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ALMOST PERIODIC FUNCTIONALS ON BANACH ALGEBRAS

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Introduction. Let A be a (real or complex) Banach algebra, A^* its dual and A_1 its closed unit ball. For f in A^* and a in A, the functional fa in A^* is defined by $fa(x) = f(ax)$. Let $H(f) = \{fa : a \in A_1\}$. The functional f is said to be (weakly) almost periodic on A if the set $H(f)$ is relatively (weakly) compact in A^*. The spaces of ap (almost periodic) and wap (weakly almost periodic) functionals on A will be denoted respectively by ap (A) and wap (A). Both ap (A) and wap (A) are (norm) closed subspaces of A^* and ap $(A) \subset$ wap (A). For the representation theory of Banach algebras, for Arens regularity theory and for independent interest (see, for example, [**11, 19**]) the following problems are relevant:

(i) Determine when ap (A) and wap (A) are nontrivial,

(ii) Determine when ap $(A) =$ wap (A),

(iii) Determine ap (A) and wap (A) for known classes of Banach algebras,

(iv) Determine the behavior of ap (A) under the standard Banach algebra constructions.

It seems to be a difficult problem to characterize when ap (A) (or wap (A)) is nontrivial. The sufficient conditions we give rely either on the existence of multiplicative functionals or on the compactness properties of the associated multiplication operators $L_a(x) = ax$ or $R_a(x) = xa$. In particular, we show that if Φ_A (the set of multiplicative functional on A) separates the points of A and if L_a is weakly compact for some a in A, then fa is ap for every f in A^*. For the second problem, we show that if A has a bounded right (or left) approximate identity and if A is a left (or right) ideal in the Arens second dual A^{**} (with either product), then we have ap $(A) =$ wap (A) if and only if Φ_A separates the points of A. For examples we calculate ap (A) for $C(K)$ (K compact Hausdorff), l^p $(1 \leq p \leq \infty)$, $L^p(G)$ $(1 \leq p < \infty$,

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G compact) and $K(X)$ (the compact operators on a Banach space X). Finally, we establish the expected behavior of ap under continuous homomorphism, quotient, and direct sum. As corollaries to some of our results, we give very short proofs of some known results for Banach algebras. We conclude with some general remarks and conjectures.

1. Notation and preliminaries. Wherever possible, we follow standard notation and terminology. Given Banach spaces X and Y we denote their injective and projective tensor products by $X \overset{\vee}{\otimes} Y$ and $X \overset{\wedge}{\otimes} Y$, respectively (see [**6**, Chapter VIII]). By $L(X,Y)$ and $K(X,Y)$ we denote, respectively, the spaces of bounded linear and compact linear operators $u : X \to Y$. When $X = Y$ we abbreviate to $L(X)$ and $K(X)$, respectively. By c_0 and l^p $(1 \leq p \leq \infty)$ we denote the usual sequence spaces, regarded as Banach algebras with coordinatewise multiplication. We define the l^p-sum of a sequence of Banach spaces (X_n) by

$$\left(\sum_{n=1}^{\infty} \oplus X_n\right)_p = \left\{x = (x_n) : x_n \in X_n \text{ and } ||x||^p = \sum_{n=1}^{\infty} ||x_n||^p\right\}.$$

For $p = 0$ or ∞, the definition is modified in the usual way. The natural duality between X and X^* is denoted by $\langle x, x^* \rangle$. We regard X as naturally embedded into its second dual.

For Banach algebras we use the terminology of [**3**]. For a Banach algebra A and for a in A, the element a is called left (weakly) compact if the left multiplication operator L_a is (weakly) compact. The adjoint of L_a is given by $L_a^*(f) = fa$. A net $(e_\alpha)_{\alpha \in I}$ is an LAI (left approximate identity) if, for each a in A, $||e_\alpha a - a|| \to 0$. Similarly, we define RAI and two-sided AI. We recall that A is Arens regular if the two Arens products on A^{**} coincide (see [**2, 5, 7**]). This is equivalent to the condition that $\operatorname{wap}(A) = A^*$. We also recall [**14**] that f in A^* is wap if and only if, for any two bounded sequences (a_n), (b_m) in A, we have

$$\lim_n \lim_m \langle f, a_n b_m \rangle = \lim_m \lim_n \langle f, a_n b_m \rangle$$

whenever these iterated limits exist.

2. Almost periodic functional on a Banach algebra. Let A be a Banach algebra with LAI $(e_\alpha)_{\alpha\in I}$. We write

$$c(A^*) = \{f \in A^* : ||fe_\alpha - f|| \to 0\}$$

and

$$A^*A = \{fa : f \in A^*, a \in A\}.$$

For f in A^* we define the operator $\Phi_f : A \to A^*$ by $\Phi_f(a) = fa$. Clearly, the operator Φ_f is weakly compact if and only if f is wap on A. Observe that $\Phi_f L_a = \Phi_{fa}$. We recall [**7**, Lemma 3] that the algebra A is a right ideal in its second dual A^{**}, for either Arens product, if and only if L_a is weakly compact for each a in A. Our first lemma compares the subspaces $c(A^*)$, A^*A, ap(A) and wap(A) of A^* for a Banach algebra A with LAI. For a subset D of A^*, $\overline{D}$ denotes the closure of D in the norm topology.

Lemma 2.1. *Let A be a Banach algebra with* LAI $(e_\alpha)_{\alpha\in I}$.

(i) $A^*A \subset c(A^*) \subset \overline{A^*A}$.

(ii) *If $(e_\alpha)_{\alpha\in I}$ is bounded, then A^*A is closed and $A^*A = c(A^*) = \overline{A^*A}$.*

(iii) *If A is a right ideal in its second dual A^{**}, then $\overline{A^*A} \subset$* wap(A)*. Moreover, the equalities* wap$(A) = A^*A = \overline{A^*A}$ *hold if $(e_\alpha)_{\alpha\in I}$ is bounded.*

(iv) *If A is reflexive, then $\overline{A^*A} = A^*$.*

(v) *If each a in A is left compact then $\overline{A^*A} \subset$* ap(A)*. Moreover, if $(e_\alpha)_{\alpha\in I}$ is bounded, then the equalities* ap$(A) = c(A^*) = A^*A$ *hold.*

Proof. (i) Let f be in A^* and a in A. As $(fa)e_\alpha = fae_\alpha$ and

$$||fae_\alpha - fa|| \le ||f||\,||ae_\alpha - a|| \to 0,$$

we see that the inclusion $A^*A \subset c(A^*)$ holds. The inclusion $c(A^*) \subset \overline{A^*A}$ is obvious from the definition of $c(A^*)$.

(ii) If $(e_\alpha)_{\alpha\in I}$ is bounded, then the space A^*A is closed in A^* by [**9**, Section 32.22].

(iii) Assume A is a right ideal in its second dual A^{**}. Then L_{e_α} is weakly compact. For any f in A^* we have $\Phi_{fe_\alpha} = \Phi_f L_{e_\alpha}$ and so fe_α

is in wap(A). For f in $c(A^*)$ we have $||\Phi_{fe_\alpha} - \Phi_f|| \leq ||fe_\alpha - f|| \to 0$. It follows that $c(A^*) \subset$ wap(A). Since $A^*A \subset c(A^*)$ and wap(A) is closed in A^*, we have $\overline{A^*A} \subset$ wap(A). Now suppose that $(e_\alpha)_{\alpha \in I}$ is bounded. Let f be in wap(A) and a^{**} in A^{**}. By Goldstine's Theorem there is a bounded net (a_β) in A with $a_\beta \to a^{**}$ in the weak* topology. Since (e_α) is a bounded LAI and f is wap, we get

$$\begin{aligned}\lim_\alpha \langle fe_\alpha, a^{**}\rangle &= \lim_\alpha \lim_\beta \langle f, e_\alpha a_\beta\rangle = \lim_\beta \lim_\alpha \langle f, e_\alpha a_\beta\rangle \\ &= \lim_\beta \langle f, a_\beta\rangle = \langle f, a^{**}\rangle\end{aligned}$$

so that $fe_\alpha \to f$ in the weak topology. This proves that f is in A^*A, and we have wap$(A) = c(A^*) = A^*A$.

(iv) Assume that A is reflexive, so that, for each f in A^*, $fe_\alpha \to f$ in the weak topology. Hence, $A^* \subset \overline{A^*A}$ and so $A^* = \overline{A^*A}$.

(v) If each a in A is compact, then as in (iii) above, we get $\overline{A^*A} \subset$ ap(A). The rest is clear. □

For a Banach algebra, commutative or otherwise, we denote by Φ_A the set of (nonidentically zero) multiplicative functionals. For f in Φ_A and a in A, we have of course that $fa = f(a)f$. The next lemma was somewhat of a surprise to us.

Lemma 2.2. *Assume that Φ_A separates the points of A. Let a be a weakly compact element of A and let f be in A^*. Then the functional fa is almost periodic.*

Proof. Actually more is true: fa is in $\overline{\text{span}}\,\Phi_A$. Assume not. Then by the Hahn-Banach theorem there is a^{**} in A^{**} such that $\langle fa, a^{**}\rangle$ is nonzero but $\langle g, a^{**}\rangle = 0$ for all g in span Φ_A. In particular, for g in Φ_A, we have $ga = g(a)g$ in span Φ_A and so $\langle ga, a^{**}\rangle = \langle g, aa^{**}\rangle = 0$. But aa^{**} is in A since a is weakly compact. Since Φ_A separates the points of A, it follows that $aa^{**} = 0$. This contradicts the fact that $\langle fa, a^{**}\rangle \neq 0$, and so fa is in $\overline{\text{span}}\,\Phi_A$. □

The main result of this section is the following theorem.

Theorem 2.3. *Let A be a Banach algebra with a bounded* LAI (e_α)*. Assume that A is a right ideal in its second dual. Then the equality $\overline{\operatorname{span}}\,\Phi_A = \operatorname{wap}(A)$ holds if and only if Φ_A separates the points of A. When this happens we have*

$$\overline{\operatorname{span}}\,\Phi_A = \operatorname{wap}(A) = A^*A = c(A^*) = \operatorname{ap}(A).$$

Proof. The inclusions $\overline{\operatorname{span}}\,\Phi_A \subset \operatorname{ap}(A) \subset \operatorname{wap}(A)$ are obvious. Suppose that Φ_A separates the points of A. Let f be in $\operatorname{wap}(A)$. By the preceding lemma, fe_α is in $\overline{\operatorname{span}}\,\Phi_A$ for each α in I. The proof of Lemma 2.1 (iii) shows that $fe_\alpha \to f$ weakly. Hence, f is also in $\overline{\operatorname{span}}\,\Phi_A$, and the equality $\overline{\operatorname{span}}\,\Phi_A = \operatorname{wap}(A)$ holds.

Conversely, suppose that this last equality holds. Let a be in A such that $\langle g, a\rangle = 0$ for each g in Φ_A. Then $\langle f, a\rangle = 0$ for each f in $\operatorname{wap}(A)$. Since e_α is weakly compact, for each f in A^*, we have fe_α is in $\operatorname{wap}(A)$ and so $\langle f, e_\alpha a\rangle = 0$. This gives $e_\alpha a = 0$ for each α and hence $a = 0$, i.e., Φ_A separates the points of A. The rest is clear from Lemma 2.1. □

We note that all the hypotheses of the above theorem are satisfied when $A = L^1(G)$ with G a compact group. In this case the conclusion of the theorem is that the span of the characters of G is dense in $C(G) = \operatorname{wap}(A)$ ([**4**], see also [**18**] and the references there). This is essentially the Peter-Weyl theorem ([**9**, Section 27.40]).

For the next result we recall that a Banach space X has the DPP (Dunford-Pettis property) if any weakly compact linear operator u from X to another Banach space Y maps weakly compact subsets of X into compact subsets of Y. For example, for any compact space K and any measure space (Ω, Σ, μ), the Banach spaces $C(K)$ and $L^1(\mu)$ have the DPP (see [**8**]). The next result points out one class of Banach algebras for which the equality $\operatorname{wap}(A) = \operatorname{ap}(A)$ holds.

Theorem 2.4. *Let A be a Banach algebra with a bounded* LAI *(e_α). If A has the* DPP *and is a right ideal in its second dual, then $\operatorname{wap}(A) = \operatorname{ap}(A)$.*

Proof. By Lemma (2.1) (iii), the equalities $\text{wap}\,(A) = c(A^*) = A^*A$ hold. Let f be in $\text{wap}\,(A)$. Since L_{e_α} and Φ_f are weakly compact, by the DPP the operator $\Phi_{fe_\alpha} = \Phi_f L_{e_\alpha}$ is compact. Since $||\Phi_{fe_\alpha} - \Phi_f|| \leq ||fe_\alpha - f|| \to 0$, we conclude that f is in $\text{ap}\,(A)$. Thus $\text{wap}\,(A) = \text{ap}\,(A)$. □

As an application of this theorem we give a short proof of the following well-known result, see [**18**] for references and another proof.

Proposition 2.5. *Let $A = L^1(G)$ where G is a locally compact group. Then A is a right ideal in its second dual if and only if G is compact.*

Proof. Assume A is a right ideal in its second dual. By the preceding theorem, $\text{ap}\,(A) = \text{wap}\,(A)$, i.e., $\text{WAP}\,(G) = \text{AP}\ (G)$ (see [**17**,**4**]). It follows from [**4**, Section 2.24] that G is compact. The converse is immediate (see [**18**]). □

3. The space $\text{ap}\,(A)$ **for some classical Banach algebras.** Recall that on c_0, l^p $(1 \leq p \leq \infty)$ the multiplication is defined coordinatewise.

Proposition 3.1. *The following equalities hold.*

(i) $\text{ap}\,(c_0) = l^1$.

(ii) $\text{ap}\,(l^1) = c_0$.

(iii) $\text{ap}\,(l^p) = l^q$ $(1 < p < \infty, p^{-1} + q^{-1} = 1)$.

Proof. (i) c_0 has a bounded approximate identity, the DPP, and it is an ideal in its second dual and is Arens regular. Hence, $l^1 = \text{wap}\,(A) = \text{ap}\,(A)$.

(ii) l^1 has an approximate identity and each element of l^1 is compact. Hence, by Lemma 2.1 (v), with $A = l^1$ we have $\overline{A^*A} \subset \text{ap}\,(A)$. But $\overline{A^*A} = c_0$ and so $c_0 \subset \text{ap}\,(l^1)$. To prove the reverse inclusion we first recall that a bounded subset H of c_0 is relatively compact if and only if $\lim_{n\to\infty} \sup_{x\in H} |x_n| = 0$. Now let f be in $\text{ap}\,(l^1)$. Then the set $H = \{fa : a \in l^1, ||a|| \leq 1\}$ is contained in c_0 and is relatively compact.

Since $\sup\{|(fa)_n| : ||a||_1 \leq 1\} = |f_n|$, it follows that $f_n \to 0$, i.e., f is in c_0, as required.

(iii) For $1 < p < \infty$, each element of l^p is compact. It follows from Lemma 2.1 (iv), (v) that $\mathrm{ap}\,(l^p) = l^q$. □

Proposition 3.2. *Let G be a compact group, $1 < p < \infty$ and let $A = L^p(G)$ with the usual convolution product. Then* $\mathrm{ap}\,(A) = L^q(G)$.

Proof. Each element of $L^p(G)$ is compact and $L^p(G)$ has an (unbounded) approximate identity. Apply Lemma 2.1 (iv),(v). □

Proposition 3.3. *Let X be an infinite dimensional Banach space with the approximation property. Then* $\mathrm{ap}\,(K(X)) = \{0\}$.

Proof. If X is not reflexive, then by [**19**, Theorem 3], $\mathrm{wap}\,(K(X)) = \{0\}$ and so $\mathrm{ap}\,(K(X)) = \{0\}$. Suppose now that X is reflexive. Then $K(X) = X^* \overset{\vee}{\otimes} X$ and $K(X)^* = X \overset{\wedge}{\otimes} X^*$ by [**6**, VIII.4.7]. Let f be in $K(X)^*$. Suppose first that $f = x \otimes x^*$ with $||x|| \leq 1$, $||x^*|| \leq 1$. We easily check that $\{fu : u \in K(X), ||u|| \leq 1\} = x \otimes X_1^*$ where X_1^* is the unit ball of X^*. Since X is infinite dimensional, $x \otimes X_1^*$ is not compact. In general, let $f = \sum_{n=1}^{\infty} x_n \otimes x_n^*$ with $\sum_{n=1}^{\infty} ||x_n||\,||x_n^*|| < \infty$. We easily check that the set $\{fu : u \in K(X), ||u|| \leq 1\}$ contains a set of the form $x \otimes X_1^*$ for some x in X. Hence, f is not ap and so $\mathrm{ap}\,(K(X)) = \{0\}$. □

Remarks 3.4. Let X be an infinite dimensional Banach space. It is well known that $K(X)$ has no nonzero compact elements although for each u in $K(X)$ the operator $Tv = uvu$ is compact on $K(X)$ [**1**]. It is also well known that $K(X)$ has no multiplicative functional. In contrast, there exists a reflexive Banach space Y with uncountably many multiplicative functionals on $L(Y)$. For, recently, P. Mankiewicz [**13**] has given an example of a separable reflexive Banach space Y with the approximation property such that there is an epimorphism from $L(Y)$ to $C(\beta\mathbf{N})$ where $\beta\mathbf{N}$ is the Stone-Čech compactification of $\mathbf{N}$. The point here is that all these multiplicative functionals annihilate $K(Y)$. We do not know whether these 2^c functionals span $K(Y)^{\perp}$. If they do, then since $L(Y)^* = Y \overset{\wedge}{\otimes} Y^* + K(Y)^\perp$, Y would furnish the first example of a non-hilbertian reflexive Banach space with $L(Y)$ Arens regular. Examples are given in [**19, 15**] of reflexive Banach spaces X for which $L(X)$ is not Arens regular.

The next proposition which determines ap$\,(C(K))$ is intuitively clear but its proof seems to be surprisingly difficult. Before this statement and proof some preparation is needed. Let K be a compact (Hausdorff) space. We have the usual identification $C(K)^* = M(K)$, where $M(K)$ is the space of regular Borel measures on K and for a in $C(K)$, μ in $M(K)$, $\langle a, \mu \rangle = \int_K a(t)\, d\mu(t)$. It follows that for μ in $M(K)$ the set $H(\mu) = \{\mu a : a \in C(K), ||a|| \leq 1\}$ is just the image in $L^1(\mu)$ of the unit ball of $C(K)$ under the natural injection operator $\iota : C(K) \to L^1(\mu)$. Since μ is finite, the set $H(\mu)$ is relatively norm compact if and only if any sequence (a_n) in the unit ball of $C(K)$ has a μ a.e. convergent subsequence. Therefore, μ, as a functional on $C(K)$, is ap if and only if the natural injection operator ι is compact. Recall also that a measure μ in $M(K)$ is said to be atomic if it is of the form $\mu = \sum_{i=1}^\infty \lambda_i \delta_{t_i}$ with $\sum_{i=1}^\infty |\lambda_i| < \infty$ and $\delta_t(a) = a(t)$. Denote by $M_a(K)$ the subspace of $M(K)$ consisting of all atomic measures. Then $M_a(K)$ is a closed complemented subspace of $M(K)$ and it is isometrically isomorphic to $l^1(K)$.

Proposition 3.5. *We have* ap$\,(C(K)) = M_a(K)$.

Proof. If μ is a point mass, then $L^1(\mu)$ is one-dimensional and so the natural injection operator $\iota : C(K) \to L^1(\mu)$ is compact. Thus, μ is ap and so $M_a(K) \subset$ ap$\,(C(K))$. The proof of the reverse inclusion is given in several steps.

Suppose that $K = [0, 1]$ and μ is Lebesgue measure. The natural injection ι is not compact since, for example, the sequence given by $a_n(t) = \sin(2\pi n t)$ has no a.e.-convergent subsequence. So for this example μ is not ap.

Suppose now that K is dispersed, i.e., K contains no nonempty perfect subset. Then $M(K) = M_a(K)$ [**16**, Theorem 19.7.6] and every μ in $M(K)$ is ap on $C(K)$.

Suppose finally that K is not dispersed and that μ is not a purely atomic measure in $M(K)$. By decomposing μ and using the fact that $M_a(K) \subset \mathrm{ap}\,(C(K))$, we may assume without loss that μ is atomless and positive. Let μ be ap. Then the unit ball of $C(K)$ is relatively compact in $L^1(\mu)$ and so separable. Since $C(K)$ is dense in $L^1(K)$, it follows that $L^1(\mu)$ is separable. By Caratheodory's classification of separable atomless measure algebras, [**12**, p. 121, Theorem 5], $L^1(\mu)$ is isometrically isomorphic to the Lebesgue space $L^1([0,1],m)$. Since K is not dispersed, by [**16**, Theorem 8.5.4] there exists a continuous onto mapping $\theta : K \to [0,1]$ and an induced linear isometry $\theta^* : C([0,1]) \to C(K)$. By simple diagram chasing we see that $\iota : C(K) \to L^1(\mu)$ is compact if and only if $\iota : C([0,1]) \to L^1(m)$ is compact. Our assumption that μ is ap now contradicts the first step in our argument. The proof is complete. □

4. The behavior of ap under standard constructions. We begin with a straightforward lemma.

Lemma 4.1. *Let X, Y be Banach spaces, $\phi : X \to Y$ continuous linear onto with adjoint ϕ^*, and let H be a bounded subset of Y^*. Then H is norm compact if $\phi^*(H)$ is norm compact.*

Proof. Since ϕ is onto it is open and so ϕ^* is one–to–one and bicontinuous onto its range. The result follows. □

Proposition 4.2. *Let A, B be Banach algebras and $\phi : A \to B$ a continuous epimorphism. Then $\phi^*(\mathrm{ap}\,(B)) = \mathrm{ap}\,(A) \cap (\ker\phi)^\perp$.*

Proof. Let g be in $\mathrm{ap}\,(B)$. Clearly, $\phi^*(g)$ is in $(\ker\phi)^\perp$. For a, b in A we have

$$\begin{aligned}\langle \phi^*(g)a, b\rangle &= \langle \phi^*(g), ab\rangle = \langle g, \phi(ab)\rangle = \langle g, \phi(a)\phi(b)\rangle \\ &= \langle g\phi(a), \phi(b)\rangle = \langle \phi^*(g\phi(a)), b\rangle\end{aligned}$$

and so $\phi^*(g)a = \phi^*(g\phi(a))$. Since $\{g\phi(a) : a \in A_1\}$ is relatively compact, we conclude that $\phi^*(g)$ is in $\mathrm{ap}\,(A)$.

For the reverse inclusion, let f be in $\mathrm{ap}\,(A) \cap (\ker\phi)^\perp$. Then $\ker\phi \subset \ker f$ and by Sard's quotient theorem [**10**, p. 176] there is a unique element g in B^* such that $g \circ \phi = f$. A simple computation gives $\phi^*(g) = f$ and so it remains to prove that g is ap. Another simple computation gives $\langle g\phi(x), \phi(y)\rangle = \langle fx, y\rangle$ and so $\phi^*(g\phi(x)) = fx$. Since f is ap, it follows that $\{\phi^*(g\phi(x)) : x \in A_1\}$ is relatively compact and so $\{g\phi(x) : x \in A_1\}$ is relatively compact by Lemma 4.1. Since ϕ is onto, by the open mapping theorem, there exists $\varepsilon > 0$ with $\varepsilon B_1 \subset \phi(A_1)$ and so $\{gb : b \in B_1\} \subset \varepsilon^{-1}\{g\phi(x) : x \in A_1\}$. Hence, g is ap as required. □

Applying the preceding proposition to the quotient mapping, we obtain the following result.

Corollary 4.3. *Let A be a Banach algebra and let M be a closed bi-ideal of A. Then* $\mathrm{ap}\,(A/M) = \mathrm{ap}\,(A) \cap M^{\perp}$.

Now let (A_n) be a sequence of Banach algebras and let $A = (\sum_{n=1}^{\infty} \oplus A_n)_0$. With coordinatewise multiplication A is a Banach algebra and $A^* = (\sum_{n=1}^{\infty} \oplus A_n^*)_1$. In the natural way we identify A_n with a subalgebra of A and A_n with a subspace of A^*.

Proposition 4.4. *If $A = (\sum_{n=1}^{\infty} \oplus A_n)_0$, then* $\mathrm{ap}\,(A) = (\sum_{n=1}^{\infty} \oplus \mathrm{ap}\,(A_n))_1$.

Proof. Given $a = (a_n)$ in A and $f = (f_n)$ in A^*, we have $fa = (f_n a_n)$. It follows that $\mathrm{ap}\,(A) \subset (\sum_{n=1}^{\infty} \oplus \mathrm{ap}\,(A_n))_1$. On the other hand, it is clear that $\mathrm{ap}\,(A_n) \subset \mathrm{ap}\,(A)$ and so the result follows since $\mathrm{ap}\,(A)$ is a closed subpsace of A^*. □

Proposition 4.5. *Let $A = (\sum_{n=1}^{\infty} \oplus A_n)_1$. Assume that, for each n in $\mathbf{N}$ and each f in A_n, $||f|| = \sup\{||fa|| : a \in A_n, ||a|| \le 1\}$. Then* $\mathrm{ap}\,(A) = (\sum_{n=1}^{\infty} \oplus \mathrm{ap}\,(A_n))_0$.

Proof. Note that $A^* = (\sum_{n=1}^{\infty} \oplus A_n^*)_\infty$. For $a = (a_n)$ in A and $f = (f_n)$ in A^* we have $fa = (f_n a_n)$ is in $(\sum_{n=1}^{\infty} \oplus A_n^*)_0$. Let $H_n = \{f_n a_n : a_n \in A_n, ||a_n|| \le 1\}$. Exactly as in the scalar case, $H(f)$ is relatively compact in $(\sum_{n=1}^{\infty} \oplus A_n^*)_0$ if and only if each H_n is relatively compact and $\lim_{n\to\infty} \sup\{||f_n a_n|| : a \in A_1\} = \lim_{n\to\infty} ||f_n|| = 0$. This is the case if and only if f is in $(\sum_{n=1}^{\infty} \oplus \mathrm{ap}\,(A_n))_0$. $\square$

5. Concluding remarks. It follows from Propositions 3.5 and 4.4 that, if A is the c_0-direct sum of a family of C^*-algebras each of which is either finite dimensional or $C(K)$ for some compact dispersed K, then $\mathrm{ap}\,(A) = A^*$. We conjecture that the converse is also true; the difficulty lies in the fact that the dual space of a C^*-algebra does not determine (up to isomorphism) the algebra itself. Can we replace the C^* condition with a geometrical condition that will force the same conclusion? When A is a commutative unital C^*-algebra with $\mathrm{ap}\,(A) = A^*$, then $A = C(K)$ with K dispersed. We conjecture that if A is a function algebra with $\mathrm{ap}\,(A) = A^*$, then the Gelfand space of A must be dispersed.

It is not clear in general what the implication is for the structure of A when A has a rich supply of almost periodic functionals. Certainly A can be infinite dimensional and radical even if $\mathrm{ap}\,(A) = A^*$; simply take A_n to be a two-dimensional algebra with all products zero and A to be the c_0-direct sum of the A_n. Even if A is semisimple the prospects are not encouraging. One might hope that the extreme points of the unit ball of $\mathrm{ap}\,(A)$ would give rise to irreducible (even finite dimensional) modules X_f under the usual construction (see [**3**]). Proposition 3.1 (iii) destroys any such hope. Almost periodicity is a topological property rather than a geometrical property and hence is preserved under equivalent renorming. Even so, it seems unlikely that one could renorm l^p so that the extreme points of its unit ball are just the multiples of point masses. It may be that any irreducible module associated with an almost periodic functional must be finite dimensional; this is true for all the examples known to us.

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REFERENCES

1. J.C. Alexander, *Compact Banach algebras*, Proc. London Math. Soc. **18** (1968), 1–18.

2. R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839–848.

3. F.F. Bonsall and J. Duncan, *Complete normed algebras*, New York, Springer Verlag, 1974.

4. R.B. Burckel, *Weakly almost periodic functions on semigroups*, New York, Gordon and Breach, 1970.

5. P. Civin and B. Yood, *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math. **11** (1961), 847–870.

6. J. Diestel and J.J. Uhl, Jr., *Vector measures*, Math. Surveys No. **15**, Amer. Math. Soc., Providence, 1977.

7. J. Duncan and S.A.R. Hosseiniun, *The second dual of a Banach algebra*, Proc. Roy. Soc. Edinburgh, Sec. A **84** (1979), 309–325.

8. A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. **5** (1953), 129–173.

9. E. Hewitt and K.A. Ross, *Abstract harmonic analysis*, volume 2, Berlin, Springer-Verlag, 1970.

10. R. Holmes, *Geometric functional analysis and its applications*, New York, Springer-Verlag, 1975.

11. S.A. McKilligan and A.J. White, *Representations of L-algebras*, Proc. London Math. Soc. **25** (1972), 655–674.

12. H. Lacey, *The isometric theory of classical Banach spaces*, New York, Springer-Verlag, 1974.

13. P. Mankiewicz, *A superreflexive Banach space X with $L(X)$ admitting a homomorphism onto the Banach algebra $C(\beta\mathbf{N})$*, Israel J. Math. **65** (1989), 1–16.

14. J.S. Pym, *The convolution of functionals on spaces of bounded functions*, Proc. London Math. Soc. **15** (1965), 84–104.

15. J.S. Pym and A. Ülger, *Arens regularity of inductive limits and related matters*, Quart. J. Math. (Oxford) (2), **40** (1989), 101–109.

16. Z. Semadeni, *Banach spaces of continuous functions*, PWN, Warsaw, 1971.

17. A. Ülger, *Continuity of weakly almost periodic functionals on $L^1(G)$*, Quart. J. Math. (Oxford) (2) **37** (1986), 495–497.

18. ———, *Arens regularity sometimes implies the RNP*, Pacific J. Math., **143** (1990), 377–399.

19. N. Young, *Periodicity of functionals and representations of normed algebras on reflexive spaces*, Proc. Edinburgh Math. Soc. **20** (1976), 99–120.

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