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ANALYTIC CONTINUATION OF FUNCTIONS GIVEN BY CONTINUED FRACTIONS, REVISITED

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ABSTRACT. Let the continued fraction $K(a_n(z)/b_n(z))$ converge to a meromorphic function $f(z)$ in a domain D. This function $f(z)$ may very well be meromorphic in a larger domain D_0^* containing D, even if the continued fraction itself diverges in $D_0^*\backslash D$. The value of $f(z)$ for points $z \in D_0^*\backslash D$ can then be obtained by using modified approximants. This technique is known. We shall give sufficient conditions for obtaining a continuous extension of $f(z)$ to the boundary of D_0^* by the same method.

1. Introduction. Let us illustrate the idea with some very simple examples which are taken from Thron and Waadeland's paper [**11**].

Example 1.1. The periodic, regular C-fraction

$$K(az/1) = \frac{az}{1} \underset{+}{} \frac{az}{1} \underset{+}{} \frac{az}{1} \underset{+}{} \cdots; \qquad a \in \mathbf{C}\backslash\{0\} \tag{1.1}$$

converges in the cut plane $D := \{z \in \mathbf{C}; |\arg(1+4az)| < \pi\}$ to the holomorphic function

$$w(z) := \frac{1}{2}((1+4az)^{1/2} - 1) \qquad \text{for } z \in D. \tag{1.2}$$

Clearly, $\Re((1+4az)^{1/2}) > 0$ for $z \in D$ in $w(z)$, and clearly, $w(z)$ can be extended analytically to the function

$$W(z) := \frac{1}{2}((1+4az)^{1/2} - 1) \qquad \text{for } z \in D^*, \tag{1.3}$$

where D^* is the two-sheeted Riemann surface on which $W(z)$ is holomorphic. (That is, D^* is a Riemann surface with branch points of

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order 1 at $z=-1/4a$ and at $z=\infty$.) There is no way that the classical approximants of (1.1) can converge to $W(z)$ at points $z\in D^*\backslash\bar{D}$, since they are rational functions (with no branch points). The modified approximants

$$\tilde{S}_n(W(z),z):=\frac{az}{1}+\frac{az}{1}+\cdots+\frac{az}{1+W(z)},\qquad (n\text{ terms}); \tag{1.4}$$

however, converge for all $z\in D^*$ to the *right* function, $W(z)$. In fact, $\tilde{S}_n(W(z),z)=W(z)$ for all n and all $z\in D^*$.

These properties are to a certain extent inherited by the limit periodic regular C-fraction

$$K(a_nz/1)=\frac{a_1z}{1}+\frac{a_2z}{1}+\frac{a_3z}{1}+\cdots\qquad\text{where } a_n\in\mathbf{C}\backslash\{0\},\ a_n\to a. \tag{1.5}$$

Firstly, (1.5) converges to a meromorphic function $f(z)$ in the cut plane D, [**7**, p. 95]. That is, its classical approximants

$$S_n(0,z)=\frac{a_1z}{1}+\frac{a_2z}{1}+\cdots+\frac{a_nz}{1}$$

converge to $f(z)$ in D. Its modified approximants

$$S_n(W(z),z):=\frac{a_1z}{1}+\frac{a_2z}{1}+\cdots+\frac{a_nz}{1+W(z)}\quad\text{for } z\in D^*,\ n=1,2,3,\dots \tag{1.6}$$

are well-defined, meromorphic functions in D^*. For $z\in D$ we know that $\lim S_n(W(z),z)=f(z)$, [**9**]. Moreover, the convergence is uniform on compact subsets of $\{z\in D; f(z)\neq\infty\}$, and $1/S_n(W(z),z)$ converges uniformly to $1/f(z)$ on compact subsets of $\{z\in D; f(z)\neq 0\}$. We shall in the following say that $S_n(W(z),z)$ converges $\hat{\mathbf{C}}$-*uniformly* on compact subsets of D, or that $S_n(W(z),z)$ *converges locally* $\hat{\mathbf{C}}$-*uniformly* to $f(z)$ in D to describe this kind of convergence. ($\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$.)

The point now is that if $a_n\to a$ sufficiently fast, then $\{S_n(W(z),z)\}$ converges locally $\hat{\mathbf{C}}$-uniformly not only in D, but in a larger domain $\subset D^*$ to a function $F(z)$. That is, the modified approximants $S_n(W(z),z)$ converge in a larger domain than the classical approximants $S_n(0,z)$.

Since $F(z) = f(z)$ in D, it follows that $F(z)$ represents a meromorphic extension of $f(z)$. Without loss of generality, we let $a = 1/4$:

Theorem A [**11**]. *Let $C > 0$, $0 \le r < 1$ and*

$$D_r^* := \left\{ z \in D^*; \left| \frac{W(z)}{1+W(z)} \right| < \frac{1}{r} \right\} \tag{1.7a}$$

where

$$W(z) := \frac{1}{2}((1+z)^{1/2} - 1). \tag{1.7b}$$

If $|a_n - 1/4| \le Cr^n$ for $n = 2, 3, 4, \ldots$, then $\{S_n(W(z), z)\}$ converges locally $\hat{\mathbf{C}}$-uniformly in D_r^*.

Remarks. 1. D_r^* is a domain in D^*. It increases if r decreases. $D_1^* = D$ and $D_0^* = D^* \backslash \{0 \text{ in the sheet } D^* \backslash \bar{D}\}$.

2. D_r^* is all of D^* except for a bounded hole $H := D^* \backslash D_r^*$ which lies in the sheet $D^* \backslash \bar{D}$, symmetric about the axis $\arg z = 2\pi$. Since $z \in H$ if and only if $v(z) := (1+z)^{1/2}$ satisfies $|v(z) - 1|/|v(z) + 1| \ge 1/r$; i.e., if and only if $\Re(v(z)) < 0$ and

$$|v(z) - C| \le R; \quad \text{where } C := -\frac{1+r^2}{1-r^2} \quad \text{and} \quad R := \frac{2r}{1-r^2},$$

it follows that H is contained in the circular disk with center at $C^2 - 1$ and radius $R(2|C| + R)$ in the sheet $D^* \backslash \bar{D}$.

3. The branch points $z = -1$ and $z = \infty$ of $W(z)$ are bounded away from the hole H. Hence, $F(z) = \lim S_n(W(z), z)$ also has branch points of order 1 at $z = -1$ and $z = \infty$.

The question we are interested in in this paper can in this context be phrased as follows: Under what conditions will $\{S_n(W(z), z)\}$ converge $\hat{\mathbf{C}}$-uniformly in the closure $\overline{D_r^*}$ of D_r^* in D^*?

Theorem 1. *Let $0 \le r \le 1$, and let $C_n > 0$ be such that*

$$\sum C_n < \infty \quad \textit{if } r < 1, \qquad \sum nC_n < \infty \qquad \textit{if } r = 1, \tag{1.8}$$

and let D_r^* and $W(z)$ be given by (1.7). *If* $|a_n - 1/4| \le C_n r^n$ *from some* n *on, then* $\{S_n(W(z), z)\}$, *given by* (1.6), *converges* $\hat{\mathbf{C}}$*-uniformly in* $\overline{D_r^*}$.

A similar result is proved in [**8**] and [**10**]. We shall generalize this result in Section 2. Our main result contains both Theorem 1 and the following example:

Example 1.2. The periodic general T-fraction

(1.9)

$$K(Fz/(1-Fz)) := \frac{Fz}{1-Fz} + \frac{Fz}{1-Fz} + \frac{Fz}{1-Fz} + \cdots \qquad \text{where } F \neq 0$$

converges to the function $w(z) = Fz$ for $|z| < 1/|F|$ and to the function $\tilde{w}(z) = -1$ for $|z| > 1/|F|$. The limit periodic general T-fraction

$$(1.10)\quad K(F_n z/(1+G_n z)) := \frac{F_1 z}{1+G_1 z} + \frac{F_2 z}{1+G_2 z} + \frac{F_3 z}{1+G_3 z} + \cdots,$$

where

$$F_n \in C\backslash\{0\}, \qquad F_n \to F \in \mathbf{C}\backslash\{0\}, \qquad G_n \in \mathbf{C}\backslash\{0\}, \qquad G_n \to -F$$

converges to a meromorphic function $f(z)$ in the disk $|z| < 1/|F|$ and to another meromorphic function $g(z)$ in its exterior $|z| > 1/|F|$. Its modified approximants $S_n(Fz, z)$ or $S_n(-1, z)$ may converge in larger domains, though. Without loss of generality, we let $F = 1$:

Theorem B [**11**]. *Let* $C > 0$, $0 \le r < 1$, *and let* $|F_n - 1| \le Cr^n$ *for* $n = 2, 3, 4, \ldots$ *and* $|G_n + 1| \le Cr^n$ *for* $n = 1, 2, 3, \ldots$. *Then* $\{S_n(z, z)\}$ *converges locally* $\hat{\mathbf{C}}$*-uniformly for* $|z| < 1/r$ *and* $\{S_n(-1, z)\}$ *converges locally* $\hat{\mathbf{C}}$*-uniformly for* $|z| > r$.

Since $\lim S_n(z, z) = \lim S_n(0, z)$ for $|z| < 1$ and $\lim S_n(-1, z) = \lim S_n(0, z)$ for $|z| > 1$, we see that Theorem B gives us meromorphic extensions of $f(z)$ to $|z| < 1/r$ and of $g(z)$ to $|z| > r$. The continuous continuation to the boundaries is then secured under conditions similar to the ones in Theorem 1:

(1.11)

$$|F_n - 1| \le C_n r^n, \qquad |G_n + 1| \le C_n r^n, \qquad \text{where } 0 \le r \le 1,\ C_n \ge 0,$$
$$\sum C_n < \infty \quad \text{if } r < 1 \quad \text{and} \quad \sum nC_n < \infty \quad \text{if } r = 1.$$

This method for analytic continuation originated with Waadeland, [**13, 14**]. It is described for limit periodic continued fractions in [**10, 11**] and for more general continued fractions in [**1**]. The main results of this paper will be presented in Section 2. In Section 3 we consider the question of sharpness. The proofs are given in Section 4.

Throughout this paper we shall consider continued fractions $K(a_n(z)/b_n(z))$ where z lies in a domain. (Domain = open, connected set, possibly including (parts of) its boundary.) The convergence results we obtain are also valid in more general sets, but that is beyond the scope of this paper.

2. The main result. The previous examples can be regarded as nearness (or perturbation) results: The continued fraction $K(a_n z/1)$ in Example 1.1 is near the periodic continued fraction $K((z/4)/1)$ (or it is a perturbation of this one). $K((z/4)/1)$ converges to

$$w(z) := \frac{1}{2}((1+z)^{1/2}-1), \qquad \text{where } \Re\left((1+z)^{1/2}\right) > 0$$

in the cut plane $D := \{z \in \mathbf{C}; |\arg(1+z)| < \pi\}$, and it can be continued analytically to $W(z)$ in D^*, as given in (1.3). It has modified approximants

$$(2.1)\quad \tilde{S}_n(W(z),z) := \frac{z/4}{1} + \frac{z/4}{1} + \cdots \frac{z/4}{1+W(z)} \equiv W(z) \qquad \text{in } D^*.$$

When $K(a_n z/1)$ is near enough to $K((z/4)/1)$, then the modified approximants $S_n(W(z),z)$ of $K(a_n z/1)$ also converge to the right function in a larger domain. The nearer $K(a_n z/1)$ is to $K((z/4)/1)$, i.e., the smaller $r \geq 0$ is in Theorem A, the larger is this domain D_r^*.

Similarly, $K(F_n z/(1+G_n z))$ in Theorem B is near the periodic general T-fraction $K(z/(1-z))$ which converges to z for $|z| < 1$ and to -1 for $|z| > 1$. In this case, we have a choice: Either we can regard $w(z) := z$ and $\tilde{w}(z) := -1$ as two separate functions in $D^* = \mathbf{C}$, or we can choose D^* to be a Riemann surface consisting of two separate sheets (which both are isomorphic to $\mathbf{C}$), and let $W(z) := z$ in one sheet, $W(z) := -1$ in the other sheet, and thus regard $w(z)$ and $\tilde{w}(z)$ as one holomorphic function $W(z)$ in D^*. For simplicity, we choose the latter version.

The idea of Theorem A and B was extended in [**1**]. Let $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ be a continued fraction converging to a holomorphic function $w_0(z)$ in a domain D. Let $w_n(z)$ be the value of its nth tail; i.e.,

$$(2.2) \qquad w_n(z) := \frac{\tilde{a}_{n+1}(z)}{\tilde{b}_{n+1}(z)} \, \genfrac{}{}{0pt}{}{}{+} \, \frac{\tilde{a}_{n+2}(z)}{\tilde{b}_{n+2}(z)} \, \genfrac{}{}{0pt}{}{}{+} \, \cdots, \qquad n = 0, 1, 2, \dots,$$

and assume that all $w_n(z)$ are holomorphic in D. Assume further that all $w_n(z)$ can be continued analytically to functions $W_n(z)$ in a larger domain D^* containing D. This is now the reference continued fraction (playing the role of $K((z/4)/1)$ in Theorem A and $K(z/(1-z))$ in Theorem B), and it has modified approximants

$$\tilde{S}_n(W_n(z), z) := \frac{\tilde{a}_1(z)}{\tilde{b}_1(z)} \, \genfrac{}{}{0pt}{}{}{+} \, \cdots \, \genfrac{}{}{0pt}{}{}{+} \, \frac{\tilde{a}_n(z)}{\tilde{b}_n(z) + W_n(z)} \equiv W_0(z) \qquad \text{in } D^*.$$

Now, let $K(a_n(z)/b_n(z))$ be near to $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ in the sense that

$$(a_n(z) - \tilde{a}_n(z)) \to 0, \qquad (b_n(z) - \tilde{b}_n(z)) \to 0 \quad \text{fast enough.}$$

For simplicity, we assume that $\tilde{a}_n, \tilde{b}_n, a_n$ and b_n are holomorphic in our universe, that is, in the domain D^*. This is no severe restriction since the elements of continued fractions are normally entire functions, in fact, they are usually polynomials. If D^* is a Riemann surface with more than one sheet (such as in our examples), then this means that we redefine $\tilde{a}_n, \tilde{b}_n, a_n$ and b_n to be functions in D^*: For $z \in D^*$ we let for instance $a_n(z)$ be the value of the original a_n evaluated at the projection of z in $\mathbf{C}$.

Under such conditions, the modified approximants $S_n(W_n(z), z)$ of $K(a_n(z)/b_n(z))$ converge locally $\hat{\mathbf{C}}$-uniformly in a domain $D_0^* \subseteq D^*$, [**1**]. Let $F(z)$ denote this limit function. If $K(a_n(z)/b_n(z))$ converges to a function $f(z)$ on an infinite set $E \subseteq D_0^* \cap D$, then $f(z) = F(z)$ on E, [**3**]. Hence, $F(z)$ is a meromorphic extension of $f(z)$ to D_0^*. We refer to [**4**] for an illustration of how this turns out for limit k-periodic continued fractions.

To state this more precisely, we define some convenient concepts:

Definition 1. The reference continued fraction $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ is said to belong to the class $\operatorname{Ref}(D^*, \{M_n(z)\})$ for a given domain D^*

and a given sequence $\{M_n(z)\}$ of real-valued, nonnegative functions on D^* ($M_n(z) = \infty$ is also allowed) if

(i) all $\tilde{a}_n(z)$ and $\tilde{b}_n(z)$ are holomorphic in D^* with all $\tilde{a}_n(z) \neq 0$ in D^*;

(ii) $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ converges in a nonempty domain $D \subseteq D^*$, and all its tail values $w_n(z)$ as given by (2.2) are holomorphic in D,

(iii) all $w_n(z)$ have analytic continuations $W_n(z)$ to D^*, and

(iv)

$$\prod_{j=k+1}^{k+n} \left| \frac{W_j(z)}{\tilde{b}_j(z) + W_j(z)} \right| \leq M_n(z) \qquad \text{for all } z \in D^*,\ k, n \in \mathbf{N}_0. \tag{2.3}$$

Remarks. 1. D^* is the universe in which we operate. $\mathbf{N}_0$ denotes the set of all nonnegative integers, i.e., $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

2. In the rest of this paper, we shall always let $w_n(z)$ and $W_n(z)$ be as defined here.

3. There is no problem if we relax the requirements in (i) and (iii) to be $\tilde{a}_n, \tilde{b}_n$ and W_n holomorphic in the interior of D^* and continuous in D^*. So also in the following definition.

Definition 2. The continued fraction $K(a_n(z)/b_n(z))$ is said to belong to the class $\operatorname{Near}(K(\tilde{a}_n(z)/\tilde{b}_n(z)),\ \{C_n(z)\}, D_0^*)$ for a given domain D_0^*, a given continued fraction $K(\tilde{a}_n(z)/\tilde{b}_n(z))$, and a given sequence $\{C_n(z)\}$ of real-valued, nonnegative functions on D_0^* if

(i) all $a_n(z)$ and $b_n(z)$ are holomorphic in D_0^* with all $a_n(z) \neq 0$ in D_0^*; and

(ii)

$$\left| \frac{a_{n+1}(z) - \tilde{a}_{n+1}(z)}{\tilde{b}_{n+1}(z) + W_{n+1}(z)} \right| \leq C_n(z), \qquad |b_n(z) - \tilde{b}_n(z)| \leq C_n(z) \tag{2.4}$$
$$\text{for all } z \in D_0^* \text{ and } n \in \mathbf{N}.$$

Definition 3. A sequence $\{f_n(z)\}$ of functions is said to belong to the class $\operatorname{Uniconv}(D)$ for a given set D if

(i) all $f_n(z)$ are defined on the set $D(f_n(z) = \infty$ is allowed), and

(ii) $\{f_n(z)\}$ converges locally $\hat{\mathbf{C}}$-uniformly in D.

A slightly stronger result than the one in [**1**] can now be formulated with the notation introduced above:

Theorem C [**1**]. *Let $M(z), r(z), C(z)$ and $r_1(z)$ be continuous, real-valued, nonnegative functions in the domain D^* with $r(z) \neq 0$. Further, let $K(\tilde{a}_n(z)/\tilde{b}_n(z)) \in \operatorname{Ref}(D^*, \{M(z)r(z)^{-n}\})$ and $K(a_n(z)/b_n(z)) \in$ Near $(K(\tilde{a}_n(z)/\tilde{b}_n(z)), \{C(z)r_1(z)^n\}, D^*)$, and assume that $\liminf|\tilde{b}_n(z) + W_n(z)|$ has positive lower bounds on compact subsets of D^*. Then $\{S_n(W_n(z), z)\} \in \operatorname{Uniconv}(D_0^*)$ where*

$$D_0^* := \{z \in D^*; r_1(z) < \min\{r(z), 1\}\}. \tag{2.5}$$

Remarks. 1. The result in [**1**] also depended on some conditions relating $M(z)$ and $C(z)$. These can now be disposed of. Theorem C is a simple corollary of Theorem 3 to follow.

2. For information on how to treat possible poles of $W_n(z)$ or zeros of $a_n(z)$ we refer to [**1**].

3. If $K(a_n(z)/b_n(z))$ converges to some meromorphic function $f(z)$ in some nonempty domain $D_0 \subseteq D_0^*$, $F(z) := \lim S_n(W_n(z), z)$ is meromorphic in D_0^* and $f(z) = F(z)$ in D_0, then $F(z)$ is a meromorphic extension of $f(z)$ to D_0^*.

A natural question in connection with Remark 3 above is the following. When/where will $K(a_n(z)/b_n(z))$ converge to a function $f(z)$ which satisfies $f(z) = F(z)$ in D_0? The next theorem can give an answer to this question. But instead of basing this on classical convergence, we shall use the more general concept of *general convergence*. This concept was introduced in [**2**], and it is defined as follows: We say that a continued fraction $K(\alpha_n/\beta_n)$ converges generally to the value ψ if its modified approximants

$$T_n(q_n) := \frac{\alpha_1}{\beta_1} \, {}_{+} \, \frac{\alpha_2}{\beta_2} \, {}_{+} \cdots {}_{+} \, \frac{\alpha_n}{\beta_n + q_n} \qquad \text{for } n = 1, 2, 3, \dots \tag{2.6}$$

are such that there exist two sequences $\{p_n\}$ and $\{q_n\}$ from $\hat{\mathbf{C}}$ such that

$$(2.7)\qquad \lim T_n(p_n) = \lim T_n(q_n) = \psi \quad \text{and} \quad \liminf d(p_n, q_n) > 0.$$

Here $d(\dots,\dots)$ denotes the chordal metric on the Riemann sphere $\hat{\mathbf{C}}$. It is also a result from [**2**] that if $K(\alpha_n/\beta_n)$ converges generally to ψ, then $\lim T_n(u_n) = \psi$ for every sequence $\{u_n\}$ from $\hat{\mathbf{C}}$ which satisfies $\liminf d(u_n, T_n^{-1}(p)) > 0$ for some $p \neq \psi$. Hence, general convergence is a very natural concept. We also see that if we choose $p_n = 0$ and $q_n = \infty$ for all n, then $T_{n+1}(q_{n+1}) = T_n(p_n) = T_n(0)$, so that classical convergence to ψ implies general convergence to ψ.

We shall also use the notion of *tail sequences* for continued fractions: A sequence $\{t_n\}_{n=0}^{\infty}$ of numbers from $\hat{\mathbf{C}}$ is a tail sequence for the continued fraction $K(\alpha_n/\beta_n)$ with all $\alpha_n \neq 0$, if

$$(2.8)\qquad t_{n-1} = \alpha_n/(\beta_n + t_n) \qquad \text{for } n = 1, 2, 3, \dots .$$

(The interpretation of (2.8) is the obvious one if $t_{n-1} = 0$ or ∞, or if $t_n = \infty$.) Clearly then $t_0 = T_n(t_n)$ for all n. In particular, $\{W_n(z)\}$ and $\{w_n(z)\}$ are tail sequences for $K(\tilde{a}_n(z)/\tilde{b}_n(z))$.

Theorem 2. *Let $\{\tilde{t}_n\}$ be a tail sequence for $K(\tilde{\alpha}_n/\tilde{\beta}_n)$; all $\tilde{\alpha}_n \neq 0$, such that* $\liminf |\tilde{\beta}_n + \tilde{t}_n| > 0$, $\limsup |\tilde{t}_n| < \infty$ *and*

$$(2.9)\qquad \prod_{j=n+1}^{n+m} \left| \frac{\tilde{t}_j}{\tilde{\beta}_j + \tilde{t}_j} \right| \le M_m \qquad \textit{for all } m \in \mathbf{N}_0, \ \textit{where } \sum M_m < \infty,$$

for all n from some n on. Then every continued fraction $K(\alpha_n/\beta_n)$ with $\lim(\alpha_n - \tilde{\alpha}_n) = 0$ *and* $\lim(\beta_n - \tilde{\beta}_n) = 0$ *converges generally to some value* $\psi := \lim T_n(\tilde{t}_n)$, *where T_n is given by* (2.6).

Remarks. 1. In particular, $K(\tilde{\alpha}_n/\tilde{\beta}_n)$ converges generally to the value $\tilde{t}_0$ under the conditions of Theorem 2. Comparing (2.9) and (2.3), we see that this means that $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ in Theorem C converges generally to $W_0(z)$ in

$$(2.10)\qquad D_0 := \{z \in D_0^*; r(z) > 1 \text{ and } \limsup |W_n(z)| < \infty\}.$$

2. Also, $K(a_n(z)/b_n(z))$ in Theorem C converges generally in D_0 to the value $f(z) := \lim S_n(W_n(z), z)$ by Theorem 2. If $D_0 \neq \varnothing$, then $F(z) := \lim S_n(W_n(z), z)$ in D_0^* is a meromorphic continuation of $f(z)$ by Theorem C.

3. The conclusion of Theorem 2 still holds if $(\alpha_n - \tilde{\alpha}_n)$ and/or $(\beta_n - \tilde{\beta}_n)$ no longer converge to 0, as long as $\limsup |\alpha - \tilde{\alpha}_n|$ and $\limsup |\beta_n - \tilde{\beta}_n|$ are *small enough*.

We are now ready to state the main theorem of this paper.

Theorem 3. *Let* $K(\tilde{a}_n(z)/\tilde{b}_n(z)) \in \mathrm{Ref}\,(D^*, \{M_n(z)\})$ *and* $K(a_n(z)/b_n(z)) \in Near\,(K(\tilde{a}_n(z)/\tilde{b}_n(z)), \{C_n(z)\}, D^*)$ *for a given domain* D^* *and given sequences* $\{M_n(z)\}$ *and* $\{C_n(z)\}$ *of real-valued, nonnegative functions on* D^*. *Then* $\{S_n(W_n(z), z)\} \in \mathrm{Uniconv}\,(D_0^*)$, *where* $D_0^* \subseteq D^*$ *is such that*

$$\sum_{n=1}^{\infty} C_n(z) M_{n-1}^*(z) |\tilde{b}_n(z) + W_n(z)|^{-1} < \infty, \tag{2.11}$$

converges locally uniformly in D_0^*, *where*

$$M_k^*(z) := \sum_{n=0}^{k} M_n(z). \tag{2.12}$$

For the situation in Theorem C this means in particular that we can reach points where $r_1(z) = \min\{1, r(z)\}$ under additional conditions:

Corollary 4. *Let* $M(z)$, $r(z)$ *and* $C_n(z)$ *be real-valued, nonnegative functions in a domain* D^* *with* $r(z) \neq 0$, *and assume that* $M(z)$ *and* $r(z)$ *are continuous in* D^*. *Further, let* $K(\tilde{a}_n(z)/\tilde{b}_n(z)) \in \mathrm{Ref}\,(D^*, \{M(z)r(z)^{-n}\})$ *and* $K(a_n(z)/b_n(z)) \in Near\,(K(\tilde{a}_n(z)/\tilde{b}_n(z)), \{C_n(z)r_1(z)^n\}, D^*)$, *where* $r_1(z) := \min\{1, r(z)\}$ *for all* $z \in D^*$. *Then* $\{S_n(W_n(z), z)\} \in \mathrm{Uniconv}\,(D_0^*)$, *where* $D_0^* \subseteq D^*$ *is such that* $\liminf |\tilde{b}_n(z) + W_n(z)|$ *has positive lower bounds in compact subsets of* D_0^*,

$$\sum C_n(z) \quad \text{converges locally uniformly in } \{z \in D_0^*; r(z) \neq 1\},$$

and

$$\sum nC_n(z) \quad \textit{converges locally uniformly in } \{z \in D_0^*; r(z) = 1\}.$$

3. Sharpness. The conditions in Theorem 3 are sufficient conditions. In what way are they necessary for the existence of an analytic continuation of $f(z)$ beyond D? This is naturally a question of which reference continued fraction $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ we choose to use. If we insist on using a certain reference continued fraction, the conditions are in no way necessary. One has for instance

$$\log(1+z) = \frac{z}{1} {}_{+} \frac{\frac{1^2}{1\cdot 2}z}{1} {}_{+} \frac{\frac{1^2}{2\cdot 3}z}{1} {}_{+} \frac{\frac{2^2}{3\cdot 4}z}{1} {}_{+} \frac{\frac{2^2}{4\cdot 5}z}{1} {}_{+} \frac{\frac{3^2}{5\cdot 6}z}{1} {}_{+} \cdots$$

for $z \in D := \{z \in \mathbf{C}; |\arg(1+z) < \pi\}$. That is, $\log(1+z) = K(a_n z/1)$ where $a_n \to a := 1/4$; i.e., $K(a_n z/1)$ is *close to* $K((z/4)/1)$, and

$$a_{2n} - a = \frac{1}{4(2n-1)}, \qquad a_{2n+1} - a = -\frac{1}{4(2n+1)}.$$

We see that $K(a_n z/1)$ does not satisfy the conditions for closeness to $K((z/4)/1)$ in Theorem 3. Still, $\log(1+z)$ can be continued analytically to a Riemann surface with a logarithmic branch point at $z = -1$.

On the other hand, we could, as an extreme choice, let the reference continued fraction be the continued fraction $K(a_n(z)/b_n(z))$ itself. Then $K(a_n(z)/b_n(z))$ *inherits* all its properties. However, the conditions are necessary in the sense that if we let $a_n := 1/4 + r^n$ for some r, $0 \le r < 1$, then the modified approximants $S_n(W(z), z)$ of $K(a_n z/1)$ diverge for a $z \in \overline{D_r^*}$ (the closure of D_r^* in D^*), where $W(z)$ and D_r^* are given by (1.7):

Theorem 5. *Let $0 < r < 1$ be fixed. Then the sequence*

$$S_n(W(z), z) := \frac{(\frac{1}{4}+r)z}{1} {}_{+} \frac{(\frac{1}{4}+r^2)z}{1} {}_{+} \cdots {}_{+} \frac{(\frac{1}{4}+r^n)z}{1+W(z)}$$
$$\textit{for } n = 1, 2, 3, \ldots,$$

diverges for $z := 4r/(1-r)^2$, where $W(z) := -(\sqrt{1+z}+1)/2$ with $\Re\sqrt{1+z} > 0$.

Observe that $z = 4r/(1-r)^2 \in \partial D_r^*$ in the sheet $D^*\backslash\overline{D}$ since for this value of z we have

$$W(z) = -\frac{1}{2}(\sqrt{1+z}+1) = -\frac{1}{2}\left(\frac{1+r}{1-r}+1\right) = -\frac{1}{1-r}$$

so that

$$\left|\frac{W(z)}{1+W(z)}\right| = \left|\frac{-1/(1-r)}{1-1/(1-r)}\right| = \frac{1}{r}.$$

4. Proofs. We shall first prove Theorem 2 and Theorem 3.

Proof of Theorem 2. Because of (2.8), we have under our conditions that $\tilde{t}_n \neq \infty$, $\tilde{t}_n \neq 0$ and $\tilde{\beta}_n + \tilde{t}_n \neq 0$ from some n on. Without loss of generality, we assume that this and (2.9) hold for all $n \in \mathbf{N}_0$. (Otherwise, we could have first proved convergence for a tail (2.2) of $K(\tilde{\alpha}_n/\tilde{\beta}_n)$.) We can therefore define

$$d_n := \frac{1}{|\tilde{\beta}_n + \tilde{t}_n|}\sum_{m=0}^{n-1}\prod_{j=n-m}^{n-1}\left|\frac{\tilde{t}_j}{\tilde{\beta}_j+\tilde{t}_j}\right| \qquad \text{for } n = 0, 1, 2, \dots . \tag{4.1}$$

(We follow the standard convention that an empty sum has the value 0 and an empty product has the value 1.) Then

$$d_n|\tilde{\beta}_n + \tilde{t}_n| - d_{n-1}|\tilde{t}_{n-1}| = 1. \tag{4.2}$$

Let $1/2 < r < 2/3$. According to [**3**, Theorem 4.1B] (with $D = 1$, $\mu = 1-r < 1/2$ and $c_n = (2-r)/4$), we have that if

(4.3)

$$|\alpha_n - \tilde{\alpha}_n| \leq \frac{c_n}{2d_nd_{n-1}} = \frac{2-r}{8d_nd_{n-1}} \quad \text{and} \quad |\beta_n - \tilde{\beta}_n| \leq \frac{2-r}{4d_n(2d_{n-1}|\tilde{t}_{n-1}|+r)}$$

from some n on, then the limit

$$\lim_{n\to\infty} T_n(w_n) \qquad \text{where } |w_n - \tilde{t}_n| \leq \frac{1-\mu}{2d_n} = \frac{r}{2d_n} \tag{4.4}$$

exists if $\prod_{n=1}^{\infty} Q_n = 0$, where

$$Q_n = \frac{(d_{n-1}|\tilde{t}_{n-1}| + r/2)^2}{d_n|\tilde{\beta}_n + \tilde{t}_n|d_{n-1}|\tilde{t}_{n-1}| + r(2-r)/8}.$$

This limit (4.4) is independent of the actual choice of $\{w_n\}$. In our case it follows by (4.2) that

$$Q_n = 1 - \frac{(1-r)G_n + r/4 - 3r^2/8}{(1+G_n)G_n + r(2-r)/8} \qquad \text{where } G_n = d_{n-1}|\tilde{t}_{n-1}|.$$

We have by (4.1) and (2.9) that

$$G_{n+1} = d_n|\tilde{t}_n| = \sum_{m=0}^{n-1} \prod_{j=n-m}^{n} \left| \frac{\tilde{t}_j}{\tilde{\beta}_j + \tilde{t}_j} \right| \leq \sum_{m=0}^{n-1} M_{m+1} \leq M,$$

where

$$M = \sum_{m=0}^{\infty} M_m < \infty. \tag{4.5}$$

Further, $1 - r > 0$ and $r/4 - 3r^2/8 > 0$. This means that $\prod Q_n = 0$ under our conditions. Further, by (4.2),

$$\limsup d_n = \limsup \frac{1 + G_n}{|\tilde{\beta}_n + \tilde{t}_n|} =: \lambda < \infty$$

since $\liminf |\tilde{\beta}_n + \tilde{t}_n| > 0$. This again means that (4.3) holds, at least from some n on, and the limit (4.4) exists. Moreover, we can choose two sequences $\{u_n\}$ and $\{v_n\}$ such that

$$|u_n - \tilde{t}_n| < \frac{r}{2d_n}, \qquad |v_n - \tilde{t}_n| < \frac{r}{2d_n} \quad \text{and} \quad |u_n - v_n| > \frac{r}{4\lambda} > 0$$

from some n on. By (4.4), it follows that $\lim T_n(u_n) = \lim T_n(v_n) = \psi$ for some $\psi \in \hat{\mathbf{C}}$. Hence, the continued fraction $K(\tilde{\alpha}_n/\tilde{\beta}_n)$ converges generally to ψ. $\square$

To prove Theorem 3, we shall use the notions of canonical numerators A_n and canonical denominators B_n of a continued fraction $K(\alpha_n/\beta_n)$. That is, $\{A_n\}$ and $\{B_n\}$ are solutions of the recurrence relation

$$X_n = \beta_n X_{n-1} + \alpha_n X_{n-2} \qquad \text{for } n = 1, 2, 3, \ldots \tag{4.6}$$

with initial values $A_{-1}=1$, $A_0=0$, $B_{-1}=0$, $B_0=1$, so that

$$S_n(w) := \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_n}{\beta_n + w} = \frac{A_n + A_{n-1}w}{B_n + B_{n-1}w}. \tag{4.7}$$

Similarly, we shall use $A_n^{(k)}$ and $B_n^{(k)}$ to denote the canonical numerators and denominators of the kth tail

$$\mathop{\mathbf{K}}_{n=1}^{\infty}(\alpha_{n+k}/\beta_{n+k}) = \frac{\alpha_{k+1}}{\beta_{k+1}} + \frac{\alpha_{k+2}}{\beta_{k+2}} + \frac{\alpha_{k+3}}{\beta_{k+3}} + \cdots \tag{4.8}$$

of $K(\alpha_n/\beta_n)$. For later reference, we state the following standard results which can be proved by straightforward induction using the relations (4.6) and (4.7). Here $\{t_n\}$ denotes a tail sequence for $K(\alpha_n/\beta_n)$.

$$B_n + B_{n-1}t_n = \prod_{k=1}^{n}(\beta_k + t_k) \qquad \text{if all } t_k \neq \infty, \text{ [}\mathbf{6}\text{]} \tag{4.9}$$

$$B_n = \sum_{m=0}^{n}\left(\prod_{k=1}^{m}(\beta_k + t_k)\prod_{k=m+1}^{n}(-t_k)\right) \qquad \text{if all } t_k \neq \infty, \text{ [}\mathbf{6}\text{]} \tag{4.10}$$

$$A_n = t_0\sum_{m=1}^{n}\left(\prod_{k=1}^{m}(\beta_k + t_k)\prod_{k=m+1}^{n}(-t_k)\right) \qquad \text{if all } t_k \neq \infty, \text{ [}\mathbf{6}\text{]} \tag{4.11}$$

$$B_{n+1}^{(k)} = \beta_{k+1}B_n^{(k+1)} + \alpha_{k+2}B_{n-1}^{(k+2)} \qquad \text{(induction on } n\text{)}, \tag{4.12}$$

$$A_{n+1}^{(k)} = \alpha_{k+1}B_n^{(k+1)}. \tag{4.13}$$

In the following, we let $A_n, B_n, A_n^{(k)}, B_n^{(k)}$ and S_n refer to the continued fraction $K(a_n(z)/b_n(z))$ and $\tilde{A}_n, \tilde{B}_n, \tilde{A}_n^{(k)}, \tilde{B}_n^{(k)}$ and $\tilde{S}_n$ refer to $K(\tilde{a}_n(z)/\tilde{b}_n(z))$.

Lemma 1. *Let $K(\tilde{a}_n(z)/\tilde{b}_n(z)) \in \mathrm{Ref}\,(D^*, \{M_n(z)\})$ for a given domain D^* and a given sequence $\{M_n(z)\}$ of real-valued, nonnegative functions on D^*. Then*

$$\begin{aligned}\left|\frac{\tilde{A}_n(z)}{\prod_{k=1}^{n}(\tilde{b}_k(z) + W_k(z))}\right| &\leq |W_0(z)|\sum_{k=0}^{n-1}M_k(z),\\ \left|\frac{\tilde{B}_n(z)}{\prod_{k=1}^{n}(\tilde{b}_k(z) + W_k(z))}\right| &\leq \sum_{k=0}^{n}M_k(z)\end{aligned} \tag{4.14}$$

for all $n \in \mathbf{N}_0$ *and* $z \in D^*$.

Proof. $\{W_k(z)\}$ is a tail sequence for $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ with all $W_k(z) \neq \infty$ in D^*. Hence, the result follows from (4.10), (4.11) and (2.3) $\square$

The next lemma is well known. See, for instance, [**12**]. We include the proof for completeness.

Lemma 2. *Let* $A > 0$, $c_n \geq 0$ *and* $d_n \geq 0$ *satisfy*

$$c_n \leq A + \sum_{k=0}^{n-1} d_k c_k \qquad \textit{for } n = 0, 1, 2, \dots .$$

Then

$$c_n \leq A \exp\left(\sum_{k=0}^{n-1} d_k\right) \qquad \textit{for } n = 0, 1, 2, \dots .$$

Proof. The result is trivially true for $n = 0$. Assume that it holds for all n-values up to and including $n - 1$. Then

$$\begin{aligned} c_n &\leq A + \sum_{k=0}^{n-1} d_k \left(A \exp\left(\sum_{j=0}^{k-1} d_j\right)\right) \\ &\leq A + A \sum_{k=0}^{n-1} \left((\exp(d_k) - 1) \exp\left(\sum_{j=0}^{k-1} d_j\right)\right) \\ &= A \exp\left(\sum_{j=0}^{n-1} d_j\right). \qquad \square \end{aligned}$$

Lemma 3. *Let* $K(\tilde{a}_n(z)/\tilde{b}_n(z)) \in \operatorname{Ref}(D^*, \{M_n(z)\})$ *and* $K(a_n(z)/b_n(z)) \in$ *Near*$(K(\tilde{a}_n(z)/\tilde{b}_n(z)), \{C_n(z)\}, D^*)$ *for a given domain* D^* *and given sequences* $\{M_n(z)\}$ *and* $\{C_n(z)\}$ *of real-valued, nonnegative* *functions on D^*. Then*

$$\text{(4.15)}\quad \frac{|A_n(z)|}{\prod_{k=1}^n |\tilde{b}_k(z)+W_k(z)|} \le \left|\frac{a_1(z)}{\tilde{a}_1(z)}W_0(z)\right| M_{n-1}^*(z)\exp\left(2\sum_{k=2}^n \frac{C_k(z)M_{k-2}^*(z)}{|\tilde{b}_k(z)+W_k(z)|}\right)$$

and

$$\text{(4.16)}\quad \frac{|B_n(z)|}{\prod_{k=1}^n |\tilde{b}_k(z)+W_k(z)|} \le M_n^*(z)\exp\left(2\sum_{k=1}^n \frac{C_k(z)M_{k-1}^*(z)}{|\tilde{b}_k(z)+W_k(z)|}\right)$$

for all $z\in D^*$ and for all $n\in\mathbf{N}$, where $M_n^*(z)=\sum_{k=0}^n M_k(z)$.

Proof. We first prove (4.16). Let k and m be nonnegative integers with $k<m$. Then it follows from (4.12) (suppressing the variable z) that

$$\text{(4.17)}\quad \tilde{B}_{m-k}^{(k)} = \tilde{b}_{k+1}\tilde{B}_{m-k-1}^{(k+1)} + \tilde{a}_{k+2}\tilde{B}_{m-k-2}^{(k+2)}.$$

Further, $\{B_k\}$ satisfies the recurrence relation

$$\text{(4.18)}\quad B_{k+1} = b_{k+1}B_k + a_{k+1}B_{k-1}.$$

Multiplying (4.17) by B_k and (4.18) by $\tilde{B}_{m-k-1}^{(k+1)}$, and then subtracting the two equations give

$$\text{(4.19)}\quad \tilde{B}_{m-k}^{(k)}B_k - B_{k+1}\tilde{B}_{m-k-1}^{(k+1)} = (\tilde{b}_{k+1}-b_{k+1})B_k\tilde{B}_{m-k-1}^{(k+1)} + \tilde{a}_{k+2}\tilde{B}_{m-k-2}^{(k+2)}B_k - a_{k+1}B_{k-1}\tilde{B}_{m-k-1}^{(k+1)}.$$

Summing equation (4.19) for $k=0,1,\dots,m-1$ then gives

$$\begin{aligned}\tilde{B}_m^{(0)}B_0 - B_m\tilde{B}_0^{(m)} &= \sum_{k=0}^{m-1}(\tilde{b}_{k+1}-b_{k+1})\tilde{B}_{m-k-1}^{(k+1)}B_k \\ &\quad + \sum_{k=0}^{m-1}(\tilde{a}_{k+1}-a_{k+1})\tilde{B}_{m-k-1}^{(k+1)}B_{k-1} \\ &\quad + \tilde{a}_{m+1}\tilde{B}_{-1}^{(m+1)}B_{m-1} - \tilde{a}_1B_{-1}\tilde{B}_{m-1}^{(1)}.\end{aligned}$$

Finally, solving for B_m and using the initial values for the B_n-sequences, give the following basic formula:

(4.20)

$$B_m = \tilde{B}_m^{(0)} + \sum_{k=0}^{m-1} \Big((b_{k+1} - \tilde{b}_{k+1}) \tilde{B}_{m-k-1}^{(k+1)} + (a_{k+2} - \tilde{a}_{k+2}) \tilde{B}_{m-k-2}^{(k+2)} \Big) B_k.$$

Dividing (4.20) by $\prod_{j=1}^{m+1} (\tilde{b}_j + W_j)$ and using the notation

$$\Delta_m := \left| \frac{B_m}{\prod_{j=1}^{m+1} (\tilde{b}_j + W_j)} \right|, \qquad \tilde{\Delta}_m^{(k)} := \left| \frac{\tilde{B}_m^{(k)}}{\prod_{j=k+1}^{k+m+1} (\tilde{b}_j + W_j)} \right|$$

give the inequality

(4.21)

$$\Delta_m \leq \tilde{\Delta}_m^{(0)} + \sum_{k=0}^{m-1} \left(|b_{k+1} - \tilde{b}_{k+1}| \tilde{\Delta}_{m-k-1}^{(k+1)} + \left| \frac{a_{k+2} - \tilde{a}_{k+2}}{\tilde{b}_{k+2} + W_{k+2}} \right| \tilde{\Delta}_{m-k-2}^{(k+2)} \right) \Delta_k.$$

Since Lemma 1 also applies to the kth tail of $K(\tilde{a}_n/\tilde{b}_n)$ (tail as in (4.8)), we get from (4.14) that

$$|\tilde{b}_{n+k+1} + W_{n+k+1}| \tilde{\Delta}_n^{(k)} \leq M_n^* \qquad \text{for all } n, k \in \mathbf{N}_0. \tag{4.22}$$

Further, we have assumed (2.4). Inserting (2.4) and (4.22) into (4.21) gives

$$|\tilde{b}_{m+1} + W_{m+1}| \Delta_m \leq M_m^* + \sum_{k=0}^{m-1} 2C_{k+1} M_{m-k-1}^* \Delta_k$$

since $\tilde{\Delta}_{-1}^{(m+1)} = 0 < M_0^*$ and $M_{m-k-2}^* \leq M_{m-k-1}^*$, that is,

$$c_m \leq 1 + \sum_{k=0}^{m-1} \frac{2C_{k+1} M_{m-k-1}^*}{M_m^*} c_k \frac{M_k^*}{|\tilde{b}_{k+1} + W_{k+1}|}$$

where

$$c_k := \frac{|\tilde{b}_{k+1} + W_{k+1}| \Delta_k}{M_k^*}.$$

Observe that $M_{m-k-1}^* \leq M_m^*$ so that $M_{m-k-1}^*/M_m^* \leq 1$, and we can use Lemma 2 on the inequality

$$c_m \leq 1 + \sum_{k=0}^{m-1} \frac{2C_{k+1} M_k^*}{|\tilde{b}_{k+1} + W_{k+1}|} c_k$$

to obtain

$$c_m = \frac{|\tilde{b}_{m+1} + W_{m+1}|}{M_m^*} \Delta_m \le \exp\left(\sum_{k=0}^{m-1} \frac{2C_{k+1}M_k^*}{|\tilde{b}_{k+1} + W_{k+1}|} \right),$$

and thus,

$$\frac{|B_m|}{\prod_{k=1}^m |\tilde{b}_k + W_k|} = |\tilde{b}_{m+1} + W_{m+1}|\Delta_m \le M_m^* \exp\left(\sum_{k=0}^{m-1} \frac{2C_{k+1}M_k^*}{|\tilde{b}_{k+1} + W_{k+1}|} \right)$$

which gives (4.16).

To prove (4.15), we observe that applying (4.16) to the first tails of the two continued fractions involved gives

$$\frac{|B_{n-1}^{(1)}|}{\prod_{k=2}^n |\tilde{b}_k + W_k|} \le M_{n-1}^* \exp\left(2 \sum_{k=2}^{n} \frac{C_k M_{k-2}^*}{|\tilde{b}_k + W_k|} \right), \tag{4.23}$$

where $B_n^{(1)}$ denotes the nth canonical denominator of the first tail of $K(a_n/b_n)$. Hence, (4.15) follows from (4.13) and (4.23) since $\tilde{a}_1 = W_0(\tilde{b}_1 + W_1)$ by (2.8). □

These new estimates in Lemma 3 improve the ones in [**1, 2, 11**]. They are the basis for the proof of Theorem 3:

Proof of Theorem 3. We want to prove locally uniform convergence of

$$S_n(W_n(z), z) = \frac{A_n(z) + A_{n-1}(z)W_n(z)}{B_n(z) + B_{n-1}(z)W_n(z)}.$$

Suppressing the variable z we have that a solution $\{X_n\}$ of the recurrence relation (4.18) (such as $\{A_n\}$ and $\{B_n\}$) satisfies

(4.24)
$$\begin{aligned}X_n + X_{n-1}W_n &= (b_n + W_n)X_{n-1} + a_nX_{n-2}\\ &= (\tilde{b}_n + W_n)(X_{n-1} + W_{n-1}X_{n-2}) + (b_n - \tilde{b}_n)X_{n-1}\\ &\quad + (a_n - \tilde{a}_n)X_{n-2}\\ &= (X_0 + W_0X_{-1})\prod_{k=1}^{n}(\tilde{b}_k + W_k)\\ &\quad + \sum_{m=1}^{n}\left((b_m - \tilde{b}_m)X_{m-1}\prod_{k=m+1}^{n}(\tilde{b}_k + W_k)\right)\\ &\quad + \sum_{m=1}^{n}\left((a_m - \tilde{a}_m)X_{m-2}\prod_{k=m+1}^{n}(\tilde{b}_k + W_k)\right),\end{aligned}$$

since $\tilde{a}_m = W_{m-1}(\tilde{b}_m + W_m)$. For $X_n = A_n$ we have $A_{-1} = 1$, $A_0 = 0$, so that

$$\begin{aligned}A_n + A_{n-1}W_n &= W_0\prod_{k=1}^{n}(\tilde{b}_k + W_k)\\ &\quad + \sum_{m=2}^{n}\left((b_m - \tilde{b}_m)A_{m-1}\prod_{k=m+1}^{n}(\tilde{b}_k + W_k)\right)\\ &\quad + \sum_{m=1}^{n}\left((a_m - \tilde{a}_m)A_{m-2}\prod_{k=m+1}^{n}(\tilde{b}_k + W_k)\right).\end{aligned} \tag{4.25}$$

Similarly, for $X_n = B_n$ we have $B_{-1} = 0$, $B_0 = 1$, which gives

$$\begin{aligned}B_n + B_{n-1}W_n &= \prod_{k=1}^{n}(\tilde{b}_k + W_k)\\ &\quad + \sum_{m=1}^{n}\left((b_m - \tilde{b}_m)B_{m-1}\prod_{k=m+1}^{n}(\tilde{b}_k + W_k)\right)\\ &\quad + \sum_{m=2}^{n}\left((a_m - \tilde{a}_m)B_{m-2}\prod_{k=m+1}^{n}(\tilde{b}_k + W_k)\right).\end{aligned} \tag{4.26}$$

Hence,

$$(4.27)\quad S_n(W_n) = \frac{W_0+\sum_{m=2}^{n}(b_m-\tilde b_m)\frac{A_{m-1}}{\prod_{k=1}^{m}(\tilde b_k+W_k)}+\sum_{m=1}^{n}(a_m-\tilde a_m)\frac{A_{m-2}}{\prod_{k=1}^{m}(\tilde b_k+W_k)}}{1+\sum_{m=1}^{n}(b_m-\tilde b_m)\frac{B_{m-1}}{\prod_{k=1}^{m}(\tilde b_k+W_k)}+\sum_{m=2}^{n}(a_m-\tilde a_m)\frac{B_{m-2}}{\prod_{k=1}^{m}(\tilde b_k+W_k)}}.$$

This formula was also used in [**1**].

We shall first see that the four sums in (4.27) converge absolutely in D_0^* as $n\to\infty$. We have by (2.11) that

$$(4.28)\qquad \sum_{k=1}^{\infty}\frac{C_kM_{k-1}^*}{|\tilde b_k+W_k|}=:\Lambda<\infty\qquad\text{for } z\in D_0^*$$

(where Λ depends on z), and so, since $M_{k-2}^*\le M_{k-1}^*$, that

$$\sum_{k=2}^{\infty}\frac{C_kM_{k-2}^*}{|\tilde b_k+W_k|}\le\Lambda\qquad\text{for } z\in D_0^*,$$

so that, by (2.4), (4.15) and (4.16),

$$\begin{aligned}\sum_{m=2}^{\infty}|b_m-\tilde b_m|\frac{|A_{m-1}|}{\prod_{k=1}^{m}|\tilde b_k+W_k|}&\le\sum_{m=2}^{\infty}C_m\left|\frac{a_1}{\tilde a_1}W_0\right|\frac{M_{m-2}^*}{|\tilde b_m+W_m|}\exp(2\Lambda)\\&=\left|\frac{a_1}{\tilde a_1}W_0\right|\exp(2\Lambda)\sum_{m=2}^{\infty}\frac{C_mM_{m-2}^*}{|\tilde b_m+W_m|}\\&\le\left|\frac{a_1}{\tilde a_1}W_0\right|\exp(2\Lambda)\Lambda<\infty\qquad\text{for } z\in D_0^*,\end{aligned}$$

$$\begin{aligned}\sum_{m=1}^{\infty}|a_m-\tilde a_m|&\frac{|A_{m-2}|}{\prod_{k=1}^{m}|\tilde b_k+W_k|}\\&=\frac{|a_1-\tilde a_1|}{|\tilde b_1+W_1|}+\sum_{m=3}^{\infty}\frac{|a_m-\tilde a_m|}{|\tilde b_m+W_m|}\frac{|A_{m-2}|}{|\prod_{k=1}^{m-1}\tilde b_k+W_k|}\\&\le C_1+\sum_{m=3}^{\infty}C_{m-1}\left|\frac{a_1}{\tilde a_1}W_0\right|\frac{M_{m-2}^*}{|\tilde b_{m-1}+W_{m-1}|}\exp(2\Lambda)\\&=C_1+\left|\frac{a_1}{\tilde a_1}W_0\right|\exp(2\Lambda)\sum_{m=3}^{\infty}\frac{C_{m-1}M_{m-2}^*}{|\tilde b_{m-1}+W_{m-1}|}\\&\le C_1+\left|\frac{a_1}{\tilde a_1}W_0\right|\exp(2\Lambda)\Lambda<\infty\qquad\text{for } z\in D_0^*,\end{aligned}$$

$$\sum_{m=1}^{\infty}|b_m-\tilde{b}_m|\frac{|B_{m-1}|}{\prod_{k=1}^{m}|\tilde{b}_k+W_k|}\le\sum_{m=1}^{\infty}\frac{C_mM_{m-1}^*}{|\tilde{b}_m+W_m|}\exp(2\Lambda)$$
$$=\Lambda\exp(2\Lambda)\qquad\text{for } z\in D_0^*$$

and

$$\sum_{m=2}^{\infty}|a_m-\tilde{a}_m|\frac{|B_{m-2}|}{\prod_{k=1}^{m}|\tilde{b}_k+W_k|}\le\sum_{m=2}^{\infty}C_{m-1}\frac{M_{m-2}^*}{|\tilde{b}_{m-1}+W_{m-1}|}\exp(2\Lambda)$$
$$\le\Lambda\exp(2\Lambda)\qquad\text{for } z\in D_0^*.$$

Note also that M_k^* is nondecreasing as k increases. Hence, by (2.11),

$$\sum_{n=1}^{\infty}\frac{C_n}{|\tilde{b}_n+W_n|}<\infty\qquad\text{for } z\in D_0^*.$$

Let $E\subseteq D_0^*$ be compact, and let $\varepsilon>0$ be arbitrarily chosen. Then the series

$$\sum_{n=1}^{\infty}\frac{C_nM_{n-1}^*}{|\tilde{b}_n+W_n|}\quad\text{and}\quad\sum_{n=1}^{\infty}\frac{C_n}{|\tilde{b}_n+W_n|}$$

converge uniformly in D_0^*, and there exists an $\tilde{N}\in\mathbf{N}$ such that

$$\sum_{n=1}^{\infty}\frac{C_{\tilde{N}+n}M_{n-1}^*}{|\tilde{b}_{\tilde{N}+n}+W_{\tilde{N}+n}|}<\varepsilon\qquad\text{for all } z\in E.$$

This in turn means that to a given q, $0<q<1$, there exists an $N\in\mathbf{N}$ such that

$$\sum_{m=1}^{\infty}|b_{N+m}-\tilde{b}_{N+m}|\frac{|B_{m-1}^{(N)}|}{\prod_{k=N+1}^{N+m}|\tilde{b}_k+W_k|}$$
$$+\sum_{m=2}^{\infty}|a_{N+m}-\tilde{a}_{N+m}|\frac{|B_{m-2}^{(N)}|}{\prod_{k=N+1}^{N+m}|\tilde{b}_k+W_k|}\le q<1$$

for all $z\in E$. Hence, $S_n^{(N)}(W_{N+n})$ (with the obvious interpretation) converges uniformly to a holomorphic function in E. Since $S_{N+n}(W_{N+n})=S_N(S_n^{(N)}(W_{N+n}))$ where S_N is a linear fractional transformation; this proves the convergence of $S_n(W_n)$. □

Proof of Theorem C. Again we suppress the variable z in the computation. The result follows from Theorem 3 if we can prove that

$$\sum_{n=1}^{\infty} \frac{Cr_1^n M_{n-1}^*}{|\tilde{b}_n + W_n|} < \infty \tag{4.29}$$

locally uniformly in D_0^*, where

$$M_{n-1}^* = \sum_{k=0}^{n-1} Mr^{-k} = \begin{cases} M\frac{1-r^{-n}}{1-r^{-1}} & \text{if } r \neq 1, \\ nM & \text{if } r = 1. \end{cases} \tag{4.30}$$

Clearly (4.29) holds since $\sum (r_1/r)^n < \infty$ if $r < 1$, $\sum nr_1^n < \infty$ if $r = 1 > r_1$ and $\sum r_1^n < \infty$ if $r > 1$, and since $M(z)$, $r(z)$, $C(z)$ and $r_1(z)$ are continuous in D^*. $\square$

Proof of Theorem 1. With the notation from Theorem 3, we have $|\tilde{b}_n(z) + W_n(z)| = |1 + W(z)|$, so that $\liminf |\tilde{b}_n(z) + W_n(z)| > 0$ for all z except at $z = 0$ in the sheet $D^* \setminus \overline{D}$. Further, as in the proof of Theorem C, (4.29) holds if $\sum C_n r^n M_{n-1}^* < \infty$ where M_{n-1}^* is as given in (4.30). $\square$

The sufficiency of the conditions (1.11) follows similarly.

Proof of Corollary 4. Again we suppress the variable z in the computation. The result follows (as in the proof of Theorem C) if we can prove that

$$\sum_{n=1}^{\infty} \frac{C_n r_1^n M_{n-1}^*}{|\tilde{b}_n + W_n|} \tag{4.31}$$

converges locally uniformly in D_0^*, where M_n^* is given by (4.30). We see immediately that (4.31) holds under our conditions. $\square$

To prove Theorem 5, we shall use the Bauer-Muir transformation: Let $\{w_n\}_{n=0}^{\infty}$ be a sequence of complex numbers ($w_n \neq \infty$). A Bauer-Muir transform of a continued fraction $K(\alpha_n/\beta_n)$ with respect to $\{w_n\}$ is then a continued fraction $\delta_0 + K(\gamma_n/\delta_n)$ whose classical approximants

$$S_n(0) := \delta_0 + \frac{\gamma_1}{\delta_1} \, {}_{+} \, \frac{\gamma_2}{\delta_2} \, {}_{+} \, \cdots \, {}_{+} \, \frac{\gamma_n}{\delta_n} \qquad \text{for } n = 0, 1, 2, \ldots$$

are exactly the modified approximants $T_n(w_n)$ of $K(\alpha_n/\beta_n)$ (T_n as given by (2.6)) and $S_0(0) = \delta_0 = w_0$. Such a continued fraction exists if and only if

$$\lambda_n := \alpha_n - w_{n-1}(\beta_n + w_n) \neq 0 \qquad \text{for } n = 1, 2, 3, \dots . \tag{4.32}$$

It can then be written

$$\delta_0 + K(\gamma_n/\delta_n) = w_0 + \frac{\lambda_1}{\beta_1 + w_1} + \frac{\alpha_1 \frac{\lambda_2}{\lambda_1}}{\beta_2 + w_2 - \frac{\lambda_2}{\lambda_1} w_0} + \frac{a_2 \frac{\lambda_3}{\lambda_2}}{\beta_3 + w_3 - \frac{\lambda_3}{\lambda_2} w_1} + \cdots . \tag{4.33}$$

(Information on the Bauer-Muir transformation can be found in [**7**, p. 25].)

Proof of Theorem 5. The Bauer-Muir transformation with respect to $W_n(z) = W(z) = ((1+z)^{1/2} - 1)/2$ applied to $K((1/4 + r^n)z/1)$ gives

$$W(z) + \frac{rz}{1 + W(z)} + \frac{(\frac{1}{4} + r)rz}{1 + (1-r)W(z)} + \frac{(\frac{1}{4} + r^2)rz}{1 + (1-r)W(z)} + \frac{(\frac{1}{4} + r^3)rz}{1 + (1-r)W(z)} + \cdots . \tag{4.34}$$

This is a limit periodic continued fraction which converges in D_r^* as given in (1.7). Its classical approximants are $\equiv T_n(W(z), z)$. For $z = 4r/(1-r)^2$ (in the sheet $D^* \backslash \overline{D}$), it gets the form

$$-\frac{1}{1-r} + \frac{4r^2/(1-r)^2}{-r/(1-r)} + \frac{(1+4r)r^2/(1-r)^2}{0} + \frac{(1+4r^2)r^2/(1-r)^2}{0} + \cdots \tag{4.35}$$

which diverges. This proves the result. □

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