and Lemma 3.4, we have

$$\{1, \alpha_0' = \gamma_2' = 1, \alpha_1' = v_2', \alpha_2' = v_1'\gamma_2' = v_1'\} \\ = \{1, \alpha_0 = v_0, \alpha_1 = v_1, \alpha_2 = v_2\}.$$

Note that $n'_1 = n'_2 = n_0 = n_1 = n_2 = 1$ (see Table 1). By the second part of Lemma 3.5, we see that one of v_0, v_1 or v_2 must be one. We have three subcases.

(i). $v_0 = 1$. From Table 2, we see that $v_0 = 1$ implies $(a_1a_2 - a_2 + 1)/(a_0a_1a_2+1)$. From Table 1 and Lemma 3.4, we have either $v'_1 = v_1$, $v'_2 = v_2$ or $v'_1 = v_2$, $v'_2 = v_1$. If $v'_1 = v_1$, $v'_2 = v_2$, we have $w'_0 = w_0$, $w'_1 = w_1$, $w'_2 = w_2$, i.e.,

$$a'_0 = \frac{a_0a_1a_2+1}{a_1a_2-a_2+1}, \qquad \frac{a'_0a'_1}{a'_0-1} = \frac{a_0a_1a_2+1}{a_0a_2-a_0+1}, \\ \frac{a'_0a'_1a'_2}{a'_0a'_1-a'_0+1} = \frac{a_0a_1a_2+1}{a_0a_1-a_1+1}.$$

From $u'_0 = u'_1 = u'_2$, we have $a'_1/(a'_0-1)$, and $a'_1a'_2/(a'_0a'_1 - a'_0+1)$. Set $a'_1 = k$, $a'_0 = mk+1$ and $a'_2 = a$ with $a/(mk-m+1)$. Then $a_0 = mk-m+1$, $a_1 = ((mk+1)a-(mk-m+1))/ma$ and $a_2 = amk/(mk-m+1)$. To have $a_1 \in \mathbf{Z}$, we need $m/(a-1)$. Set $a = sm+1$. We have $a_1 = ((mk+1)s+1)/(sm+1) = k+((s-k+1)/(sm+1))$. Set $s-k+1 = -t(sm+1)$, i.e., $k = (tm+1)s+(t+1)$. It is easy to see that $t \geq 0$. If $t = 0$, we have $k = s+1$. Then $a'_0 = sm+m+1$, $a'_1 = s+1$, $a'_2 = sm+1$, $a_0 = sm+1$, $a_1 = s+1$ and $a_2 = m(s+1)$. So we are in case (1) of the theorem. If $t > 0$, we have $mk-m+1 = (sm+1)(tm+1)$. Thus, $a_2 = m[(tm+1)s+(t+1)]/(tm+1) \in \mathbf{Z}$. Hence, we need $(tm+1)/m(t+1)$. Note $(m, tm+1) = 1$. We see $(tm+1)/(t+1)$. This forces $m = 1$ since $t > 0$. Thus, we have $a'_0 = st+s+t+2$, $a'_1 = st+s+t+1$, $a'_2 = s+1$, $a_0 = st+s+t+1$, $a_1 = st+s+1$ and $a_2 = s+1$. It leads to case (2) of the theorem.

If $v'_1 = v_2$, $v'_2 = v_1$. Then we have $w'_0 = w_0$, $w'_1 = w_2$, $w'_2 = w_1$, i.e.,

$$a'_0 = \frac{a_0a_1a_2+1}{a_1a_2-a_2+1}, \qquad \frac{a'_0a'_1}{a'_0-1} = \frac{a_0a_1a_2+1}{a_0a_1-a_1+1}, \\ \frac{a'_0a'_1a'_2}{a'_0a'_1-a'_0+1} = \frac{a_0a_1a_2+1}{a_0a_2-a_0+1}.$$

Set $a'_1 = k$, $a'_0 = mk+1$ and $a'_2 = a$. We have $a_0 = ((mk+1)a-(mk-m+1))/a$, $a_1 = a$ and $a_2 = k$. This leads to case (3) of the proposition.

(ii). $v_1 = 1$. By Lemma 3.4 and Table 1, we have either $v'_1 = v_0$, $v'_2 = v_2$ or $v'_1 = v_2$, $v'_2 = v_0$.

If $v_1' = v_0$, $v_2' = v_2$, we have $w_0' = w_1$, $w_1' = w_0$, $w_2' = w_2$. In the argument of (i), case $v_1' = v_2$, $v_2' = v_1$, we replace a_0 by a_1, a_1 by a_2, a_2 by a_0. Then the same argument leads to case (6) of the theorem.

If $v_1' = v_2$, $v_2' = v_0$, we have $w_0' = w_1$, $w_1' = w_2$, $w_2' = w_0$. In the argument of (i), case $v_1' = v_1$, $v_2' = v_2$, we replace a_0 by a_1, a_1 by a_2, a_2 by a_0. Then the same argument leads to case (4) or (5) of the theorem.

(iii). $v_2 = 1$. By Lemma 3.4 and Table 1, we have either $v_1' = v_0$, $v_2' = v_1$ or $v_1' = v_1$, $v_2' = v_0$. A similar argument as in (ii) leads to case (7), (8), or (9) of the theorem. □

Theorem 3.23. *Suppose that* $f = z_0^{a_0'} + z_0 z_1^{a_1'} + z_1 z_2^{a_2'}$ *is of type IV and* $g = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$ *is of type VI. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

$$(1) \qquad \begin{cases} f = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{\frac{a_2(a_0 a_1 - a_0 + 1)}{a_1(a_0 - 1)}} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{cases}$$

with $(a_1(a_0-1)/(a_0-1, a_1))/a_2$, *or*

$$(2) \qquad \begin{cases} f = z_0^{a_0} + z_0 z_1^{a_2} + z_1 z_2^{\frac{a_1(a_0 a_2 - a_0 + 1)}{a_2(a_0 - 1)}} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{cases}$$

with $(a_2(a_0-1)/(a_0-1, a_2))/a_1$.

Proof. From Table 2, we have $u_0'/u_1'/u_2'$, u_0/u_1 and u_0/u_2. By Lemma 3.2, Lemma 3.3 and Lemma 3.5, we have $u_0 = u_0'$ and either (i) $u_1 = u_1'/u_2 = u_2'$ or (ii) $u_2 = u_1'/u_1 = u_2'$.

Case (i). We consider the following subcases.

(α) $u_2' = u_2 > u_1' = u_1$. By (ii) of Theorem 2.3 with $u = u_2 = u_2'$, we have $w_2' = w_2$. Since $v_0 = v_0' = 1$, we have $w_0' = w_0$. Note $(1-w_0)(1-w_1)(1-w_2) = (1-w_0')(1-w_1')(1-w_2')$ by (ii) of Proposition 2.3. We see that $w_1' = w_1$. Thus, we have

$$a_0' = a_0, \qquad \frac{a_0' a_1'}{a_0' - 1} = \frac{a_0 a_1}{a_0 - 1}, \qquad \frac{a_0' a_1' a_2'}{a_0' a_1' - a_0' + 1} = \frac{a_0 a_2}{a_0 - 1}.$$

These imply that $a_1' = a_1$, $a_2' = a_2(a_0a_1 - a_0 + 1)/a_1(a_0 - 1)$. To have $a_2' \in \mathbf{Z}$, we need $(a_1(a_0 - 1)/(a_0 - 1, a_1))/a_2$. This leads to case (1) of the theorem.

(β) $u_2' = u_2 = u_1' = u_1$. In this case we have $\gamma_1 = \gamma_2 = \gamma_2' = 1$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_2' = 1, \alpha_1' = v_2', \alpha_2' = v_1'\gamma_2' = v_1'\}$$
$$= \{1, \alpha_0 = \gamma_1 = 1, \alpha_1 = \gamma_2 = 1, \alpha_2 = v_1\gamma_2 = v_1, \alpha_3 = v_2\gamma_1 = v_2\}$$

and $n_1' = n_2' = n_2 = n_3 = 1$. So we have either $v_1 = v_1'$, $v_2 = v_2'$ or $v_1 = v_2'$, $v_2 = v_1'$. If $v_1 = v_1'$, $v_2 = v_2'$, we have $w_0' = w_0$, $w_1' = w_1$, $w_2' = w_2$. Then the same argument as in (α) applies. If $v_1' = v_2$, $v_2' = v_1$, we have $w_0' = w_0$, $w_1' = w_2$, $w_2' = w_1$, i.e.,

$$a_0' = a_0, \qquad \frac{a_0'a_1'}{a_0' - 1} = \frac{a_0a_2}{a_0 - 1}, \qquad \frac{a_0'a_1'a_2'}{a_0'a_1' - a_0' + 1} = \frac{a_0a_1}{a_0 - 1}.$$

This leads to case (2) of the theorem.

(ii). In this case, if we replace a_1 by a_2, a_2 by a_1 in the argument of (i), then a similar argument leads to case (1) or (2) of the theorem again. □

Theorem 3.24. *Suppose that $f = z_0^{a_0'} + z_0z_1^{a_1'} + z_1z_2^{a_2'}$ is of type IV and $g = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$ is of type VII. Then $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ have the same topological type if and only if*

$$(1) \qquad \left\{ \begin{aligned} f &= z_0^{mk+m+1} + z_0z_1^{k+1} + z_1z_2^{\frac{(mk+1)a}{m(k+1)}} \\ g &= z_0^{mk+1}z_1 + z_0z_1^{k+1} + z_0z_1^{a} + z_1^{b_1}z_2^{b_2} \end{aligned} \right\}$$

with $m(k+1)/(mk+1)a$, or

$$(2) \qquad \left\{ \begin{aligned} f &= z_0^{mk+m+1} + z_0z_1^{k+1} + z_1z_2^{a} \\ g &= z_0^{k+1}z_1 + z_0z_1^{mk+1} + z_0z_2^{a} + z_1^{b_1}z_2^{b_2} \end{aligned} \right\}$$

or

$$(3) \qquad \left\{ \begin{aligned} f &= z_0^{mst+1} + z_0z_1^{tm} + z_1z_2^{k+1-\frac{s-1}{m}} \\ g &= z_0^{m(st-1)+1}z_1 + z_0z_1^{st} + z_0z_2^{tm} + z_1^{b_1}z_2^{b_2} \end{aligned} \right\}$$

or

$$(4)\qquad \left\{\begin{array}{l} f = z_0^{mk+m+1} + z_0 z_1^{m(k+1)} + z_1 z_2^{k+1} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{mk+1} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

or

$$(5)\qquad \left\{\begin{array}{l} f = z_0^{mnk+m+n} + z_0 z_1^{\frac{mnk+m+n-1}{n}} + z_1 z_2^{nk+1} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{nk+1} + z_0 z_2^{m(nk+1)} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

with $n/(m-1)$, *or*

$$(6)\qquad \left\{\begin{array}{l} f = z_0^{mnk+m+n} + z_0 z_1^{\frac{mnk+m+n-1}{m}} + z_1 z_2^{mk+1} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{nk+1} + z_0 z_2^{m(nk+1)} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

with $m/(n-1)$.

Proof. From Table 2, we have $u_0 = u_1/u_2$ and $u_0'/u_1'/u_2'$. By Lemma 3.2 and Lemma 3.3, we have $u_0 = u_1 = u_0' = u_1'$ and $u_2 = u_2'$. From $u_0 = u_1 = u_0' = u_1'$, we have $a_0' = (a_0a_1 - 1/(a_0-1, a_1-1))$ and $a_1'/(a_0'-1)$. We consider two subcases.

(i). $u_0 = u_1 = u_0' = u_1' < u_2 = u_2'$. Hence $\gamma_2 = \gamma_2' > 1$. By (ii) of Theorem 2.3, we have $1 = w_2 = 1 - w_2'$, then $w_2 = w_2'$ and hence $v_2 = v_2'$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_2', \alpha_1' = v_2', \alpha_2' = v_1'\gamma_2'\} \\ = \{1, \alpha_0 = \gamma_2, \alpha_1 = v_2, \alpha_2 = v_1\gamma_2, \alpha_3 = v_0\gamma_2\}.$$

Note that $(\gamma_2', v_2') = (\gamma_2, v_2) = 1$, $\gamma_2 = \gamma_2' > 1$ and $v_2' = v_2$. We have $\{1, \alpha_0' = \gamma_2', \alpha_2' = v_1'\gamma_2'\} = \{1, \alpha_0 = \gamma_2, \alpha_2 = v_1\gamma_2, \alpha_3 = v_0\gamma_2\}$. If both v_0 and v_1 were bigger than one, we would have $\alpha_2' = \alpha_2 = \alpha_3$. By the second part of Lemma 3.4, one sees that $n_2' = n_2 + n_3$, which gives a contradiction because $n_2' = n_2 = n_3 = 1$. Hence, we conclude that one of v_0 or v_1 must be one.

If $v_0 = 1$, we have $w_0' = w_0$, $w_1' = w_1$, $w_2' = w_2$ and $(a_1-1)/(a_0-1)$, i.e.,

$$a_0' = \frac{a_0a_1 - 1}{a_1 - 1}, \qquad \frac{a_0'a_1'}{a_0' - 1} = \frac{a_0a_1 - 1}{a_0 - 1},$$
$$\frac{a_0'a_1'a_2'}{a_0'a_1' - a_0' + 1} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}.$$

Set $a_1 = k+1$ and $a_2 = a$. Then $a_0 = mk+1$, $a'_0 = mk+m+1$, $a'_1 = k+1$ and $a'_2 = (mk+1)a/m(k+1)$. This leads to case (1) of the theorem.

If $v_1 = 1$, we have $w'_0 = w_1$, $w'_1 = w_0$, $w'_2 = w_2$ and $(a_0-1)/(a_1-1)$, i.e.,

$$a'_0 = \frac{a_0a_1-1}{a_0-1}, \qquad \frac{a'_0a'_1}{a'_0-1} = \frac{a_0a_1-1}{a_1-1},$$
$$\frac{a'_0a'_1a'_2}{a'_0a'_1-a'_0+1} = \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}.$$

Set $a_0 = k+1$, $a_2 = a$. Then $a_1 = mk+1$, $a'_0 = mk+m+1$, $a'_1 = k+1$ and $a'_2 = a$. This leads to case (2) of the proposition.

(ii). $u_0 = u_1 = u'_0 = u'_1 = u_2 = u'_2$. Hence, $\gamma_2 = \gamma'_2 = 1$. By Lemma 3.4 and Table 1, we have

$$\begin{aligned}&\{1, \alpha'_0 = \gamma'_2 = 1, \alpha'_1 = v'_2, \alpha'_2 = v'_1\gamma'_2 = v'_1\}\\ &\quad = \{1, \alpha_0 = \gamma_2 = 1, \alpha_1 = v_2, \alpha_2 = v_1\gamma_1 = v_1, \alpha_3 = v_0\gamma_2 = v_0\}\end{aligned}$$

Note $n'_1 = n'_2 = n_1 = n_2 = n_3 = 1$. If v_0, v_1 and v_2 were all bigger than one, by the second part of Lemma 3.4, we would have $n'_1 + n'_2 \geq n_1 = n_2 + n_3$, which is absurd. Thus, one of v_0, v_1 or v_2 must be one. Now we consider the following three subcases.

(α) $v_0 = 1$. By Lemma 4.4, Table 1 and Table 2, we have $(a_1-1)/(a_0-1)$ and either $v_1 = v'_1$, $v_2 = v'_2$ or $v_1 = v'_2$, $v_2 = v'_1$.

If $v_1 = v'_1$, $v_2 = v'_2$, we have $w_0 = w'_0$, $w_1 = w'_1$, $w_2 = w'_2$. The same argument as in (i), with $v_0 = 1$, applies.

If $v_1 = v'_2$, $v_2 = v'_1$, we have $w_0 = w'_0$, $w_1 = w'_2$, $w_2 = w'_1$, i.e.,

$$\frac{a_0a_1-1}{a_1-1} = a'_0, \qquad \frac{a_0a_1-1}{a_0-1} = \frac{a'_0a'_1a'_2}{a'_0a'_1-a'_0+1},$$
$$\frac{a_2(a_0a_1-1)}{a_1(a_0-1)} = \frac{a'_0a'_1}{a'_0-1}.$$

Set $a_1 = k+1$, $a_0 = mk+1$, and $a_2 = a$. Then $a'_0 = mk+m+1$, $a'_1 = a$ and $a'_2 = ((mk+m+1)a - m(k+1))/ma$. To have $a'_2 \in \mathbf{Z}$, we need m/a and $a/m(k+1)$. Set $a = tm$, then from $a/m(k+1)$, we have

$t/(k+1)$. So we set $k = st - 1$. Finally, we have

$$a_1 = st, a_0 = m(st-1)+1, a_2 = tm,$$
$$a'_0 = mst+1, a'_1 = tm, a'_2 = k+1-\frac{s-1}{m}$$

with $m/(s-1)$. This leads to case (3) of the theorem.

(β) $v_1 = 1$. By Lemma 3.4, Table 1 and Table 2, we have $(a_0 - 1)/(a_1 - 1)$ and either $v_0 = v'_1$, $v_2 = v'_2$ or $v_0 = v'_2$, $v_2 = v'_1$. If $v_0 = v'_1$, $v_2 = v'_2$, we have $w_0 = w'_1$, $w_1 = w'_0$, $w_2 = w'_2$. The same argument as in (i) with $v_1 = 1$ applies.

If $v_0 = v'_2$, $v_2 = v'_1$, we have $w_0 = w'_2$, $w_1 = w'_0$, $w_2 = w'_1$, i.e.,

$$\frac{a_0a_1-1}{a_1-1} = \frac{a'_0a'_1a'_2}{a'_0a'_1-a'_0+1}, \qquad \frac{a_0a_1-1}{a_0-1} = a'_0,$$
$$\frac{a_2(a_0a_1-1)}{a_1(a_0-1)} = \frac{a'_0a'_1}{a'_0-1}.$$

Set $a_0 = k+1$, $a_1 = mk+1$ and $a_2 = a$. Then $a'_0 = mk+m+1$, $a'_1 = ma(k+1)/(mk+1)$ and $a'_2 = (a(mk+m+1)-(mk+1))/ma$. To have $a'_2 \in \mathbf{Z}$, we need $m/(a-1)$. Set $a = sm+1$. Then

$$\begin{aligned} a'_2 &= \frac{(sm+1)(mk+m+1)-(mk+1)}{m(sm+1)} \\ &= \frac{smk+sm+s+1}{sm+1} = k+1+\frac{s-k}{sm+1}. \end{aligned}$$

So $(sm+1)/(s-k)$. Set $s-k = -t(sm+1)$. Then $a'_1 = m(sm+1)(tms+s+t+1)/(m(tms+s+t)+1) = m(tms+s+t+1)/(tm+1)$. To have $a'_1 \in \mathbf{Z}$, we need $(tm+1)/(tms+s+t+1)$. This implies that $(tm+1)/(t+1)$. Note $k > 0$, $m > 0$ implies $t \geq 0$. We have two possible subcases: $t = 0$ or $t > 0$. If $t = 0$, we have $s = k$. Thus, $a_2 = a = km+1$, $a'_1 = m(k+1)$, $a'_2 = k+1$. It leads to case (4) of the theorem. If $t > 0$, then $(tm+1)/(t+1)$ implies $m = 1$. Thus, $a_0 = a_1 = k+1$. So $v_0 = 1$. We are in case (ii) (α).

(γ) $v_2 = 1$. In this case, we have $a_1(a_0-1)/a_2(a_0a_1-1)$ and either $v_0 = v'_1$, $v_1 = v'_2$ or $v_0 = v'_2$, $v_1 = v'_1$.

If $v_0 = v_1'$, $v_1 = v_2'$, we have $w_0 = w_1'$, $w_1 = w_2'$, $w_2 = w_0'$, i.e.,

$$\frac{a_0a_1 - 1}{a_1 - 1} = \frac{a_0'a_1'}{a_0' - 1}, \qquad \frac{a_0a_1 - 1}{a_0 - 1} = \frac{a_0'a_1'a_2'}{a_0'a_1' - a_0' + 1},$$
$$\frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)} = a_0'.$$

Remember that we have $u_2 = u_1 = u_0$. Thus, from Table 2, one can check that $a_2(a_0 - 1, a_1 - 1)/a_1(a_0 - 1)$. Now $v_2 = 1$ implies $a_1(a_0 - 1)/a_2(a_0 - 1, a_1 - 1)$. So $a_2(a_0 - 1, a_1 - 1) = a_1(a_0 - 1)$. Set $(a_0 - 1, a_1 - 1) = k$. Then $a_0 = mk + 1$, $a_1 = nk + 1$, with $(m, n) = 1$ and $a_2 = m(nk + 1)$. It follows that $a_0' = mnk + m + n$, $a_1' = (mnk + m + n - 1)/n$, $a_2' = nk + 1$ with $n/(m - 1)$. This leads to case (5) of the theorem.

If $v_0 = v_2'$, $v_1 = v_1'$, we have $w_0 = w_2'$, $w_1 = w_1'$, $w_2 = w_0'$, i.e.,

$$\frac{a_0a_1 - 1}{a_1 - 1} = \frac{a_0'a_1'a_2'}{a_0'a_1' - a_0' + 1}, \qquad \frac{a_0a_1 - 1}{a_0 - 1} = \frac{a_0'a_1'}{a_0' - 1},$$
$$\frac{a_2(a_0a_1 - 1)}{a_2(a_0 - 1)} = a_0'.$$

Set $(a_0 - 1, a_1 - 1) = k$, $a_0 = mk + 1$, $a_1 = nk + 1$. Then $a_2 = m(nk + 1)$. Hence, we have

$$a_0' = mnk + m + n, \qquad a_1' = \frac{mnk + m + n - 1}{m}, \qquad a_2' = mk + 1$$

with $m/(n - 1)$. This leads to case (6) of the theorem. □

Theorem 3.25. *Suppose that* $f = z_0^{a_0'}z_1 + z_1^{a_1'}z_2 + z_0z_2^{a_2'}$ *is of type V and* $g = z_0^{a_0} + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$ *is of type VI. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

(1) $$\begin{Bmatrix} f = z_0^{mk}z_1 + z_1^{mk-m+1}z_2 + z_0z_2^{k} \\ g = z_0^{mk+1} + z_0z_1^{mk} + z_0z_2^{k} + z_1^{b_1}z_2^{b_2} \end{Bmatrix}$$

or

(2) $$\begin{Bmatrix} f = z_0^{mk}z_1 + z_1^{mk-m+1}z_2 + z_0z_2^{k} \\ g = z_0^{mk+1} + z_0z_1^{k} + z_0z_2^{mk} + z_1^{b_1}z_2^{b_2} \end{Bmatrix}$$

or

$$(3)\qquad \left\{\begin{array}{l} f = z_0^k z_1 + z_1^{mk} z_2 + z_0 z_2^{mk-m+1} \\ g = z_0^{mk+1} + z_0 z_1^k + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

or

$$(4)\qquad \left\{\begin{array}{l} f = z_0^k z_1 + z_1^{mk} z_2 + z_0 z_2^{mk-m+1} \\ g = z_0^{mk+1} + z_0 z_1^{mk} + z_0 z_2^k + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

or

$$(5)\qquad \left\{\begin{array}{l} f = z_0^{mk-m+1} z_1 + z_1^k z_2 + z_0 z_2^{mk} \\ g = z_0^{mk+1} + z_0 z_1^k + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

or

$$(6)\qquad \left\{\begin{array}{l} f = z_0^{mk-m+1} z_1 + z_1^k z_2 + z_0 z_2^{mk} \\ g = z_0^{mk+1} + z_0 z_1^{mk} + z_0 z_2^k + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

Proof. From Table 2, we have $u_0' = u_1' = u_2'$, u_0/u_1 and u_0/u_2. By Lemma 3.5, we have $u_0' = u_1' = u_2' = u_0 = u_1 = u_2$. Hence, $\gamma_1 = \gamma_2 = 1$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = v_0', \alpha_1' = v_1', \alpha_2' = v_2'\} = \{1, \alpha_0 = \gamma_1 = 1, \alpha_1 = \gamma_2 = 1, \alpha_2 = v_1, \alpha_3 = v_2\}.$$

Note that $n_0' = n_1' = n_2' = n_2 = n_3 = 1$. By the second part of Lemma 3.4, if v_0', v_1' and v_2' were all bigger than one, we would have $n_0' + n_1' + n_2' \le n_2 + n_3$, a contradiction. Thus, one of v_0', v_1' or v_2' must be one.

(i). $v_0' = 1$. By Lemma 3.4 and Table 1, we have either $v_1' = v_1$, $v_2' = v_2$ or $v_1' = v_2$, $v_2' = v_1$. If $v_1' = v_1$, $v_2' = v_2$, we have $w_0' = w_0$, $w_1' = w_1$, $w_2' = w_2$, i.e.,

$$a_0 = \frac{a_0'a_1'a_2' + 1}{a_1'a_2' - a_2' + 1}, \qquad \frac{a_0 a_1}{a_0 - 1} = \frac{a_0'a_1'a_2' + 1}{a_0'a_2' - a_0' + 1},$$
$$\frac{a_0 a_2}{a_0 - 1} = \frac{a_0'a_1'a_2' + 1}{a_0'a_1' - a_1' + 1}.$$

These imply that

$$a_0' = \frac{a_0a_1 - a_0 + 1}{a_1}, \qquad a_1' = \frac{a_1(a_0a_2 - a_0 + 1)}{a_2(a_0 - 1)}, \qquad a_2' = a_2.$$

To have $a_0' \in \mathbf{Z}$, we need $a_1/(a_0 - 1)$. From Table 2, $u_1 = u_2$ implies that $a_2(a_0 - 1, a_1) = a_1(a_0 - 1, a_2)$, i.e., $a_1a_2 = a_1(a_0 - 1, a_2)$. Hence, $a_2/(a_0 - 1)$. To have $a_1' \in \mathbf{Z}$, we need $a_2(a_0 - 1)/a_1(a_0 - 1, a_2)$. It follows that $(a_0 - 1)/a_1$, hence $a_0 - 1 = a_1$. Set $a_2 = k$, $a_0 = mk + 1$, $a_1 = mk$. Then $a_0' = mk$, $a_1' = mk - m + 1$, $a_2' = k$. It leads to case (1) of the theorem.

If $v_1' = v_2$, $v_2' = v_1$, we have $w_0' = w_0$, $w_1' = w_2$, $w_2' = w_1$, i.e.,

$$a_0 = \frac{a_0'a_1'a_2' + 1}{a_1'a_2' - a_2' + 1}, \qquad \frac{a_0a_1}{a_0 - 1} = \frac{a_0'a_1'a_2' + 1}{a_0'a_1' - a_1' + 1},$$
$$\frac{a_0a_2}{a_0 - 1} = \frac{a_0'a_1'a_2' + 1}{a_0'a_2' - a_0' + 1}.$$

Using similar arguments as above, we see that $a_0 = mk + 1$, $a_2 = mk$, $a_1 = k$, $a_0' = mk$, $a_1' = mk - m + 1$, $a_2' = k$. This leads to case (2) of the theorem.

(ii). $v_1' = 1$. In this case, we have either $v_0' = v_1$, $v_2' = v_2$ or $v_0' = v_2$, $v_2' = v_1$.

If $v_0' = v_1$, $v_2' = v_2$, we have $w_0 = w_1'$, $w_1 = w_0'$, $w_2 = w_2'$, i.e.,

$$a_0 = \frac{a_0'a_1'a_2' + 1}{a_0'a_2' - a_0' + 1}, \qquad \frac{a_0a_1}{a_0 - 1} = \frac{a_0'a_1'a_2' + 1}{a_1'a_2' - a_2' + 1},$$
$$\frac{a_0a_2}{a_0 - 1} = \frac{a_0'a_1'a_2' + 1}{a_0'a_1' - a_1' + 1}.$$

These imply $a_0' = a_1$, $a_1' = (a_0a_2 - a_0 + 1)/a_2$, $a_2' = a_2(a_0a_1 - a_0 + 1)/a_1(a_0 - 1)$. A similar argument as case $v_1' = v_2$, $v_2' = v_1$ in (i) shows that $a_0 = mk + 1$, $a_1 = k$, $a_2 = mk$, $a_0' = k$, $a_1' = mk$, $a_2' = mk - m + 1$. So we are in case (3) of the theorem.

If $v_0' = v_2$, $v_2' = v_1$, we have $w_0 = w_1'$, $w_1 = w_2'$, $w_2 = w_0'$, i.e.,

$$a_0 = \frac{a_0'a_1'a_2' + 1}{a_0'a_2' - a_0' + 1}, \qquad \frac{a_0a_1}{a_0 - 1} = \frac{a_0'a_1'a_2' + 1}{a_0'a_1' - a_1' + 1},$$
$$\frac{a_0a_2}{a_0 - 1} = \frac{a_0'a_1a_2' + 1}{a_1'a_2' - a_2' + 1}.$$

Similar as in the case $v_1' = v_1$, $v_2' = v_2$, these will lead to $a_0 = mk + 1$, $a_1 = mk$, $a_2 = k$, $a_0' = k$, $a_1' = mk$, $a_2' = mk - m + 1$. So we are in case (4) of the theorem.

(iii). $v_2' = 1$. A similar argument as above leads to case (5) or case (6) of the theorem. □

Theorem 3.26. *Suppose that* $f = z_0^{a_0'} z_1 + z_1^{a_1'} z_2 + z_0 z_2^{a_2'}$ *is of type V and* $g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$ *is of type VII. Then* $(f^{-1})(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

(1)
$$\begin{Bmatrix} f = z_0^{mk+1} z_1 + z_1^{a_1'} z_2 + z_0 z_2^{tm} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{nk+1} + z_0 z_2^{tm} + z_1^{b_1} z_2^{b_2} \end{Bmatrix}$$

where $a_1' = (nk + 1) + ((tn - nk - 1)/tm)$ *with* $t/(nk + 1)$ *and* $tm/(tn - nk - 1)$; *or*

(2)
$$\begin{Bmatrix} f = z_0^{tm} z_1 + z_1^{mk+1} z_2 + z_0 z_2^{a_2'} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{nk+1} + z_0 z_2^{tm} + z_1^{b_1} z_2^{b_2} \end{Bmatrix}$$

where $a_2' = (nk + 1) + ((tn - nk - 1)/tm)$ *with* $t/(nk + 1)$ *and* $tm/(tn - nk - 1)$; *or*

(3)
$$\begin{Bmatrix} f = z_0^{z_0'} z_1 + z_1^{tm} z_2 + z_0 z_2^{mk+1} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{nk+1} + z_0 z_2^{tm} + z_1^{b_1} z_2^{b_2} \end{Bmatrix}$$

where $a_0' = (nk + 1) + ((tn - nk - 1)/tm)$ *with* $t/(nk + 1)$ *and* $tm/(tn - nk - 1)$; *or*

(4)
$$\begin{Bmatrix} f = z_0^{a_0'} z_1 + z_1^{a_1'} z_2 + z_0 z_2^{a_1} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{Bmatrix}$$

where $a_0' = (a_2(a_0 a_1 - 1) - a_1(a_0 - 1))/a_2(a_1 - 1) \in \mathbf{Z}$, $a_1' = a_0 a_2 (a_1 - 1)/a_1(a_0 - 1) \in \mathbf{Z}$; *or*

(5)
$$\begin{Bmatrix} f = z_0^{a_1} z_1 + z_1^{a_1'} z_2 + z_0 z_2^{a_2'} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{Bmatrix}$$

where $a_1' = (a_2(a_0a_1-1) - a_1(a_0-1))/a_2(a_1-1) \in \mathbf{Z}$, $a_2' = a_0a_2(a_1-1)/a_1(a_0-1) \in \mathbf{Z}$; *or*

$$(6) \qquad \begin{Bmatrix} f = z_0^{a_0'}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2'} \\ g = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2} \end{Bmatrix}$$

where $a_0' = a_0a_2(a_1-1)/a_1(a_0-1) \in \mathbf{Z}$, $a_2' = (a_2(a_0a_1-1) - a_1(a_0-1))/a_2(a_1-1) \in \mathbf{Z}$.

Proof. From Table 2, we have $u_0' = u_1' = u_2'$. By Lemma 3.5, we have $u_0' = u_1' = u_2' = u_0 = u_1 = u_2$. It follows $\gamma_2 = 1$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = v_0', \alpha_1' = v_1', \alpha_2' = v_2'\} = \{1, \alpha_0 = \gamma_2 = 1, \alpha_1 = v_2, \alpha_2 = v_1, \alpha_3 = v_0\}.$$

Note that $n_0' = n_1' = n_2' = n_1 = n_2 = n_3 = 1$. By the second part of Lemma 3.4, we have the following six subcases.

(i). $v_0' = v_0$, $v_1' = v_1$, $v_2' = v_2$. It follows that $w_0' = w_0$, $w_1' = w_1$, $w_2' = w_2$, i.e.,

$$\frac{a_0'a_1'a_2'+1}{a_1'a_2'-a_2'+1} = \frac{a_0a_1-1}{a_1-1}, \qquad \frac{a_0'a_1'a_2'+1}{a_0'a_2'-a_0'+1} = \frac{a_0a_1-1}{a_0-1},$$
$$\frac{a_0'a_1'a_2'+1}{a_0'a_1'-a_1'+1} = \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}.$$

These imply that $a_0' = a_0$, $a_1' = (a_2(a_0a_1-1) - a_1(a_0-1))/a_2(a_0-1)$, $a_2' = a_2$. Set $(a_0-1, a_1-1) = k$, $a_0 = mk+1$, $a_1 = nk+1$ with $(m,n) = 1$ and $a_2 = a$. From Table 2, $u_1 = u_2$ implies $a_2(a_0-1, a_1-1)/a_1(a_0-1)$, i.e., $ak/mk(nk+1)$, so $a/m(nk+1)$. Hence, $a_1' = (a(mnk+m+n) - m(nk+1))/ma$. To have $a_1' \in \mathbf{Z}$, we need m/na. Since $(m,n) = 1$, we have m/a. Set $a = tm$. Then $a_1' = (nk+1) + ((tn-nk-1)/tm)$. Finally, we have $a_0 = mk+1$, $a_1 = nk+1$, $a_2 = tm$, $a_0' = mk+1$, $a_1' = nk+1+((tn-nk-1)/tm)$, $a_2' = tm$ where $tm/(tn-nk-1)$. This leads to case (1) of the theorem.

(ii). $v_0' = v_1$, $v_1' = v_2$, $v_2' = v_0$, i.e., $w_2' = w_0$, $w_0' = w_1$, $w_1' = w_2$. A similar argument as in (i), by replacing a_1' by a_2', a_1' by a_0', a_2' by a_1', leads to case (2) of the theorem.

(iii). $v_0' = v_2$, $v_1' = v_0$, $v_2' = v_1$, i.e., $w_1' = w_0$, $w_2' = w_1$, $w_0' = w_2$. A similar argument as in (i), by replacing a_0' by a_1', a_1' by a_2', a_2' by a_0', leads to case (3) of the theorem.

(iv). $v_0' = v_0$, $v_1' = v_2$, $v_2' = v_1$. We have $w_0' = w_0$, $w_1' = w_2$, $w_2' = w_1$, i.e.,

$$\frac{a_0'a_1'a_2'+1}{a_1'a_2'-a_2'+1} = \frac{a_0a_1-1}{a_1-1}, \qquad \frac{a_0'a_1'a_2'+1}{a_0'a_2'-a_0'+1} = \frac{a_2(a_0a_1-1)}{a_1(a_0-1)},$$
$$\frac{a_0'a_1'a_2'+1}{a_0'a_1'-a_1'+1} = \frac{a_0a_1-1}{a_0-1}.$$

These imply $a_0' = (a_2(a_0a_1-1) - a_1(a_0-1))/a_2(a_0-1)$, $a_1' = a_0a_2(a_1-1)/a_1(a_0-1)$, $a_2' = a_1$. This leads to case (4) of the theorem.

(v). $v_0' = v_2$, $v_1' = v_1$, $v_2' = v_0$. Then we have $w_0' = w_2$, $w_1' = w_1$, $w_2' = w_0$. A similar argument as in (iv), by replacing a_0' by a_2', a_1' by a_0', a_2' by a_1', leads to case (5) of the theorem.

(vi). $v_0' = v_1$, $v_1' = v_0$, $v_2' = v_2$. Then we have $w_1' = w_0$, $w_0' = w_1$, $w_2' = w_2$. A similar argument as in (iv), by replacing a_0' by a_1', a_1' by a_2', a_2' by a_0', leads to case (6) of the theorem. □

Theorem 3.27. *Suppose that* $f = z_0^{a_0'} + z_0z_1^{a_1'} + z_0z_2^{a_2'} + z_1^{b_1}z_2^{b_2}$ *is of type VI and* $g = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$ *is of type VII. Then* $(f^{-1}(0),0)$ *and* $(g^{-1}(0),0)$ *have the same topological type if and only if*

$$(1) \qquad \left\{ \begin{array}{l} f = z_0^{mk+m+1} + z_0z_1^{k+1} + z_0z_2^{a} + z_1^{b_1'}z_2^{b_2'} \\ g = z_0^{mk+1} + z_1 + z_0z_1^{k+1} + z_0z_2^{a} + z_1^{b_1}z_2^{b_2} \end{array} \right\}$$

or

$$(2) \qquad \left\{ \begin{array}{l} f = z_0^{mk+m+1} + z_0z_1^{k+1} + z_0z_2^{\frac{ma(k+1)}{mk+1}} + z_1^{b_1'}z_2^{b_2'} \\ g = z_0^{k+1}z_1 + z_0z_1^{mk+1} + z_0z_2^{a} + z_1^{b_1}z_2^{b_2} \end{array} \right\}$$

with $(mk+1)/a(k+1)$, *or*

$$(3) \qquad \left\{ \begin{array}{l} f = z_0^{mk+m+1} + z_0z_1^{a} + z_0z_2^{k+1} + z_1^{b_1'}z_2^{b_2'} \\ g = z_0^{mk+1}z_1 + z_0z_1^{k+1} + z_0z_2^{a} + z_1^{b_1}z_2^{b_2} \end{array} \right\}$$

or

$$(4)\qquad \left\{\begin{array}{l} f = z_0^{mk+m+1} + z_0 z_1^{\frac{ma(k+1)}{mk+1}} + z_0 z_2^{k+1} + z_1^{b_1'} z_2^{b_2'} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$$

with $(mk+1)/a(k+1)$.

Proof. From Table 2, we have u_0'/u_1', u_0'/u_2' and $u_0 = u_1/u_2$. By Lemma 3.2 and Lemma 3.3, we have two possible cases: (i) $u_0' = u_1' = u_0 = u_1/u_2' = u_2$ or (ii) $u_0' = u_2' = u_0 = u_1/u_1' = u_2$.

In case (i), we consider two subcases.

(α). $u_2 = u_2' > u_0' = u_1' = u_0 = u_1$. By (ii) of Theorem 2.3, we have $w_2 = w_2'$. The assumption of (α) implies $\gamma_2' = \gamma_2 > 1$, $\gamma_1' = 1$ and $v_2' = v_2$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_0' = 1, \alpha_1' = \gamma_2', \alpha_2' = v_1'\gamma_2', \alpha_3' = v_2'\gamma_1' = v_2'\} \\ = \{1, \alpha_0 = \gamma_2, \alpha_1 = v_2, \alpha_2 = v_1\gamma_2, \alpha_3 = v_0\gamma_2\}.$$

Note that $\gamma_2' = \gamma_2 = 1$, $(\gamma_2', v_2') = (\gamma_2, v_2) = 1$. We have

$$\{1, \alpha_1' = \gamma_2', \alpha_2' = v_1'\gamma_2'\} = \{1, \alpha_0 = \gamma_2, \alpha_2 = v_1\gamma_2, \alpha_3 = v_0\gamma_2\}.$$

If v_0 and v_1 were both bigger than one, then $\alpha_2 > \alpha_0 = \alpha_1'$, $\alpha_3 > \alpha_0 = \alpha_1'$. Thus, we would have $\alpha_2' = \alpha_2 = \alpha_3$; hence, by the second part of Lemma 3.4, we have $n_2' = n_2 + n_3$ which contradicts the fact that $n_2' = n_2 = n_3 = 1$. It follows that either $v_0 = 1$ or $v_1 = 1$.

If $v_0 = 1$, then we have $v_1 = v_1'$. Thus, $w_0 = w_0'$, $w_1 = w_1'$, $w_2' = w_2$, i.e.,

$$a_0' = \frac{a_0 a_1 - 1}{a_1 - 1}, \qquad \frac{a_0' a_1' - 1}{a_0' - 1} = \frac{a_0 a_1 - 1}{a_0 - 1}, \qquad \frac{a_0' a_2'}{a_0' - 1} = \frac{a_1(a_0 a_1 - 1)}{a_1(a_0 - 1)}.$$

These imply that $a_1' = a_1$, $a_2' = a_2$. $v_0 = 1$ implies $(a_1 - 1)/(a_0 - 1)$. Set $a_1 = k+1$, $a_0 = mk+1$, $a_2 = a$. Then $a_0' = mk+m+1$, $a_1' = k+1$, $a_2' = a$. This leads to case (1) of the theorem.

If $v_1 = 1$, then we have $v_0 = v_1'$. Thus, $w_0 = w_1'$, $w_1 = w_0'$, $w_2 = w_2'$, i.e.,

$$a_0' = \frac{a_0a_1 - 1}{a_0 - 1}, \qquad \frac{a_0'a_1'}{a_0' - 1} = \frac{a_0a_1 - 1}{a_1 - 1}, \qquad \frac{a_0'a_2'}{a_0' - 1} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}.$$

These imply that $a_1' = a_0$, $a_2' = a_0a_2(a_1 - 1)/a_1(a_0 - 1)$. $v_1 = 1$ implies that $(a_0 - 1)/(a_1 - 1)$. Set $a_0 = k + 1$, $a_1 = mk + 1$, $a_2 = a$. Then $a_0' = mk + m + 1$, $a_1' = k + 1$, $a_2' = ma(k + 1)/(mk + 1)$ with $(mk + 1)/a(k + 1)$. This leads to case (2) of the theorem.

(β). $u_0 = u_1 = u_2 = u_0' = u_1' = u_2'$. Then $\gamma_1' = \gamma_2' = \gamma_2 = 1$. By Lemma 3.4 and Table 1, we have

$$\begin{aligned}&\{1, \alpha_0' = \gamma_1' = 1, \alpha_1' = \gamma_2' = 1, \alpha_2' = v_1'\gamma_2' = v_1', \alpha_3' = v_2'\gamma_1' = v_2'\}\\ &\qquad = \{1, \alpha_0 = \gamma_2 = 1, \alpha_1 = v_2, \alpha_2 = v_1\gamma_2 = v_1, \alpha_3 = v_0\gamma_2 = v_0\}.\end{aligned}$$

Note that $n_2' = n_3' = n_1 = n_2 = n_3 = 1$. If v_0, v_1 and v_2 were all bigger than one, by the second part of Lemma 3.4, we would have $n_2' + n_3' \geq n_1 + n_2 + n_3$. This is a contradiction. Thus, one of v_0, v_1 or v_2 must be one.

(a). $v_0 = 1$. By Lemma 3.4 and Table 1, we have either $v_1 = v_1'$, $v_2 = v_2'$ or $v_1 = v_2'$, $v_2 = v_1'$. If $v_1 = v_1'$, $v_2 = v_2'$, then we have $w_0' = w_0$, $w_1' = w_1$, $w_2' = w_2$. The same argument as in case (i) (α), $v_0 = 1$, applies. It leads to case (1) of the theorem. If $v_1 = v_2'$, $v_2 = v_1'$, we have $w_0' = w_0$, $w_1' = w_2$, $w_2' = w_1$. In the argument of (i) (α), $v_0 = 1$, if we replace a_1' by a_2', a_2' by a_1', then a similar argument leads to case (3) of the theorem.

(b). $v_1 = 1$. By Lemma 3.4 and Table 1, we have either $v_1' = v_0$, $v_2' = v_2$ or $v_1' = v_2$, $v_2' = v_0$.

If $v_0 = v_1'$, $v_2 = v_2'$, then we have $w_0' = w_1$, $w_1' = w_0$, $w_2' = w_2$. The same argument as in case (i) (α), $v_1 = 1$, applies. This leads to case (2) of the theorem. If $v_0 = v_2'$, $v_2 = v_1'$, then we have $w_0' = w_1$, $w_1' = w_2$, $w_2' = w_0$. Replace a_1' by a_2', a_2' by a_1' in the argument of case (i) (α), $v_1 = 1$. A similar argument leads to case (4) of the proposition.

(c). $v_2 = 1$. By Lemma 3.4 and Table 1, we have either $v_1 = v_1'$, $v_0 = v_2'$ or $v_1 = v_2'$, $v_0 = v_1'$. If $v_1 = v_1'$, $v_0 = v_2'$, we have $w_0' = w_2$,

$w_1' = w_1$, $w_2' = w_0$, i.e.,

$$a_0' = \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}, \qquad \frac{a_0'a_1'-1}{a_0'-1} = \frac{a_0a_1-1}{a_0-1}, \qquad \frac{a_0'a_2'}{a_0'-1} = \frac{a_0a_1-1}{a_1-1}.$$

These imply that $a_1' = (a_0a_1a_2 - a_0a_1 + a_1 - a_2)/a_2(a_0-1)$, $a_2' = (a_0a_1a_2 - a_0a_1 + a_1 - a_2)/a_2(a_1-1)$. Now $u_0 = u_1 = u_2$ implies $\gamma_2 = 1$. By Table 2, $\gamma_2 = 1$ implies $a_2 = a_1(a_0-1)/(a_0-1, a_1-1)$. Set $(a_0-1, a_1-1) = k$, $a_0 = nk+1$, $a_1 = mk+1$. Then $a_2 = n(mk+1)$, $a_1' = (mnk+m+n-1)/n$, $a_2' = (mnk+m+n-1)/m$. To have $a_1' \in \mathbf{Z}$ and $a_2' \in \mathbf{Z}$, we need $m/(n-1)$ and $n/(m-1)$. It forces one of m or n to be one. Hence, we have either $a_0 = nk+1$, $a_1 = k+1$ or $a_0 = k+1$, $a_1 = mk+1$, i.e., either $v_0 = 1$ or $v_1 = 1$. So we are in case (a) or (b) again. If $v_1 = v_2'$, $v_0 = v_1'$, a similar argument shows either $v_0 = 1$ or $v_1 = 1$ again.

(ii). $u_0' = u_2' = u_0 = u_1/u_1' = u_2$. In the argument of case (i), exchange the indices 1 and 2; a similar argument leads to cases (1), (2), (3), (4) of the theorem. □

Theorem 3.28. *Suppose that $f = z_0^{a_0'} + z_1^{a_1'} + z_2^{a_2'}$ and $g = z_0^{a_0} + z_1^{a_1} + z_2^{a_2}$ are of type I. Then $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ have the same topological type if and only if $a_0' = a_{i_0}$, $a_1' = a_{i_1}$, $a_2' = a_{i_2}$ where $\{i_0, i_1, i_2\} = \{0, 1, 2\}$.*

Proof. We may assume that $a_0' \le a'1 \le a_2'$, $a_0 \le a_1 \le a_2$ without loss of generality. By Lemma 3.5, we have $a_0' = a_0$, $a_1' = a_1$, $a_2' = a_2$ as required. □

Theorem 3.29. *Suppose that $f = z_0^{a_0'} + z_1^{a_1'} + z_1 z_2^{a_2'}$ and $g = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2}$ are of type II. Then $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ have the same topological type if and only if* (1) *$a_i' = a_i$, $i = 0, 1, 2$, or*

$$\text{(2)} \qquad \left\{ \begin{array}{l} f = z_0^{a_1} + z_1^{a_0} + z_1 z_2^{\frac{a_1a_2(a_0-1)}{a_0(a_1-1)}} \\ g = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} \end{array} \right\}$$

with $a_0(a_1-1)/a_1a_2(a_0-1)$, or

$$\text{(3)} \qquad \left\{ \begin{array}{l} f = z_0^{k+1} + z_1^{m(k+1)} + z_1 z_2^{t(mk+m-1)} \\ g = z_0^{tm(k+1)} + z_1^{k+1} + z_1 z_2^{mk} \end{array} \right\}$$

or

$$f = z_0^{m(k+1)} + z_1^{k+1} + z_1 z_2^{nk}$$

(4) $$\left\{ \begin{array}{l} f = z_0^{m(k+1)} + z_1^{k+1} + z_1 z_2^{nk} \\ g = z_0^{n(k+1)} + z_1^{k+1} + z_1 z_2^{mk} \end{array} \right\}.$$

Proof. From Table 2, we have u_1/u_1, u_1'/u_2'. By Lemma 3.2 and Lemma 3.3, we have the following subcases.

(i). $u_1 = u_1'$, $u_2 = u_2'$ and $u_0 = u_0'$. Note that $v_0 = v_0' = v_1 = v_1' = 1$. By (ii) of Theorem 2.3, set $u = u_2 = u_2'$. We have $1 - w_2 = 1 - w_2'$, i.e., $w_2 = w_2'$. Thus, $u_0 = u_0'$, $u_1 = u_1'$, and $w_2 = w_2'$, imply that $a_i = a_i'$, $i = 0, 1, 2$. This is case (1) of the theorem.

(ii). $u_0' = u_1/u_2 = u_2'$. Then $u_0 = u_1'$. By (ii) of Theorem 2.3, setting $u = u_2 = u_2'$, we have $w_2 = w_2'$ again. $u_0' = u_1$, $u_1' = u_0$, $w_2' = w_2$ imply that $a_0' = a_1$, $a_1' = a_0$, $a_2' = a_2 a_1(a_0 - 1)/a_0(a_1 - 1)$. This leads to case (2) of the theorem.

(iii). $u_0' = u_1/u_2 = u_1'$. Then $u_0 = u_2'$ and $u_1/u_2/u_0$, $u_0'/u_1'/u_2'$. We may assume that $u_0 \neq u_0'$ (here we just exclude case (i)). Now we consider two subcases: (a) $u_2 < u_0$ or (b) $u_2 = u_0$.

Case (a). By Lemma 3.5, we have $u_0 = u_2'$. By (ii) of Theorem 2.3, we have $1 - w_0 = 1 - w_2'$. Thus, $w_0 = w_2'$, so $v_2' = 1$. Using (ii) of Theorem 2.3 again, we have $(1 - w_1)(1 - w_2) = (1 - w_0')(1 - w_1')$. Note that $w_1 = u_1 = u_0' = w_0'$. We have $w_2 = w_1'$. Thus $v_2 = 1$. Now $u_0' = u_1$, $w_1' = w_2$, $w_2' = w_0$ imply $a_0' = a_1$, $a_1' = a_1 a_2/(a_1 - 1)$ and $a_1' a_2'/(a_1' - 1) = a_0$. It follows that $a_2' = a_0(a_1 a_2 - a_1 + 1)/a_1 a_2$. Note that $v_2 = 1$ implies $(a_1 - 1)/a_2$. Set $a_1 = k + 1$, $a_2 = mk$. Then $u_2 = w_2 = m(k + 1)$, hence, from u_2/u_0, we set $a_0 = tm(k + 1)$. Now we have $a_0' = k + 1$, $a_1' = m(k + 1)$, $a_2' = t(mk + m - 1)$. This leads to case (3) of the theorem.

Case (b). In this case, we have $u_2 = u_0 = u_1' = u_2'$ and $u_0' = u_1$. This is in case (ii).

(iv). $u_1 = u_2'/u_2 = u_0'$. Then $u_0 = u_1'$ and $u_1'/u_2'/u_0'$, $u_0/u_1/u_2$. A similar argument as in case (iii) applies (we just interchange f and g in case (iii)). This leads to case (2) or (3) of the theorem.

(v). $u_1 = u_1'/u_2 = u_0'$ Then $u_0 = u_2'$. We may assume $u_0 \neq u_0'$ to exclude case (i). Because of (ii) of Theorem 2.3, setting $u = u_2 = u_0'$ and $u = u_0 = u_2'$, respectively, we get $w_2 = w_0'$, $w_0 = w_2'$. Hence, $v_2 = v_2' = 1$ since $v_0 = v_0' = 1$. $u_1 = u_1'$ implies that $w_1 = w_1'$. We have $a_0' = a_1a_2/(a_1-1)$, $a_0 = a_1'a_2'/(a_1'-1)$, $a_1 = a_1'$. Note that $v_2 = v_2' = 1$ implies that $(a_1-1)/a_2$, $(a_1'-1)/a_2'$. Set $a_1 = a_1' = k+1$, $a_2 = mk$. Then $a_0' = m(k+1)$, $a_2' = nk$, $a_0 = n(k+1)$. This leads to case (4) of the proposition. □

Theorem 3.30. *Suppose that* $f = z_0^{a_0'} + z_1^{a_1'}z_2 + z_1z_2^{a_2'}$ *and* $g = z_0^{a_0} + z_1^{a_1}z_2 + z_1z_2^{a_2}$ *are of type III. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if* $a_0 = a_0'$ *and either* $a_1 = a_1'$, $a_2 = a_2'$ *or* $a_1 = a_2'$, $a_2 = a_1'$.

Proof. From Table 2, we have $u_1' = u_2'$, $u_1 = u_2$. By Lemma 3.2, we have $u_1' = u_2' = u_1 = u_2$. Hence, $u_0 = u_0'$, i.e., $a_0 = a_0'$. From Proposition 2.2 (iii), the degree of ζ_f and ζ_g are $(a_0'-1)(a_1'a_2'-1)+a_0'$ and $(a_0-1)(a_1a_2-1)+a_0$, respectively. $\zeta_f = \zeta_g$ implies $(a_0'-1)(a_1'a_2'-1)+a_0' = (a_0-1)(a_1a_2-1)+a_0$. Note $a_0 = a_0' \geq 2$. We have $a_1'a_2'-1 = a_1a_2-1$. From Table 2, $u_1 = u_1'$ implies that $(a_1a_2-1)/(a_1-1, a_2-1) = (a_1'a_2'-1)/(a_1'-1, a_2'-1)$. Hence, $(a_1-1, a_2-1) = (a_1'-1, a_2'-1)$. From Table 1, this is $n_0' = n_0$. Note that $\gamma_0' = \gamma_0$, i.e., $\alpha_0' = \alpha_0$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_0', \alpha_1' = v_2'\gamma_0', \alpha_2' = v_1'\gamma_0'\} = \{1, \alpha_0 = \gamma_0, \alpha_1 = v_2\gamma_0, \alpha_2 = v_1\gamma_0\}.$$

Note that $n_0' = n_0$, $n_1' = n_2' = n_1 = n_2 = 1$. By the second part of Lemma 3.4, we have either $v_1' = v_1$, $v_2' = v_2$ or $v_1' = v_2$, $v_2' = v_1$. Thus, either $w_1' = w_1$, $w_2' = w_2$ or $w_1' = w_2$, $w_2' = w_1$, i.e.,

$$\frac{a_1'a_2'-1}{a_2'-1} = \frac{a_1a_2-1}{a_2-1} \quad \text{and} \quad \frac{a_1'a_2'-1}{a_1'-1} = \frac{a_1a_2-1}{a_1-1}$$

or

$$\frac{a_1'a_2'-1}{a_2'-1} = \frac{a_1a_2-1}{a_1-1} \quad \text{and} \quad \frac{a_1'a_2'-1}{a_1'-1} = \frac{a_1a_2-1}{a_2-1}.$$

These together with $a_1'a_2'-1 = a_1a_2-1$ imply that $a_1 = a_1'$, $a_2 = a_2'$ or $a_1 = a_2'$, $a_2 = a_1'$, as required. □

Theorem 3.31. *Suppose that* $f = z_0^{a'_0} + z_0 z_1^{a'_1} + z_1 z_2^{a'_2}$ *and* $g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2}$ *are of type IV. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if we have* $a_i = a'_i$, $i = 0, 1, 2$.

Proof. From Table 2, we have $u'_0/u'_1/u'_2$ and $u_0/u_1/u_2$. By Lemma 3.5, we have $u'_i = u_i$, $i = 0, 1, 2$. We consider the following subcases.

(α). $u'_2 = u_2 > u'_1 = u_1$. By (ii) of Theorem 2.3, we have $1 - w'_2 = 1 - w_2$, i.e., $w'_2 = w_2$. Note $w'_0 = u'_0 = u_0 = w_0$. Thus, from $(1 - w'_0)(1 - w'_1)(1 - w'_2) = (1 - w_0)(1 - w_1)(1 - w_2)$ we have $w'_1 = w_1$. So we have

$$a'_0 = a_0, \qquad \frac{a'_0 a'_1}{a'_0 - 1} = \frac{a_0 a_1}{a_0 - 1}, \qquad \frac{a'_0 a'_1 a'_2}{a'_0 a'_1 - a'_0 + 1} = \frac{a_0 a_1 a_2}{a_0 a_1 - a_0 + 1}.$$

These imply that $a'_i = a_i$, $i = 0, 1, 2$, as required.

(β). $u'_2 = u_2 = u'_1 = u_1$. In this case, $\gamma'_2 = \gamma_2 = 1$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha'_0 = \gamma'_2 = 1, \alpha'_1 = v'_2, \alpha'_2 = v'_1\gamma'_2 = v'_1\} = \{1, \alpha_0 = \gamma_2 = 1, \alpha_1 = v_2, \alpha_2 = v_1\gamma_2 = v_1\}.$$

Note that $n'_1 = n'_2 = n_1 = n_2 = 1$. By the second part of Lemma 3.4, we have either $v'_1 = v_1$, $v'_2 = v_2$ or $v'_1 = v_2$, $v'_2 = v_1$. If $v'_1 = v_1$, $v'_2 = v_2$, we have $w'_0 = w_0$, $w'_1 = w_1$, $w'_2 = w_2$. The same argument as in case (α) applies. If $v'_1 = v_2$, $v'_2 = v_1$, then $w'_1 = w_2$, $w'_2 = w_1$. By (ii) of Theorem 2.3, we have $w'_0 = w_0$. From Table 2, $u'_1 = u'_2$ implies that $a'_2(a'_1, a'_0 - 1)/(a'_0 a'_1 - a'_0 + 1)$. Similarly, $u_1 = u_2$ implies $a_2(a_1, a_0 - 1)/(a_0 a_1 - a_0 + 1)$. Hence $v'_2 = (a'_0 a'_1 - a'_0 + 1)/a'_2(a'_1, a_0 - 1)$ and $v_2 = (a_0 a_1 - a_0 + 1)/a_2(a_1, a_0 - 1)$. Now $v_1 = v'_2$ implies $(a_0 - 1)/(a_0 - 1, a_1) = (a'_0 a'_1 - a'_0 + 1)/a'_2(a'_1, a'_0 - 1)$. Note that $(a'_1, a'_0 - 1) = (a'_0 a'_1 - a'_0 + 1, a'_0 - 1)$ and $a'_0 = a_0$. We see that $(v'_2, a_0 - 1) = 1$. This implies that $(a_0 - 1)/(a_0 - 1, a_1) = 1$, i.e., $(a_0 - 1)/a_1$. Similarly, $v_2 = v'_1$ implies $(a'_0 - 1)/a'_1$. So we have $v'_1 = v_1 = 1$. Hence, $w'_1 = w_1$. It follows that $a'_1 = a_1$ as in case (α). So $a'_2 = a_2$ follows. □

Theorem 3.32. *Suppose that* $f = z_0^{a'_0} z_1 + z_1^{a'_1} z_2 + z_0 z_2^{a'_2}$ *and* $g = z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2}$ *are of type V. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$

have the same topological type if and only if (1) $a'_0 = a_{i_0}$, $a'_1 = a_{i_1}$, $a'_2 = a_{i_2}$ *where* $\{i_0, i_1, i_2\}$ *is an even permutation of* $\{0, 1, 2\}$, *or*

$$(2)\qquad \left\{\begin{array}{l} f = z_1 z_0^{\frac{a_1(a_0a_2-a_0+1)}{a_1a_2-a_2+1}} + z_2 z_1^{\frac{a_0(a_1a_2-a_2+1)}{a_0a_1-a_1+1}} + z_0 z_2^{\frac{a_2(a_0a_1-a_1+1)}{a_0a_2-a_0+1}} \\ g = z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} \end{array}\right\}$$

or

$$(3)\qquad \left\{\begin{array}{l} f = z_1 z_0^{\frac{a_0(a_1a_2-a_2+1)}{a_0a_1-a_1+1}} + z_2 z_1^{\frac{a_2(a_0a_1-a_1+1)}{a_0a_2-a_0+1}} + z_0 z_2^{\frac{a_1(a_0a_2-a_0+1)}{a_1a_2-a_1+1}} \\ g = z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} \end{array}\right\}$$

or

$$(4)\qquad \left\{\begin{array}{l} f = z_1 z_0^{\frac{a_2(a_0a_1-a_1+1)}{a_0a_2-a_0+1}} + z_2 z_1^{\frac{a_1(a_0a_2-a_0+1)}{a_1a_2-a_2+1}} + z_0 z_2^{\frac{a_0(a_1a_2-a_2+1)}{a_0a_1-a_1+1}} \\ g = z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} \end{array}\right\}$$

with all the exponents being integers.

Proof. From Table 2, we have $u_0 = u_1 = u_2$, $u'_0 = u'_1 = u'_2$. By Lemma 3.5, we have $u_0 = u_1 = u_2 = u'_0 = u'_1 = u'_2$. Set $k' = (a'_0a'_1 - a'_1 + 1, a'_1a'_2 - a'_2 + 1, a'_0a'_2 - a'_0 + 1)$ and $k = (a_0a_1 - a_0 + 1, a_1a_2 - a_2 + 1, a_0a_2 - a_0 + 1)$. Then $u'_0 = (a'_0a'_1a'_2 + 1)/k' = u_0 = (a_0a_1a_2 + 1)/k$. Comparing the degrees of ζ_f and ζ_g, we see $a'_0a'_1a'_2 + 1 = a_0a_1a_2 + 1$, and hence $k' = k$. By Lemma 3.4 and Table 1, we have $v'_0 = v_{i_0}$, $v'_1 = v_{i_1}$, $v'_2 = v_{i_2}$ where $\{i_0, i_1, i_2\} = \{0, 1, 2\}$. There are six subcases.

(i). $v'_i = v_i$, $i = 0, 1, 2$. Then from Table 2 and $k' = k$, we have

$$a_1a_2 - a_2 + 1 = a'_1a'_2 - a'_2 + 1, \qquad a_0a_2 - a_0 + 1 = a'_0a'_2 - a'_0 + 1,$$
$$a_0a_1 - a_1 + 1 = a'_0a'_1 - a'_1 + 1.$$

Since

$$a'_2(a'_0a'_1 - a'_1 + 1) + (a'_1a'_2 - a'_2 + 1) = a'_0a'_1a'_2 + 1,$$
$$a_2(a_0a_1 - a_1 + 1) + (a_1a_2 - a_2 + 1) = a_0a_1a_2 + 1,$$

one sees that $a'_2 = a_2$. Similarly $a'_0 = a_0$, $a'_1 = a_1$. We are in case (1) of the proposition.

(ii). $v'_0 = v_1$, $v'_1 = v_2$, $v'_2 = v_0$. Replace a_0 by a_1, a_1 by a_2, a_2 by a_0 in the argument of (i). Then a similar argument gives $a'_0 = a_1$, $a'_1 = a_2$, $a'_2 = a_0$. This leads to case (1) of the theorem again.

(iii). $v'_0 = v_2$, $v'_1 = v_0$, $v'_2 = v_1$. Replace a_0 by a_2, a_1 by a_0, a_2 by a_1 in the argument of (i). Then a similar argument gives $a'_0 = a_2$, $a'_1 = a_0$, $a'_2 = a_1$. This leads to case (1) of the theorem.

(iv). $v'_0 = v_0$, $v'_1 = v_2$, $v'_2 = v_1$. From Table 2 and $k' = k$, we have

$$a'_1a'_2 - a'_2 + 1 = a_1a_2 - a_2 + 1, \qquad a'_0a'_2 - a'_0 + 1 = a_0a_1 - a_1 + 1,$$
$$a'_0a'_1 - a'_1 + 1 = a_0a_2 - a_0 + 1.$$

Note that $a'_1(a'_0a'_2 - a'_0 + 1) + (a'_0a'_1 - a'_1 + 1) = a'_0a'_1a'_2 + 1$. Since $a'_0a'_1a'_2 + 1 = a_0a_1a_2 + 1$, we have $a'_1(a_0a_1 - a_1 + 1) + (a_0a_2 - a_0 + 1) = a_0a_1a_2 + 1$. Hence, $a'_1(a_0a_1 - a_1 + 1) = a_0(a_1a_2 - a_2 + 1)$. A similar argument shows that $a'_0(a_1a_2 - a_2 + 1) = a_1(a_0a_2 - a_0 + 1)$ and $a'_2(a_0a_2 - a_0 + 1) = a_2(a_0a_1 - a_1 + 1)$. These give us

$$a'_0 = \frac{a_1(a_0a_2 - a_0 + 1)}{a_1a_2 - a_2 + 1}, \qquad a'_1 = \frac{a_0(a_1a_2 - a_2 + 1)}{a_0a_1 - a_1 + 1},$$
$$a'_2 = \frac{a_2(a_0a_1 - a_1 + 1)}{a_0a_2 - a_0 + 1}.$$

This leads us to case (2) of the proposition.

(v). $v'_0 = v_2$, $v'_1 = v_1$, $v'_2 = v_0$. A similar argument to case (iv) gives

$$a'_0 = \frac{a_0(a_1a_2 - a_2 + 1)}{a_0a_1 - a_1 + 1}, \qquad a'_1 = \frac{a_2(a_0a_1 - a_1 + 1)}{a_0a_2 - a_0 + 1},$$
$$a'_2 = \frac{a_1(a_0a_2 - a_0 + 1)}{a_1a_2 - a_2 + 1}.$$

This leads to case (3) of the theorem.

(vi). $v'_0 = v_1$, $v'_1 = v_0$, $v'_2 = v_2$. Again, a similar argument to case (iv) gives

$$a'_0 = \frac{a_2(a_0a_1 - a_1 + 1)}{a_0a_2 - a_0 + 1}, \qquad a'_1 = \frac{a_1(a_0a_2 - a_0 + 1)}{a_1a_2 - a_2 + 1},$$
$$a'_2 = \frac{a_0(a_1a_2 - a_2 + 1)}{a_0a_1 - a_1 + 1}.$$

This leads to case (4) of the theorem. □

Theorem 3.33. *Suppose that* $f = z_0^{a_0'} + z_0 z_1^{a_1'} + z_0 z_2^{a_2'} + z_1^{b_1'} z_2^{b_2'}$ *and* $g = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$ *are of type VI. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if* $a_0' = a_0$ *and either* $a_1' = a_1$, $a_2' = a_2$ *or* $a_1' = a_2$, $a_2' = a_1$.

Proof. From Table 2, we have u_0/u_1, u_0/u_2 and u_0'/u_1', u_0'/u_2'. By Lemma 3.5, we have $a_0' = u_0' = u_0 = a_0$, and hence $w_0' = w_0$ since $v_0' = v_0 = 1$. By Lemma 3.2 and Lemma 3.3 we have either (i) $u_i' = u_1$, $u_2' = u_2$ or (ii) $u_1' = u_2$, $u_2' = u_1$.

In case (i) we consider two subcases:

(α). $u_1' = u_1 \neq u_2' = u_2$. In this case, by (ii) of Theorem 2.3, we have $w_1' = w_1$, $w_2' = w_2$ and $w_0' = w_0$, i.e.,

$$a_0' = a_0, \qquad \frac{a_0 a_1}{a_0 - 1} = \frac{a_0' a_1'}{a_0' - 1}, \qquad \frac{a_0 a_2}{a_0 - 1} = \frac{a_0' a_2'}{a_0' - 1}.$$

These imply that $a_1' = a_1$, $a_2' = a_2$ as required.

(β). $u_1 = u_1' = u_2 = u_2'$. Thus, $\gamma_1 = \gamma_2 = \gamma_1' = \gamma_2' = 1$. From Lemma 3.4 and Table 1, we have

$$\begin{aligned} &\{1, \alpha_0' = \gamma_1' = 1, \alpha_1' = \gamma_2' = 1, \alpha_2' = v_1'\gamma_2' = v_1', \alpha_3' = v_2'\gamma_1' = v_2'\} \\ &\quad = \{1, \alpha_0 = \gamma_1 = 1, \alpha_1 = \gamma_2 = 1, \alpha_2 = v_1\gamma_2 = v_1, \alpha_3 = v_2\gamma_1 = v_2\}. \end{aligned}$$

Note that $n_2' = n_3' = n_2 = n_3 = 1$. By the second part of Lemma 3.4, we have either $v_1 = v_1'$, $v_2' = v_2$ or $v_1' = v_2$, $v_2' = v_1$. If $v_1' = v_1$, $v_2' = v_2$, the same argument as in (α) applies. If $v_1' = v_2$, $v_2' = v_1$, we have $w_1 = w_2'$, $w_2 = w_1'$. Then a similar argument gives $a_1 = a_2'$, $a_2 = a_1'$ as required.

In case (ii) we have $u_1' = u_2$, $u_2' = u_1$. Interchanging the indices 1 and 2 in the argument of (i), a similar argument leads to the required results of the theorem. □

Theorem 3.34. *Suppose that* $f = z_0^{a_0'} z_1 + z_0 z_1^{a_1'} + z_0 z_2^{a_2'} + z_1^{b_1'} z_2^{b_2'}$ *and* $g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$ *are of type VII. Then* $(f^{-1}(0), 0)$ *and* $(g^{-1}(0), 0)$ *have the same topological type if and only if*

(1) $$a_i' = a_i, \qquad i = 0, 1, 2$$

or

$$\left\{ \begin{array}{l} f = z_0^{a_1} z_1 + z_0 z_1^{a_0} + z_0 z_2^{\frac{a_0 a_2 (a_1 - 1)}{a_1 (a_0 - 1)}} + z_1^{b'_1} z_2^{b'_2} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{array} \right\} \tag{2}$$

with $a_1(a_0-1)/a_0a_2(a_1-1)$, *or*

$$\left\{ \begin{array}{l} f = z_0^{\frac{a_2(a_0a_1-1)-a_1(a_0-1)}{a_2(a_0-1)}} z_1 + z_0 z_1^{\frac{a_0 a_2 (a_1 - 1)}{a_1 (a_0 - 1)}} + z_0 z_2^{a_0} + z_1^{b'_1} z_2^{b'_2} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{array} \right\} \tag{3}$$

where all the exponents are integers and $a_2/(a_1(a_0-1)/(a_0-1, a_1-1))$, *or*

(4)

$$\left\{ \begin{array}{l} f = z_0^{a_2} z_1 + z_0 z_1^{\frac{a_2(a_0a_1-1)-a_1(a_0-1)}{a_2(a_1-1)}} + z_0 z_2^{\frac{a_2(a_0a_1-1)-a_1(a_0-1)}{a_2(a_0-1)}} + z_1^{b'_1} z_2^{b'_2} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{array} \right\}$$

where all the exponents are integers and $a_2/(a_1(a_0-1)/(a_0-1, a_1-1))$, *or*

$$\left\{ \begin{array}{l} f = z_0^{\frac{a_2(a_0a_1-1)-a_1(a_0-1)}{a_2(a_1-1)}} z_1 + z_0 z_1^{a_2} + z_0 z_2^{a_1} + z_1^{b'_1} z_2^{b'_2} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{array} \right\} \tag{5}$$

where all exponents are integers and $a_2/(a_1(a_0-1)/(a_0-1, a_1-1))$, *or*

(6)

$$\left\{ \begin{array}{l} f = z_0^{\frac{a_0 a_2 (a_1 - 1)}{a_1 (a_0 - 1)}} z_1 + z_0 z_1^{\frac{a_2(a_0a_1-1)-a_1(a_0-1)}{a_2(a_0-1)}} + z_0 z_2^{\frac{a_2(a_0a_1-1)-a_1(a_0-1)}{a_2(a_1-1)}} + z_1^{b'_1} z_2^{b'_2} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{array} \right\}$$

where all exponents are integers and $a_2/(a_1(a_0-1)/(a_0-1, a_1-1))$.

Proof. From Table 2, we have $u_0 = u_1/u_2$ and $u'_0 = u'_1/u'_2$. By Lemma 3.2, Lemma 3.3 and Lemma 3.5, we have $u_0 = u_1 = u'_0 = u'_1/u_2 = u'_2$. We consider two subcases.

(i). $u_2' = u_2 > u_0 = u_1 = u_0' = u_1'$. By (ii) of Theorem 2.3, we have $(1 - w_2) = (1 - w_2')$, i.e., $w_2 = w_2'$ and, hence, $v_2 = v_2'$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_2', \alpha_1' = v_2', \alpha_2' = v_1'\gamma_2', \alpha_3' = v_0'\gamma_2'\}$$
$$= \{1, \alpha_0 = \gamma_2, \alpha_1 = v_2, \alpha_2 = v_1\gamma_2, \alpha_3 = v_0\gamma_2\}.$$

Note that $\gamma_2 = \gamma_2' > 1$ and $(\gamma_2, v_2) = (\gamma_2', v_2') = 1$. We have

$$\{1, \alpha_0' = \gamma_2', \alpha_2' = v_1'\gamma_2', \alpha_3' = v_0'\gamma_2'\} = \{1, \alpha_0 = \gamma_2, \alpha_2 = v_1\gamma_2, \alpha_3 = v_0\gamma_2\}.$$

By Lemma 3.4, we have either $v_0' = v_0$, $v_1' = v_1$ or $v_0' = v_1$, $v_1' = v_0$.

If $v_0' = v_0$, $v_1' = v_1$, we have $w_0' = w_0$, $w_1' = w_1$, $w_2' = w_2$, i.e.,

$$\frac{a_0'a_1' - 1}{a_1' - 1} = \frac{a_0a_1 - 1}{a_1 - 1}, \qquad \frac{a_0'a_1' - 1}{a_0' - 1} = \frac{a_0a_1 - 1}{a_0 - 1},$$
$$\frac{a_2'(a_0'a_1' - 1)}{a_1'(a_0' - 1)} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}.$$

These imply $a_i' = a_i$, $i = 0, 1, 2$. We are in case (1) of the theorem.

If $v_0' = v_1$, $v_1' = v_0$, we have $w_0' = w_1$, $w_1' = w_0$, $w_2' = w_2$, i.e.,

$$\frac{a_0'a_1' - 1}{a_1' - 1} = \frac{a_0a_1 - 1}{a_0 - 1}, \qquad \frac{a_0'a_1' - 1}{a_0' - 1} = \frac{a_0a_1 - 1}{a_1 - 1},$$
$$\frac{a_2'(a_0'a_1' - 1)}{a_1'(a_0' - 1)} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}.$$

These imply $a_0' = a_1$, $a_1' = a_0$, $a_2' = a_0a_2(a_1 - 1)/a_1(a_0 - 1)$. We are in case (2) of the theorem.

(ii). $u_0 = u_1 = u_2 = u_0' = u_1' = u_2'$. Thus, $\gamma_2 = \gamma_2' = 1$. By Lemma 3.4 and Table 1, we have

$$\{1, \alpha_0' = \gamma_2' = 1, \alpha_1' = v_2', \alpha_2' = v_1', \alpha_3' = v_0'\}$$
$$= \{1, \alpha_0 = \gamma_2 = 1, \alpha_1 = v_2, \alpha_2 = v_1, \alpha_3 = v_0\}.$$

Note that $n_1' = n_2' = n_3' = n_1 = n_2 = n_3 = 1$. From the second part of Lemma 3.4, we have $v_0' = v_{i_0}$, $v_1' = v_{i_1}$, $v_2' = v_{i_2}$ where $\{i_0, i_1, i_2\} = \{0, 1, 2\}$. We have six subcases:

(a) $v_i' = v_i$, $i = 0, 1, 2$. Then $w_i' = w_i$, $i = 0, 1, 2$. This leads to case (1) of the theorem again.

(b) $v_0' = v_1$, $v_1' = v_0$, $v_2' = v_2$. Then $w_0' = w_1$, $w_1' = w_0$, $w_2' = w_2$. This leads to case (2) of the proposition again.

(c) $v_0' = v_1$, $v_1' = v_2$, $v_2' = v_0$. Then $w_0' = w_1$, $w_1' = w_2$, $w_2' = w_0$, i.e.,

$$\frac{a_0'a_1' - 1}{a_1' - 1} = \frac{a_0a_1 - 1}{a_0 - 1}, \qquad \frac{a_0'a_1' - 1}{a_0' - 1} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)},$$
$$\frac{a_2'(a_0'a_1' - 1)}{a_1'(a_0' - 1)} = \frac{a_0a_1 - 1}{a_1 - 1}.$$

These imply

$$a_0' = \frac{a_2(a_0a_1 - 1) - a_1(a_0 - 1)}{a_2(a_0 - 1)}, \qquad a_1' = \frac{a_0a_2(a_1 - 1)}{a_1(a_0 - 1)}, \qquad a_2' = a_0.$$

This leads to case (3) of the theorem.

(d) $v_0' = v_2$, $v_1' = v_0$, $v_2' = v_1$. Then $w_0' = w_2$, $w_1' = w_0$, $w_2' = w_1$, i.e.,

$$\frac{a_0'a_1' - 1}{a_1' - 1} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}, \qquad \frac{a_0'a_1' - 1}{a_0' - 1} = \frac{a_0a_1 - 1}{a_1 - 1},$$
$$\frac{a_2'(a_0'a_1' - 1)}{a_1'(a_0' - 1)} = \frac{a_0a_1 - 1}{a_0 - 1}.$$

These imply

$$a_0' = a_2, \qquad a_1' = \frac{a_2(a_0a_1 - 1) - a_1(a_0 - 1)}{a_2(a_1 - 1)},$$
$$a_2' = \frac{a_2(a_0a_1 - 1) - a_1(a_0 - 1)}{a_2(a_0 - 1)}.$$

This leads to case (4) of the theorem.

(e) $v_0' = v_0$, $v_1' = v_2$, $v_2' = v_1$. Then $w_0' = w_0$, $w_1' = w_2$, $w_2' = w_1$, i.e.,

$$\frac{a_0'a_1' - 1}{a_1' - 1} = \frac{a_0a_1 - 1}{a_1 - 1}, \qquad \frac{a_0'a_1' - 1}{a_0' - 1} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)},$$
$$\frac{a_2'(a_0'a_1' - 1)}{a_1'(a_0' - 1)} = \frac{a_0a_1 - 1}{a_0 - 1}.$$

These imply

$$a_0' = \frac{a_2(a_0a_1 - 1) - a_1(a_0 - 1)}{a_2(a_1 - 1)}, \qquad a_1' = a_2, \qquad a_2' = a_1.$$

This leads to case (5) of the theorem.

(f) $v_0' = v_2$, $v_1' = v_1$, $v_2' = v_0$. Then $w_0' = w_2$, $w_1' = w_1$, $w_2' = w_0$, i.e.,

$$\frac{a_0'a_1' - 1}{a_1' - 1} = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}, \qquad \frac{a_0'a_1' - 1}{a_0' - 1} = \frac{a_0a_1 - 1}{a_0 - 1},$$
$$\frac{a_2'(a_0'a_1' - 1)}{a_1'(a_0' - 1)} = \frac{a_0a_1 - 1}{a_1 - 1}.$$

These imply

$$a_0' = \frac{a_0a_2(a_1 - 1)}{a_1(a_0 - 1)}, \qquad a_1' = \frac{a_2(a_0a_1 - 1) - a_1(a_0 - 1)}{a_2(a_0 - 1)},$$
$$a_2' = \frac{a_2(a_0a_1 - 1) - a_1(a_0 - 1)}{a_2(a_1 - 1)}.$$

This leads to case (6) of the theorem. □

The end of the proof of Theorems 3.7–3.34. We only need to observe that the f and g listed in Theorems 3.7–3.34 have the same weights. It follows from the Theorem in Section 3 [**21**] that $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ have the same topological type.

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