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NONHOMOGENEITY OF POWERS OF COR IMAGES

MURRAY G. BELL

ABSTRACT. A space is called a Cor image if it is a Hausdorff continuous image of some compact ordered space. A space is called homogeneous if any point can be mapped to any other point by some autohomeomorphism of the space. By investigating special kinds of points, we supply necessary conditions for some power of a compact space to be homogeneous. Applying this, we prove that if some power of a Cor image is homogeneous, then the Cor image must be first countable.

1. Introduction. A space is homogeneous if any point can be mapped to any other point by an autohomeomorphism of the space. We call a space powerhomogeneous if some power of the space is homogeneous. This paper is devoted to a partial result on the basic problem: Which compact spaces are powerhomogeneous? Of course, if X is homogeneous, then all powers of X are homogeneous. A convergent sequence with limit point is a simple example of a nonhomogeneous space X such that X^ω is homogeneous. A famous connected example is the closed unit interval I. Keller [**3**] has shown that the Hilbert cube is homogeneous. A simple example of a compact space X that is not powerhomogeneous, due to van Douwen [**2**], is the free union of I and a one point space. No power X^λ is homogeneous because not all connected components of X^λ have the same cardinality. One must work a little harder to find a compact zero-dimensional space that is not powerhomogeneous. One reason is the interesting result of Motorov (cf. Arhangel'skii [**1**]), that every first countable compact zero-dimensional space X has X^ω homogeneous. Let $w(X)$ and $\pi w(X)$ stand for the weight and π-weight of the space X, respectively. The reader is encouraged to read a fundamental paper on nonhomogeneity by van Douwen [**2**] where he proves: If Y is an

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image of the compact space X and $w(X) \leq 2^{\pi w(Y)} < |Y|$, then X is not powerhomogeneous. This shows that many big spaces like βN or $\beta N - N$ are not powerhomogeneous. If κ is an infinite cardinal, then $\alpha\kappa$ denotes the Alexandroff one point compactification of the discrete space of size κ. If α is an infinite ordinal, then $\alpha + 1$ denotes the compact ordinal space of all ordinals not greater than α with the order topology. This paper generalizes the following two isolated facts.

1. van Douwen [**2**]. If $\kappa > \omega$, then $\alpha\kappa$ is not powerhomogeneous.

2. Malykin (cf. Arhangel'skii [**1**]). If $\alpha \geq \omega_1$, then $\alpha + 1$ is not powerhomogeneous.

A Cor is a space which is homeomorphic to a compact linearly ordered space with the interval topology. A Cor image is a Hausdorff continuous image of some Cor. If $\kappa > \omega$, then $\alpha\kappa$ is not a Cor, but it is a Cor image: just take ordinal space $\kappa + 1$ and collapse all limit ordinals to one point. Our main result is that every powerhomogeneous Cor image is first countable. This can also be considered a generalization of a result of Maurice [**4**] that a homogeneous Cor is first countable. The kind of points we consider in Section 3 are of independent interest in the study of arbitrary compact spaces.

2. Preliminaries. Cardinals are initial ordinals and ω and ω_1 are the first two infinite cardinals. The cofinality of the ordinal α is denoted by $cf(\alpha)$. All spaces in this paper are assumed to be Hausdorff. A continuous map f from a space X onto a space Y is said to be irreducible if for all proper closed subsets F of X, $f|F$ is not onto Y. If F is a closed subspace of X, then the character of F in X, denoted by char (F, X), is the least cardinal of a local open base at F in X. It is a fundamental fact of compact spaces that char (F, X) is the least cardinal of a family $\mathcal{A}$ of open sets of X such that $F = \sqcap\mathcal{A}$. If $F = \{p\}$, then we write char (p, X) instead of char $(\{p\}, X)$. We use terminology like open family and closed family to mean a family of open sets or closed sets, respectively. We abbreviate the word 'neighborhood' by 'nbhd.' To denote the closure of A in a space, we write $\mathrm{Cl}\,(A)$.

3. Powerhomogeneous spaces and three types of points. Let κ and λ be cardinals with $\kappa \geq \lambda \geq \omega$. Let X be a space. We say that $p \in X$ is a (κ, λ)-point of X if there exists $A \subseteq X$ with $|A| = \kappa$ such that for all nbhds W of p, $|A - W| < \lambda$. We say that $p \in X$ is a cellular (κ, λ)-point of X if there exists a disjoint open family $\mathcal{A}$ with $|\mathcal{A}| = \kappa$ such that for all nbhds W of p, $|\{A \in \mathcal{A} : A \not\subseteq W\}| < \lambda$. A family $\mathcal{A}$ of subsets of X is called separated if there exists a disjoint open family $\mathcal{R}$ and a bijection $f : \mathcal{A} \to \mathcal{R}$ such that $A \subseteq f(A)$ for each $A \in \mathcal{A}$. In this case, we say that $\mathcal{R}$ separates $\mathcal{A}$. We say that $p \in X$ is a weak cellular (κ, λ)-point of X if there exists a separated closed family $\mathcal{A}$ with $|\mathcal{A}| = \kappa$ such that for all nbhds W of p, $|\{A \in \mathcal{A} : A \cap W = \phi\}| < \lambda$. If $\kappa = \lambda$, then we use the notations κ-point, cellular κ-point, and weak cellular κ-point. Note that if p is one of these three types of points, then $\mathrm{char}\,(p, X) \geq cf(\kappa)$.

Example. Let $X = 2^{\omega_1}$ with the lexicographic order topology. Each point of X is a cellular ω_1-point and there is a dense set of points which are also cellular ω-points. Whether this space was powerhomogeneous was the motivating force behind this paper.

Theorem 3.1. *No compact space X with the following property can exist: for each $p \in X$, there exist distinct infinite regular cardinals κ_p and λ_p such that p is both a cellular κ_p-point and a cellular λ_p-point.*

Proof. Let us assume that such a space X exists. For each $p \in X$, let us fix open families $\mathcal{A}_p$ and $\mathcal{R}_p$ which witness the facts that p is a cellular κ_p-point and a cellular λ_p-point, respectively. Note that X has no isolated points. We proceed inductively to construct a strictly decreasing well-ordered chain of open sets of arbitrarily long length, a contradiction. Assume we have nonempty open sets O_γ for $\gamma < \alpha$ such that $\pi < \gamma$ implies that $\mathrm{Cl}\,(O_\gamma)$ is property contained in O_π. If $\alpha = \gamma + 1$, then use regularity and no isolated points in X to get O_α. If α is a limit ordinal, then put $\delta = cf(\alpha)$ and choose an increasing sequence S of order type δ with supremum α. By compactness of X, choose $p \in \sqcap\{\mathrm{Cl}\,(O_\gamma) : \gamma \in S\} = \sqcap\{O_\gamma : \gamma \in S\}$. Either $\kappa_p \neq \delta$ or $\lambda_p \neq \delta$; let us assume that $\lambda_p \neq \delta$. We claim that there exists $R \in \mathcal{R}_p$ such that $R \subseteq \sqcap(O_\gamma : \gamma \in S\}$. If $\lambda_p > \delta$, then this follows from p being a λ_p-point, each O_γ being an nbhd of p, and λ_p being regular. If $\lambda_p < \delta$, then putting $\mathcal{S}_\gamma = \{R \in \mathcal{R}_p : R \not\subseteq O_\gamma\}$ we see that $\langle \mathcal{S}_\gamma : \gamma \in S\rangle$ is an increasing sequence under $\subseteq$ of length δ consisting of sets of cardinality $< \lambda_p < \delta$, hence there exists $\pi < \delta$ such that for all $\pi < \gamma < \delta$, $\mathcal{S}_\gamma = \mathcal{S}_\pi$.

Choosing $R \in \mathcal{R}_p - \mathcal{S}_\pi$ we see that $R \subseteq \sqcap \{O_\gamma : \gamma \in S\}$. In either case, we put $O_\alpha = R$ to finish the inductive step. $\square$

By a proof analogous to the preceding proof, one can show that there is no compact space in which each point is a cellular (ω_1, ω)-point. A very interesting problem presents itself. Is there a compact space in which each point is a weak cellular (ω_1, ω)-point?

Turning to powers X^λ of a space X, for each finite $F \subseteq \lambda$ let π_F be the projection onto X^F and for each $\alpha \in \lambda$ let π_α be the projection onto the α factor space. By the canonical basis $\mathcal{C}$ for X^λ we mean the family of all finite unions of members of $\{\sqcap\{\pi_\alpha^{-1}[O_\alpha] : \alpha \in F\}$: F is a finite subset of λ and O_α is an open subset of X for each $\alpha \in F\}$. A subset A of X^λ lives on a subset F of λ if $A = \pi_F^{-1}[\pi_F[A]]$. Note that each member of $\mathcal{C}$ lives on a finite set and if A lives on F, then A lives on all supersets of F. Since our work on powers involves the diagonal, we use the notation (x) to represent the point in X^λ all of whose coordinates are x. One obvious deduction about a powerhomogeneous space X is that if X contains two points of different character, then X must at least be raised to the larger character power in order to become homogeneous. A reasonable conjecture is that if X is powerhomogeneous, then $X^{w(X)}$ is homogeneous. This is unsolved.

If X has more than one point and λ is infinite, then each point of X^λ is a (λ, ω)-point since each point of X^λ is contained in a copy of 2^λ. A straightforward counting argument will show that if λ is uncountable and κ is infinite, then X^λ has no cellular κ-points whatsoever. It is weak cellular κ-points that prove useful in uncountable powers.

Theorem 3.2. *Let X be a powerhomogeneous compact space and let κ be an uncountable regular cardinal. Assume that X contains a weak cellular κ-point. Then, for each $q \in X$, we have that* $\mathrm{char}\,(q, X) \geq \kappa$.

Proof. Let λ be a cardinal such that X^λ is homogeneous. Let $\mathcal{C}$ be the canonical basis for X^λ. Choose $p \in X$ such that p is a weak cellular κ-point. Choose a separated closed family $\mathcal{A}$ which witnesses this fact. Let q be an arbitrary element of X. Choose any $\alpha \in \lambda$. The constant point (p) is seen to be a weak cellular κ-point of X^λ via the separated closed family $\{\pi_\alpha^{-1}[A] : A \in \mathcal{A}\}$. Hence, the constant point (q) is a weak cellular κ-point also. Choose a separated closed family $\mathcal{R}$ which witnesses this fact. Since $\mathcal{R}$ is a separated family of compact sets and $\mathcal{C}$ is an open base for X^λ which is closed under finite unions, we can choose a disjoint open subcollection $\mathcal{S}$ of $\mathcal{C}$ which separates $\mathcal{R}$. Let f be the associated bijection from $\mathcal{R}$ to $\mathcal{S}$. For each $S \in \mathcal{S}$ choose a finite $F(S) \subseteq \lambda$ such that S lives on $F(S)$. Invoke a delta-system argument to choose $\mathcal{P} \subseteq \mathcal{S}$ with $|\mathcal{P}| = \kappa$ and such that $\{F(S) : S \in \mathcal{P}\}$ is a delta-system with root D, i.e., whenever $S, T \in \mathcal{P}$, then $F(S) \cap F(T) = D$. Since $\mathcal{S}$ is disjoint, $D \neq \phi$. Then $\{\pi_D[R] : f(R) \in \mathcal{P}\}$ is a separated closed family in X_D witnessing the fact that $\pi_D((q))$ is a weak cellular κ-point of X_D, and thus, $\pi_D((q))$ has character at least κ and so $\mathrm{char}\,(q, X) \geq \kappa$. $\square$

We do not know whether our conclusion above can be improved to show that each point in X is a weak cellular κ-point.

Theorem 3.3. *Let X be a powerhomogeneous compact space. Assume that X contains a cellular ω-point. Then, each point of X is either an isolated point or an ω-point.*

Proof. Let λ be a cardinal such that X^λ is homogeneous. Let $\mathcal{C}$ be the canonical basis for X^λ. Choose $p \in X$ such that p is a cellular ω-point and let $\{O_n : n < \omega\}$ be a disjoint open family in X witnessing this fact. Striving for a contradiction, assume that $q \in X$ and q is neither an isolated point nor an ω-point. Put $W = X - \{q\}$. Then W is an open dense countably compact subspace of X. Choose any $\alpha \in \lambda$. Put $V = \pi_\alpha^{-1}[W]$. Then we see that V is an open dense countably compact subspace of X^λ such that $(q) \in \mathrm{Cl}\,(V) - V$. Since X^λ is homogeneous, there must exist an open dense countably compact subspace D of X^λ such that $(p) \in \mathrm{Cl}\,(D) - D$. Let us choose such a D. We now construct, by induction on $n < \omega$, an increasing sequence F_n of finite subsets of λ and nonempty sets $C_n \in \mathcal{C}$ such that C_n lives on F_n and $C_n \subseteq D \cap \sqcap\{\pi_\alpha^{-1}[O_n] : \alpha \in F_{n-1}\}$. This we can do since D is open and dense. Now, for each $n < \omega$, we choose $f_n \in C_n \cap \sqcap\{\pi_\alpha^{-1}(p) : \alpha \in \lambda - F_n\}$. It follows that $\{f_n : n < \omega\} \subseteq D$, and we claim that $\langle f_n : n < \omega\rangle$ converges to the point (p); so, since D is countably compact, we get $(p) \in D$, our contradiction. To show convergence, it suffices to show that whenever $\alpha \in \lambda$ and O is an open subset of X with $p \in O$, then there exists $m < \omega$ such that for every $n \geq m$, we have $f_n \in \pi_\alpha^{-1}[O]$. Let such an α and O be given. If, for each n, α is not a member of F_n, then $f_n(\alpha) = p$ and, thus, $f_n \in \pi_\alpha^{-1}[O]$. Otherwise, we can choose $r < \omega$ such that $\alpha \in F_r$. Since p is a cellular ω-point, we now choose $m > r$ such that for every $n \geq m$ we have $O_n \subseteq O$. Then, if $n \geq m$, since $f_n \in C_n$ and $F_r \subseteq F_{n-1}$, our choice of C_n implies that $f_n \in \pi_\alpha^{-1}[O_n]$. Hence, $f_n \in \pi_\alpha^{-1}[O]$. $\square$

Example. Put X equal to the free union of $\alpha\omega$ and the Cantor cube 2^{ω_1}. Let p be the nonisolated point of $\alpha\omega$. Schepin's [**5**] characterization of 2^κ tells us that X^{ω_1} is homeomorphic to 2^{ω_1}, hence X is powerhomogeneous. For the reader's benefit, we mention Schepin's characterization: A compact zero-dimensional Dugundji space of weight κ and in which each point has character κ is homeomorphic to 2^κ. The point p is a cellular ω-point of X, whereas no point in the Cantor cube is a cellular ω-point of X. This example shows that the conclusion of our preceding theorem cannot be improved to show that each point of X is either an isolated point or a cellular ω-point.

4. Cor images. In this section L will always represent a Cor with underlying complete order $<$. A key fact about L is that any open subspace O of L can be written in a unique way as the disjoint union of open maximal intervals of L contained in O. A key fact about Cor images X is that X is the image of some Cor under an irreducible mapping. This follows from compactness and the fact that a closed subspace of a Cor is again a Cor. Let $\psi : L \to X$ be an irreducible onto mapping. If $\mathcal{A}$ is a disjoint family of nonempty open subsets of L, then put $\psi * (\mathcal{A}) = \{X - \psi[L - A] : A \in \mathcal{A}\}$. Then $\psi * (\mathcal{A})$ is a disjoint family of nonempty open subsets of X. Furthermore, if $\mathcal{A}$ witnesses the fact that p is a cellular κ-point of L, then $\psi * (\mathcal{A})$ witnesses the fact that $\psi(p)$ is a cellular κ-point of X. It is well known that every Cor image is sequentially compact; hence, if infinite, it must contain an ω-point. In fact, in Cor images, ω-points are cellular ω-points.

Lemma 4.1. *If p is an ω-point of a Cor image X, then p is a cellular ω-point.*

Proof. Let $\langle p_n : n < \omega\rangle$ be a sequence converging to p in X, and let ψ be an irreducible map of L onto X. For each n, choose $q_n \in L$ such that $\psi(q_n) = p_n$. Invoke Ramsey's theorem to choose an infinite $A \subseteq \omega$ such that $\langle q_n : n \in A\rangle$ is either an increasing or a decreasing sequence in L. Let q be the limit in L of this convergent subsequence. Then $\psi(q) = p$. Choose disjoint open intervals I_n with $q_n \in I_n$. $\mathcal{A} = \{I_n : n \in A\}$ witnesses the fact that q is a cellular ω-point. By the above, $\psi * (\mathcal{A})$ witnesses the fact that p is a cellular ω-point. $\square$

By being a little bit more careful we could show that in Cor images all three notions of κ-points coincide, but we don't need this.

If $B \subseteq A \subseteq L$, then B is cofinal (respectively, coinitial) in A if for each $a \in A$ there exists $b \in B$ with $a \leq b$ (respectively, $b \leq a$). Put $cf(A)$ equal to the least cardinality of a cofinal subset of A and put $ci(A)$ equal to the least cardinality of a coinitial subset of A. Put $\delta(A)$ equal to the maximum of $cf(A)$ and $ci(A)$. Note that $\delta(A)$ is always a regular cardinal and, if finite, equals 1. Put $\pi(A) = \sup(A)$ if $\delta(A) = cf(A)$ and put $\pi(A) = \inf(A)$ if $\delta(A) \neq cf(A)$. $\delta(A) = 1$ if and only if $\pi(A) \in A$. Obviously, if $\pi(A)$ is not an element of A, then $\pi(A)$ is a cellular $\delta(A)$-point of L. If F is a closed subspace of L and $\mathcal{A}$ is a disjoint family of open maximal intervals of L such that $L - F = \sqcup\mathcal{A}$, then $\operatorname{char}(F, L) = \operatorname{SUM}\{\delta(A) : A \in \mathcal{A}\}$ where SUM denotes cardinal sum.

Lemma 4.2. *If p is a nonisolated point of a Cor image X, then p is a cellular* $\operatorname{char}(p, X)$*-point of X. Moreover, if* $\operatorname{char}(p, X)$ *is singular, then* $\operatorname{char}(p, X) = \sup\{\kappa : \kappa$ *is a regular cardinal and p is a cellular κ-point*$\}$.

Proof. Let ψ be an irreducible map from L onto X. Let $\lambda = \operatorname{char}(p, X)$. Then $\operatorname{char}(\psi^{-1}(p), L) = \lambda$. Let $\mathcal{A}$ be a disjoint family of open maximal intervals of L such that $L - \psi^{-1}(p) = \sqcup\mathcal{A}$. Thus, $\lambda = \operatorname{SUM}\{\delta(A) : A \in \mathcal{A}\}$ and so $|\mathcal{A}| \leq \lambda$ and $\delta(A) \leq \lambda$ for each $A \in \mathcal{A}$. Since L is compact, every open set containing $\psi^{-1}(p)$ must actually contain all but finitely many members of $\mathcal{A}$. For each $A \in \mathcal{A}$ such that $\delta(A) > 1$, let $\mathcal{R}(A)$ be a disjoint family of intervals of L contained in A which witnesses the fact that $\pi(A)$ is a cellular $\delta(A)$-point. Note that if $\delta(A) > 1$, then $\pi(A) \in \psi^{-1}(p)$. Put

$$\mathcal{S} = \sqcup\{\mathcal{R}(A) : A \in \mathcal{A} \text{ and } \delta(A) > 1\} \cup \{A : A \in \mathcal{A} \text{ and } \delta(A) = 1\}.$$

$\mathcal{S}$ is a disjoint open family in L and every nbhd of $\psi^{-1}(p)$ contains all but less than λ many of the members of $\mathcal{S}$. Hence, the family $\psi * (\mathcal{S})$ witnesses the fact that p is a cellular λ-point. □

Theorem 4.3. *Every powerhomogeneous* Cor *image* X *is first countable.*

Proof. Assume X has a point p with $\operatorname{char}(p, X) = \kappa$ where $\kappa > \omega$. Invoke Lemma 4.2 and choose a regular $\lambda > \omega$ such that p is a cellular λ-point. By Theorem 3.2, all points of X have character $\geq \lambda$. Invoke Lemma 4.2 and choose, for each $q \in X$, an uncountable regular cardinal λ_q such that q is a cellular λ_q-point. Since X is an infinite Cor image, X contains a cellular ω-point. Since X has no isolated points, Theorem 3.3 implies that all points of X are ω-points. Lemma 4.1 implies that all points of X are cellular ω-points. Finally, Theorem 3.1 implies that X cannot exist. □

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Department of Mathematics, University of Manitoba, Winnipeg, Canada R3T 2N2