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ON SUBSOCLES OF PRIMARY ABELIAN GROUPS

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ABSTRACT. The study of subsocles is an important part of the theory of primary abelian groups. In [**5**], section 66, closed, dense and discrete subsocles are defined in terms of the p-adic topology and some useful results about them are given. In this article we consider open subsocles and show that they too have interesting properties. We introduce the notion of range of such subsocles and establish various facts about this concept. We close the article with a characterization of subsocles of a given range using a new and natural generalization of the notion of purity.

All groups considered are abelian primary groups for a fixed prime number p. The terminology and notation not specifically explained here can be found in [**5**].

1. Open subsocles and their range. Let S be a subsocle of a p-group G. We say that S is an *open* subsocle of G if there exists a nonnegative integer n such that $p^n G[p] \subset S$. Such subsocle is open in the topology induced by the p-adic topology of G on $G[p]$. Open subsocles admit a large number of characterizations, some of which are collected in the following:

Theorem 1.1. *Let S be a subsocle of a p-group G. The following properties are equivalent*

a) *S is an open subsocle of G.*

b) *$S \supset G^1[p] = (\cap p^n G)[p]$, and every pure subgroup of G containing S is a summand of G.*

c) *G/K is bounded for every pure subgroup K of G containing S.*

d) *G/K is reduced for every pure subgroup K of G containing S.*

e) *S supports a pure subgroup K of G such that G/K is bounded.*

The proof of Theorem 1.1 can be deduced from results in [**1**, **2**].

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Open subsocles also have the so-called strong purification property, namely, every pure subgroup K of G such that $K[p] \subset S$ can be extended to a pure subgroup H of G such that $H[p] = S$. This property of subsocles characterizes reduced quasi-complete groups, ([**5**, Theorem 74.1]).

We need the following convenient definitions. Let S be a subsocle of a p-group G, we say that the *height* of S is k if $p^kG \supset S$ and $p^{k+1}G \not\supset S$. Write $h(S) = k$. If no such k exists, we say that S is of *infinite height* and write $h(S) = \infty$. Thus, $h(S) = \infty$ if $S \subset G^1 = \cap p^n G$. Let S be an open subsocle of finite height $k \geq 0$. The *range* of S is the least nonnegative integer n such that $p^{k+n}G[p] \subset S$. We write range$(S) = n$.

Three well-known results in the theory of p-groups can be restated in terms of this concept.

Theorem 1.2. *Let S be a subsocle of finite height of a p-group G. Then*

i) (Folklore) *S supports a fully invariant subgroup of G if and only if* range $(S) = 0$ *(i.e., $S = p^nG[p]$ for some nonnegative integer n).*

ii) ([**1**]) *S supports an absolute summand of G if and only if* range $(S) \leq 1$.

iii) ([**6, 7**]) *S is a center of purity if and only if* range $(S) \leq 2$.

In view of the preceding statements, it is natural to look for some group-theoretic property which is characteristic of subsocles S such that range $(S) \leq n$ for $n > 2$. Before we give such a characterization, we establish some useful technical results.

Lemma 1.3. ([**4**, Lemma 2.10]). *Let S be a subsocle of a p-group G, and let n be a nonnegative integer. Then*

a) *$S \cap p^{n+1}G = 0$, if and only if $p^n(G/S)[p] \subset G[p]/S$.*

b) *$S + p^nG[p] = G[p]$, if and only if $G[p]/S \subset p^n(G/S)$.*

Lemma 1.4. *Let S be a subsocle of a p-group G such that $h(S) = k$ and $p^{k+n+1}G[p] \not\subset S$, for some integer $n \geq 0$. Then there exists a* *complementary subsocle T of S in $G[p]$ such that $h(G[p]/T) = k$ and $p^{k+n}(G/T)[p] \not\subset G[p]/T$.*

Proof. Since $(p^{k+n+1}G)[p] \not\subset S$, $(p^{k+n+1}G)[p] \cap S$ is a proper subsocle of $(p^{k+n+1}G)[p]$. Let T_0 be such that $T_0 \oplus (p^{k+n+1}G)[p] \cap S = (p^{k+n+1}G)[p]$. Note that $T_0 \cap S = 0$, and $0 \neq T_0 \subset p^{k+1}G$. Now $(S \cap p^{k+1}G) \oplus T_0 \subset p^{k+1}G[p]$; therefore, there exists a subsocle T_1 such that $(S \cap p^{k+1}G) \oplus T_0 \oplus T_1 = p^{k+1}G[p]$. Again, from the definition of k, $S \cap p^{k+1}G$ is a proper subsocle of S and, thus, there exists a nonzero subsocle S' such that $S \cap p^{k+1}G \oplus S' = S$. Note that $S' \subset p^kG$ and $S' \cap p^{k+1}G = 0$. Let T_2 be such that $p^{k+1}G[p] \oplus S' \oplus T_2 = p^kG[p]$. Then $(T_0 \oplus T_1 \oplus T_2) \cap S = 0$. Finally, write $G[p] = p^kG[p] \oplus T_3$, and let $T = T_0 \oplus T_1 \oplus T_2 \oplus T_3$. Clearly, $S \oplus T = G[p]$, so that $(S+T)/T = G[p]/T$ is contained in p^kG/T. Now, because $0 \neq T_0$, $T \cap (p^{k+n+1}G) \neq 0$, and by Lemma 1.3a, $(p^{k+n}(G/T))[p] \not\subset G[p]/T$. Furthermore, $T + p^{k+1}G[p] \neq G[p]$, therefore by Lemma 1.3b, $p^{k+1}(G/T) \not\supset G[p]/T$. Thus, $h(G[p]/T) = k$, and this completes the proof. □

Proposition 1.5. *Let S be an open subsocle of finite height of a p-group G, and let n be a nonnegative integer. Then,* range$(S) \leq n+1$ *if and only if* range $(G[p]/T) \leq n$, *for every subsocle T of G such that $T \oplus S = G[p]$.*

Proof. Suppose that range$(S) \leq n+1$, then there exists a nonnegative integer k such that $(p^{k+n+1}G)[p] \subset S \subset p^kG$. Let T be a complementary subsocle of S in $G[p]$. Since $T \cap S = 0$, we have $T \cap (p^{k+n+1}G) = 0$, and by Lemma 1.3a, $(p^{k+n}(G/T))[p] \subset G[p]/T$. Furthermore, since $p^kG[p] \supset S$, $p^kG[p] + T = G[p]$, and by Lemma 1.3b, $G[p]/T \subset p^k(G/T)$. Therefore, range $(G[p]/T) \leq n$. Conversely, suppose that range$(G[p]/T) \leq n$, for all complementary subsocles T of S in $G[p]$. Let $h(S) = k$, we show that $(p^{k+n+1}G)[p] \subset S$. Indeed, if $(p^{k+n+1}G)[p] \not\subset S$, by Lemma 1.4 there exists a complementary summand T of S in $G[p]$ such that $h(G[p]/T) = k$ and $p^{k+n}(G/T)[p] \not\subset G[p]/T$. This means that range $(G[p]/T)$ is not $\leq n$, but this is a contradiction. □

2. Centers of purity modulo p^n. We are now ready to give a group-theoretic characterization of open subsocles whose range is less than or equal to an integer $n \geq 2$. We need the following definition. A subgroup H of a p-group G is said to be *pure modulo* p^n if $H/H[p^n]$ is a pure subgroup of $G/H[p^n]$. A subsocle S of G is said to be a *center of purity modulo* p^n if all S-high subgroups of G are pure modulo p^n. The ordinary purity corresponds to the case where $n = 0$. Before we return to our original goal, let us make a small digression. Recall that a p-group G is said to be pure-complete if all the subsocles of G support a pure subgroup of G. It is well known that direct sums of cyclic groups are pure-complete and that there exist large classes of p-groups without elements of infinite height which are not pure-complete. However, even though purity modulo p is a seemingly small weakening of ordinary purity, it is sufficient to guarantee that all p-groups are pure-complete modulo p. We state this fact in the following

Theorem 2.1. *Let S be any subsocle of a p-group G. Then there exists a neat subgroup K of G pure modulo p such that $K[p] = S$.*

Proof. In [**3**], it was shown that for any subgroup K of a p-group G there exists a K-high subgroup of G which is pure in G. This result applied to $G[p]/S$ in G/S yields a subgroup K/S pure in G/S which is $(G[p]/S)$-high in G/S. It is easy to verify that $K[p] = S$, and thus K is pure modulo p (note that K is neat in G). □

Lemma 2.2. *Let S be a subsocle of a p-group G which is a center of purity modulo p^n, $n \geq 1$. Then $G[p]/T$ is a center of purity modulo p^{n-1} in G/T for every complementary subsocle T of S in $G[p]$.*

Proof. Let H/T be a $G[p]/T$-high subgroup of G/T. Then it is a fact that H is also an S-high subgroup of G. Therefore, $H/H[p^n]$ is pure in $G/H[p^n]$. We need to show that $(H/T)/(H/T)[p^{n-1}]$ is pure in $(G/T)/(H/T)[p^{n-1}]$. This follows from the observation that $T = H[p]$ and $(H/T)[p^{n-1}] = H[p^n]/T$. Thus, $(G/T)/(H/T)[p^{n-1}] = (G/T)/(H[p^n]/T)$, which is canonically isomorphic to $G/H[p^n]$. The isomorphism takes $(H/T)/(H/T)[p^{n-1}]$ onto $H/H[p^n]$ and purity is preserved. □

Theorem 2.3. *A subsocle S of a p-group G is a center of purity modulo p^n, $n \geq 0$, if and only if either $h(S) = \infty$, or S is an open subsocle of G such that* $\text{range}(S) \leq n + 2$.

Proof. If $h(S) = \infty$, it is well known that S is a center of ordinary purity, and therefore it is also a center of purity modulo p^n. Then let S be of finite height k. We want to show that $p^{k+n+2}G[p] \subset S$. If this were not the case, by Lemma 1.4, there exists a complementary subsocle T of S in $G[p]$ such that $h(G[p]/T) = k$ and $p^{k+n+1}(G/T)[p]$ is not contained in $G[p]/T$. By Lemma 2.2, however, $G[p]/T$ is a center of purity modulo p^{n-1}. By induction, $\text{range}\,(G[p]/T) \leq n - 1 + 2 = n + 1$. This means that $p^{k+n+1}(G/T)[p] \subset G[p]/T$. This is clearly a contradiction and $\text{range}\,(S) \leq n + 2$. Conversely, if $\text{range}\,(S) \leq n + 2$, $p^{k+n+2}G[p] \subset S \subset p^k G$. Let H be an S-high subgroup of G. We claim that $(p^{k+2}(G/H[p^n]))[p] \subset (G[p]+H[p^n])/H[p^n] \subset p^k(G/H[p^n])$. Indeed, if for $g \in G$, $p(p^{k+2}g + H[p^n]) = 0$, then $p^{k+3}g \in H[p^n]$. But H is neat; therefore, there exists $h \in H$, such that $p^{k+3}g = ph$. Now $p^{k+3+n-1}g = p^n h \in H \cap p^{k+n+2}G = 0$, so that $h \in H[p^n]$, and $(p^{k+2}g - h) \in G[p]$. It follows that $p^{k+2}g + H[p^n] = (p^{k+2}g - h) + H[p^n]$ is an element of $(G[p] + H[p^n])/H[p^n]$. This proves the first inclusion. For the other inclusion, note that $p^k(G/H[p^n]) = (p^k G + H[p^n])/H[p^n]$, and $H[p]+S = G[p]$. Therefore, $\text{range}\,((G[p]+H[p^n]/H[p^n]) \leq 2$. Thus, from Theorem 1.2iii, this is a center of purity in $G/H[p^n]$. It is not difficult to check that $H/H[p^n]$ is in fact $((G[p]+H[p^n])/H[p^n])$-high in $G/H[p^n]$. This means that $H/H[p^n]$ is pure in $G/H[p^n]$. We conclude that S is a center of purity modulo p^n. This completes the proof. □

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