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ON THE EXISTENCE OF THE CARATHÉODORY SOLUTIONS FOR SOME BOUNDARY VALUE PROBLEMS

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Dedicated to Paul Waltman on the occasion of his 60th birthday

ABSTRACT. In the paper we give some sufficient conditions for the existence of Carathéodory solutions to the Darboux-Goursat boundary value problem and the Dirichlet problem.

0. Introduction. The Darboux-Goursat and Dirichlet boundary value problems are, in general, considered in the Sobolev-type functional spaces where the differentiation of functions is understood in the generalized sense (cf., e.g., [**7, 6**]).

In the present paper we give some sufficient conditions for the existence of Carathéodory solutions to the Darboux-Goursat problem in the space of absolutely continuous functions in R^2. By a Carathéodory solution we mean here an absolutely continuous function which possesses the partial derivatives in the classical sense and satisfies the equation under consideration almost everywhere.

Making use of the properties of absolutely continuous functions, one can prove some generalization of the Du Bois-Reymond lemma (cf. [**3**]) for functions of several variables. The formulation of this lemma and an application to the Dirichlet-type boundary problems are given in the final part of the paper.

1. Absolutely continuous functions of several variables. Denote by P^k an interval in the space R^k, $k \geq 1$, of the form

$$P^k = \{x \in R^k; 0 \leq x^i \leq 1, i = 1, 2, \dots, k\}.$$

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Let $F = F(Q)$ be an additive function of an interval $Q \subset P^k$ where

$$Q = \{x \in R^k; 0 \le x_0^i \le x^i \le x_1^i \le 1, i = 1, 2, \dots, k\} \tag{1}$$

and the points x_0^i, x_1^i are fixed.

The function F is called absolutely continuous if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^n |F(Q_i)| < \varepsilon$ for any system of intervals $Q_i \subset P^2$ such that $\sum_{i=1}^n |Q_i| < \delta$ (cf. [**2, 5**]).

Let z be a real function defined on P^k. The function of an interval Q (cf. (1)) defined by the formula

$$F_z(Q) = \sum (-1)^{\sum_{i=1}^n \varepsilon_i} z(x_{\varepsilon_1}^1, x_{\varepsilon_2}^2, \dots, x_{\varepsilon_n}^n) \tag{2}$$

where $\varepsilon_i = 0, 1$ is called a function associated with z (the summation in formula (2) is taken over all systems $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$; $\varepsilon_i = 0, 1$) (cf. [**2, 5**]).

In the case $k = 2$, formula (2) takes the form

$$F_z(Q) = z(x_1^1, x_1^2) - z(x_0^1, x_1^2) + z(x_0^1, x_0^2) - z(x_1^1, x_0^2). \tag{3}$$

Definition 1. The function $z : P^k \to R$ is called an absolutely continuous function on P^k if the associated function $F_z(Q)$ is an absolutely continuous function of the interval $Q \subset P^k$ and each of the functions $z(0, x^2, \dots, x^k)$, $z(x^1, 0, x^3, \dots, x^k), \dots, z(x^1, \dots, x^{k-1}, 0)$ is an absolutely continuous function of $(k-1)$ variables. For $k = 1$, we adopt the classical definition of an absolutely continuous function of one variable.

The space of functions defined above will be denoted by $AC(P^k)$.

We shall prove

Theorem 1. *A function $z \in AC(P^2)$ if and only if*

$$z(x^1, x^2) = \int_0^{x^1} \int_0^{x^2} l^{1,2}(x^1, x^2) + \int_0^{x^1} l^1(x^1) + \int_0^{x^2} l^2(x^2) + c \tag{4}$$

where $l^{1,2} \in L^1(P^2)$, l^1, $l^2 \in L^1([0,1])$, $c \in R$.

Proof. Let $z \in AC(P^2)$. Consider the function

$$w(x^1, x^2) = z(x^1, x^2) - z(0, x^2) + z(0,0) - z(x^1, 0). \tag{5}$$

By formula (3), we have $F_z(Q) = F_w(Q)$ for any interval Q. By definition, $F_w(Q)$ is an absolutely continuous function of an interval. So there exists a function $l^{1,2} \in L^1(P^2)$ (cf. [**2, 5**]) such that

$$F_w(Q) = \iint_Q l^{1,2}(x^1, x^2) \tag{6}$$

for any interval $Q \subset P^2$. Denote by $Q(x^1, x^2)$ an interval of the form

$$Q(x^1, x^2) = \{(t^1, t^2); 0 \le t^1 \le x^1; 0 \le t^2 \le x^2\}.$$

From equalities (5) and (6), we get

$$\begin{aligned} w(x^1, x^2) = F_w(Q(x^1, x^2)) &= \iint_{Q(x^1,x^2)} l^{1,2}(t^1, t^2) \\ &= \int_0^{x^1} \int_0^{x^2} l^{1,2}(t^1, t^2) \\ &= z(x^1, x^2) - z(0, x^2) + z(0,0) - z(x^1, 0). \end{aligned}$$

The last equality and the well-known integral representation for absolutely continuous functions of one variable yield formula (4) with $c = z(0,0)$.

The second part of the theorem follows directly from the properties of the integral. □

Theorem 1′. *A function* $z \in AC(P^k)$ *if and only if*

$$\begin{aligned} z(x^1, \dots, x^k) = &\int_0^{x^1} \cdots \int_0^{x^k} l^{1,\dots,k} + \int_0^{x^2} \cdots \int_0^{x^k} l^{2,\dots,k} + \cdots \\ &+ \int_0^{x^1} \cdots \int_0^{x^{k-1}} l^{1,\dots,k-1} + \cdots + \int_0^{x^1} l^1 + \cdots \\ &+ \int_0^{x^k} l^k + c \end{aligned} \tag{7}$$

where $l^{i_1,i_2,\dots,i_s}$ are integrable functions on the respective intervals of dimensions $s = 1, 2, \dots, k$.

Proof. By Theorem 1, formula (7) holds for $k = 1$ and 2. Let us assume that any absolutely continuous function on P^{k-1}, $k > 2$, may be represented in form (7). Consider a function

$$\begin{aligned} w(x^1, x^2, \dots, x^k) = z(x^1, x^2, \dots, x^k) &- [z(0, x^2, \dots, x^k) \\ &+ z(x^1, 0, x^3, \dots, x^k) + \dots + z(x^1, x^2, \dots, x^{k-1}, 0)] + \dots \\ &+ (-1)^{k-1}[z(0, 0, \dots, x^k) + z(0, 0, \dots, x^{k-1}, 0) + \dots \\ &+ z(x^1, 0, 0, \dots, 0)] + (-1)^k z(0, 0, \dots, 0). \end{aligned}$$

By formula (2), we have $F_z(Q) = F_w(Q)$ for any interval $Q \subset P^k$. By definition 1, $F_w(Q)$ is an absolutely continuous function of an interval. So there exists a function $l^{1,2,\dots,k} \in L^1(P^k)$ such that

$$F_w(Q) = \int_Q l^{1,2,\dots,k}(x^1, x^2, \dots, x^k)$$

for any interval $Q \subset P^k$ (cf. [**2**]). Denote by $Q(x^1, x^2, \dots, x^k)$ an interval of the form

$$Q(x^1, x^2, \dots, x^k) = \{(t^1, t^2, \dots, t^k); 0 \le t^i \le x^i, i = 1, 2, \dots, k\}.$$

It is easy to notice that

$$\begin{aligned} w(x^1, x^2, \dots, x^k) &= F_w(Q(x^1, x^2, \dots, x^k)) \\ &= \int_0^{x^1} \cdots \int_0^{x^k} l^{1,2,\dots,k}(t^1, t^2, \dots, t^k). \end{aligned}$$

The above equality and the induction assumption yield formula (7). □

The space $AC(P^k)$ defined above is identical with the space of absolutely continuous functions defined in paper [**8**]. But the above definition and the proof of Theorems 1 and 1′ are more natural and simpler.

By basing oneself on integral representation (7), it is possible to prove that any $z \in AC(P^k)$ possesses all derivatives (in the classical sense) of the form

$$\frac{\partial^s z(x)}{\partial x^{i_1} \cdots \partial x^{i_s}}, \qquad s = 1, 2, \dots, k$$
$$1 \le i_1 < i_2 < \cdots < i_s \le k, \quad x \in P^k \text{ a.e.},$$

which are integrable on P^k.

Moreover, z possesses a total differential (Fréchet's differential) for almost all $x \in P^k$ (cf. [**9**]).

2. On the Darboux-Goursat boundary value problem. It is well known that the space $AC([0,1], R^m)$ is a 'good' space for ordinary differential systems

$$\dot{x} = f(t, x), \qquad x(0) = x_0. \tag{8}$$

This space is a 'good' space in the sense that system (8) possesses in $AC([0,1], R^m)$ a unique solution which continuously depends on the initial condition provided f is measurable with respect to t and lipschitzian with respect to x. The analogous role is played by the space $AC(P^2, R^m)$ for partial systems of the form

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, u\right) \tag{9}$$

with the boundary conditions of Darboux-Goursat type

$$z(x, 0) = \varphi(x), \qquad z(0, y) = \psi(y), \qquad \varphi(0) = \psi(0). \tag{10}$$

($AC(P^2, R^m)$ denotes the space of vector functions $z = z(x, y)$, $(x, y) \in P^2$, $z = (z^1, \dots, z^m)$ and z^i, $i = 1, 2, \dots, m$, is an absolutely continuous function in the sense of Definition 1). All derivatives in system (9) are understood in the classical sense.

We shall assume that:

(a) the function f is lipschitzian with respect to z, z_x, z_y, i.e., there exists a constant $L > 0$ such that

$$|f(x, y, w_0, w_1, w_2) - f(x, y, z_0, z_1, z_2)|$$
$$\le L(|w_0 - z_0| + |w_1 - z_1| + |w_2 - z_2|)$$

for $(x,y) \in P^2$ a.e. and for all points $(w_0, w_1, w_2) \in (R^m)^3$ and $(z_0, z_1, z_2) \in (R^m)^3$,

(b) for each point $(w_0, w_1, w_2) \in (R^m)^3$, the function $f(\cdot,\cdot,w_0,w_1,w_2)$ is measurable on P^2,

(c) there exists a point $(\bar{w}_0, \bar{w}_1, \bar{w}_2) \in (R^m)^3$ such that the function $f(\cdot,\cdot,\bar{w}_0,\bar{w}_1,\bar{w}_2)$ is integrable on P^2.

One can prove

Theorem 2. *If the above assumptions* (a)–(c) *hold and the functions* φ *and* ψ *are absolutely continuous on* $[0,1]$, *then the Darboux-Goursat problem* (9–10) *possesses a unique solution* $z \in AC(P^2, R^m)$. *The function* z *satisfies equation* (9) *for* $(x,y) \in P^2$ *a.e. and boundary conditions* (10) *for all* $x, y \in [0,1]$, *i.e.,* z *is a solution of system* (9–10) *in the sense of Carathéodory.*

For linear systems, the proof of Theorem 2 is given in [**8**] (for the general case, see [**1**]). Let us notice that this theorem is quite analogous to the well-known existence theorem for ordinary systems.

Sketch of the proof of Theorem 2. It is easy to observe that, by substituting

$$w(x,y) = z(x,y) - \varphi(x) - \psi(y) + c, \qquad c = \varphi(0) = \psi(0),$$

boundary value problem (9)–(10) can be reduced to a problem with homogeneous boundary conditions. Therefore, without loss of generality, one may assume that $\varphi(x) \equiv 0$ and $\psi(y) \equiv 0$. It follows from equality (4) that any function $z \in AC(P^2, R^m)$ satisfying the conditions $z(x,0) \equiv 0$ and $z(0,y) \equiv 0$ can be represented in the form

$$z(x,y) = \int_0^x \int_0^y g(x,y)\,dx\,dy$$

where $g \in L^1(P^2, R^m)$. So, a system with homogeneous boundary conditions can be represented in the form

$$g(x,y) = f\left(x, y, \int_0^x \int_0^y g, \int_0^y g, \int_0^x g\right).$$

Let $k > 1$ be a number satisfying the inequality

$$2mL\left(\frac{1}{k^2} + \frac{1}{k}\right) = \alpha < 1$$

where L is the constant from the Lipschitz condition. In the space $L^1(P^2, R^m)$ let us introduce a norm defined by the formula

$$||g||_k = \int_0^1 \int_0^1 e^{-k(x+y)} |g(x,y)|\, dx\, dy.$$

It is easy to show that

$$e^{-2k}||g||_{L^1} \le ||g||_k \le ||g||_{L^1}.$$

Consequently, the norms $||g||_k$ and $||g||_{L^1}$ are equivalent.

Denote by $F = (F^1, \dots, F^m) : L^1(P^2, R^m) \to L^1(P^2, R^m)$ an operator defined by the formula

$$F^j(g)(x,y) = f^j\left(x, y, \int_0^x \int_0^y g(x,y), \int_0^y g(x,y), \int_0^x g(x,y)\right).$$

It follows from the assumptions that the operator F is well defined. We shall demonstrate that F is a contracting operator. We have

$$\begin{aligned} ||F(g) - F(h)||_k = \int_0^1 \int_0^1 e^{-k(x+y)} \bigg| & f\left(x, y, \int_0^x \int_0^y g, \int_0^y g, \int_0^x g\right) \\ & - f\left(x, y, \int_0^x \int_0^y h, \int_0^y h, \int_0^x h\right)\bigg|\, dx\, dy \\ \le mL \int_0^1 \int_0^1 & e^{-k(x+y)} \int_0^x \int_0^y |g-h| \\ + mL \int_0^1 \int_0^1 & e^{-k(x+y)} \int_0^y |g-h| \\ + mL \int_0^1 \int_0^1 & e^{-k(x+y)} \int_0^x |g-h|. \end{aligned}$$

Integrating by parts, we obtain

$$\int_0^1 \int_0^1 \left(e^{-k(x+y)} \int_0^x |g-h|(s,y)\, ds\right) dx\, dy \le \frac{1}{k}||g-h||_k$$

and

$$\int_0^1 \int_0^1 e^{-k(x+y)} \int_0^y |g-h|(x,y) \leq \frac{1}{k}||g-h||_k.$$

Integrating by parts twice, we get the estimate

$$\int_0^1 \int_0^1 e^{-k(x+y)} \int_0^x \int_0^y |g-h|(x,y) \leq \frac{1}{k^2}||g-h||_k.$$

Consequently, we obtain

$$||F(g)-F(h)||_k \leq 2mL\left(\frac{1}{k^2}+\frac{1}{k}\right)||g-h||_k = \alpha||g-h||_k.$$

Since $\alpha < 1$, we ascertain that F is a contracting operator. So there exists exactly one point $g_0 \in L^1(P^2, R^m)$ such that $g_0 = F(g_0)$. Adopting

$$z_0(x,y) = \int_0^x \int_0^y g_0(x,y) + \varphi(x) + \psi(y) - c$$

where $c = \varphi(0) = \psi(0)$, we obtain a solution of system (9) in the space of absolutely continuous functions on P^2. $\square$

3. On a boundary value problem of Dirichlet type. Let a be any function from $L^2([0,1], R^m)$. The Du Bois-Reymond lemma (fundamental lemma) says that $a(t) = \text{const.}$ a.e. provided $\int_0^1 a(t)h'(t)\,dt = 0$ for each absolutely continuous function h such that $h(0) = h(1) = 0$ and $h' \in L^2$, i.e., $h \in W_0^{1,2}$. This lemma plays an essential role in the variational theory of ordinary differential equations with Dirichlet-type boundary conditions (cf. [**3, 4**]).

At present, we give some generalization of the Du Bois-Reymond lemma to the case of functions of several variables.

Let $AC_0^2(P^k, R^m)$ denote the space of all absolutely continuous vector functions $z = (z^1, \dots, z^m)$ such that $z^i \in AC(P^2)$, $z^i|_{\partial P^k} = 0$ and all cross derivatives

$$\frac{\partial^s z(x)}{\partial x^{i_1} \cdots \partial x^{i_s}}, \qquad s = 1, 2, \dots, k,$$
$$1 \leq i_1 < i_2 < \cdots < i_s \leq k, \quad i = 1, 2, \dots, m,$$

belong to $L^2(P^k)$.

Lemma 1. *If $a \in L^2(P^k, R^m)$ and, for any $h \in AC_0^2(P^k, R^m)$, the equality*

$$\int_{P^k} a(x) \frac{\partial^k}{\partial x^1, \dots, \partial x^k} h(x)\, dx = 0$$

holds, then the function a can be represented in the form

$$\begin{aligned} a = {} & (c_1 + \cdots + c_k) - (c_{1,2} + \cdots + c_{1,k} + c_{2,3} + \cdots + c_{2,k} + \cdots + c_{k-1,k}) \\ & + (c_{1,2,3} + \cdots + c_{1,2,k} + \cdots + c_{k-2,k-1,k}) \\ & + \cdots + (-1)^k (c_{1,2,\dots,k-1} + \cdots + c_{2,\dots,k}) + (-1)^{k+1} c_{1,2,\dots,k} \end{aligned}$$

where

$$c_{i_1,i_2,\dots,i_s} = \int_0^1 \int_0^1 \cdots \int_0^1 a(x)\, dx^{i_1}\, dx^{i_2} \dots dx^{i_s}, \\ 1 \le i_1 < i_2 < \cdots < i_s \le k.$$

The functions $c_{i_1,\dots,i_s}$ are square-integrable and constant with respect to the variables $x^{i_1}, \dots, x^{i_s}$. For $k = 2$, equality (10) takes the form

$$a(x^1, x^2) = c_1(x^2) + c_2(x^1) - c_{1,2}$$

where $c_1, c_2 \in L^2([0,1], R^m)$, $c_{1,2}$ is a constant vector.

The proof of Lemma 1 will be published in Bull. Soc. Math. Belgique (June 1993).

Further, we shall consider the case of two variables. Making use of Lemma 1, we shall prove

Lemma 2. *If functions $a_1, a_2, b \in L^2(P^2, R^m)$ and, for any function $h \in AC_0^2(P^2, R^m)$,*

$$\int_0^1 \int_0^1 (a_1 h_x + a_2 h_y + bh)\, dx\, dy = 0,$$

then

$$-\int_0^y a_1(x,y)\, dy - \int_0^x a_2(x,y)\, dx + \int_0^x \int_0^y b(x,y)\, dx\, dy = c_1(y) + c_2(x) - c$$

for some functions $c_1, c_2 \in L^2([0,1], R^m)$ *and* $c \in R^m$.

Proof. Integrating by parts, we have

$$\int_0^1\int_0^1 a_1 h_x\,dx\,dy = -\int_0^1\int_0^1 \left(\int_0^y a_1(x,v)\,dv\right) h_{xy}(x,y)\,dx\,dy,$$
$$\int_0^1\int_0^1 a_2 h_y\,dx\,dy = -\int_0^1\int_0^1 \left(\int_0^x a_2(u,y)\,du\right) h_{xy}(x,y)\,dx\,dy,$$
$$\int_0^1\int_0^1 bh\,dx\,dy = \int_0^1\int_0^1 \left(\int_0^x\int_0^y b(u,v)\,du\,dv\right) h_{xy}(x,y)\,dx\,dy$$

where $h \in AC_0^2(P^2, R^m)$.

By the assumption,

$$\int_0^1\int_0^1 \left[-\int_0^y a_1 - \int_0^x a_2 + \int_0^x\int_0^y b\right] h_{xy}\,dx\,dy = 0$$

for any function $h \in AC_0^2(P^2, R^m)$.

Making use of Lemma 1 we get the assertion of Lemma 2. □

Let f be a functional of the form

$$f(z) = \int_{P^k} F(x, z_x, z_y, z)\,dx = \int_{P^k} F(x, \nabla z, z)\,dx$$

defined in the Sobolev space $H_0^{1,2}(P^2, R^m)$.

We shall assume that

(a) f is a Gâteaux-differentiable functional, and the differential is defined by the formula

$$Df(z)\bar{z} = \int_{P^k} \left(\frac{\partial F}{\partial z_k}\bar{z}_x + \frac{\partial F}{\partial z_y}\bar{z}_y + \frac{\partial F}{\partial z}\bar{z}\right) dx\,dy, \tag{11}$$

(b) $\partial F/\partial z_x$, $\partial F/\partial z_y$ and $\partial F/\partial z$ are functions of the space $L^2(P^k, R^m)$, with any $z \in H_0^{1,2}$.

By making use of Lemma 2, it is easy to prove

Theorem 3. *If conditions* (11) *are satisfied and* $z^0 \in H_0^{1,2}$ *is a critical point of the functional* f, *that is,* $Df(z^0) = 0$, *then the point* z^0 *satisfies the Euler equation of the form*

$$-\int_0^y \frac{\partial F}{\partial z_x}(p^0)\,dy - \int_0^x \frac{\partial F}{\partial z_y}(p^0)\,dx + \int_0^x \int_0^y \frac{\partial F}{\partial z}(p^0)\,dx\,dy = c_1(y) + c_2(x) - c$$

where $p^0 = (x, y, z_x^0, z_y^0, z^0)$, c_1, c_2 *are some functions from* $L^2(P^2, R^m)$, *and* $c \in R^m$.

Now let us consider a particular case when f is of quadratic form. Namely, let

$$f(z) = \iint_{P^2} [z_x^T A_{11} z_x + 2z_x^T A_{12} z_y + z_y^T A_{22} z_y - 2bz]\,dx\,dy \tag{12}$$

where $z \in H_0^{1,2}(P^2, R^m)$. We shall assume that

(c) the matrices A_{ij}, $i, j = 1, 2$, are essentially bounded on P^2, $b \in L^2(P^2, R^m)$, $A_{ij}(x, y) \in R^{m\times m}$,

(d) there exists an $\alpha > 0$ such that

$$\xi_1^T A_{11}(x, y)\xi_1 + 2\xi_1^T A_{12}(x, y)\xi_2 + xi_2^T A_{22}(x, y)\xi_2 \geq \alpha(|\xi_1|^2 + |\xi_2|^2)$$

for $(x, y) \in P^2$ a.e. and any $\xi_1, \xi_2 \in R^m$.

It is known that the functional f given by formula (12), under assumptions (c) and (d) attains its minimum at exactly one point $z^0 \in H_0^{1,2}$ and f is differentiable (cf. [**6**]). This fact and Theorem 3 imply the following

Theorem 4. *If assumptions* (c) *and* (d) *are satisfied, then there exists exactly one function* $z^0 \in H_0^{1,2}(P^2, R^m)$ *such that*

(i) z^0 *is a minimizer of the functional* f *given by* (12),

(ii) z^0 *satisfies the following Euler equation*

$$-\int_0^y (A_{11}z_x^0 + A_{12}z_y^0)\,dy - \int_0^x (A_{22}z_y^0 + A_{12}z_x^0)\,dx + \int_0^x \int_0^y b(x, y)\,dx\,dy = c_1(y) + c_2(x) - c \qquad (x, y) \in P^2 \text{ a.e.}, \tag{13}$$

where c_1 and c_2 are some functions from $L^2([0,1],R^m)$, $c \in R^m$.

Let the matrices A_{ij}, $i,j = 1,2$, satisfy assumptions (c) and (d). Consider an elliptic system of the form

$$\frac{\partial}{\partial x}(A_{11}z_x + A_{12}z_y) + \frac{\partial}{\partial y}(A_{22}z_y + A_{12}z_x) = b \tag{14}$$

with the boundary condition

$$z(x,y) = 0 \qquad \text{for } (x,y) \in \partial P^2. \tag{15}$$

It is easy to notice that, in the case when $A_{12} = 0$ and A_{11} and A_2 are constant matrices, the problem of solving system (14)–(15) reduces to the classical Dirichlet problem.

Now let us consider an integro-differential system of the form

$$\begin{aligned} \int_0^y (A_{11}z_x + A_{12}z_y)\,dy + \int_0^x (A_{22}z_y + A_{12}z_x)\,dx \\ = \int_0^x \int_0^y b\,dx\,dy + a_1(x) + a_2(y) \end{aligned} \tag{16}$$

with the boundary condition

$$z(x,y) = 0 \qquad \text{for } (x,y) \in \partial P^2 \tag{17}$$

where a_1 and a_2 are some functions from the $L^2([0,1],R^m)$-space.

We shall say that a function $z^0 \in AC_0^2(P^2,R^m)$ is a solution of system (16)–(17) in the sense of Carathéodory if there exist some functions a_1 and $a_2 \in L^2$ such that z^0 satisfies system (16) for $(x,y) \in P^2$ almost everywhere.

It is easy to notice that any function $z^0 \in AC_0^2(P^2,R^m)$ which satisfies system (14) for $(x,y) \in P^2$ a.e. is a solution of system (16) for some a_1, a_2. The converse implication holds only under the additional assumptions.

Problem (16)–(17) will be referred to as the integro-differential Dirichlet problem.

Theorem 4 immediately implies the following

Theorem 5. *If the matrices A_{ij} are essentially bounded and satisfy conditions* (c) *and* (d), *then integro-differential Dirichlet problem* (16)–(17) *possesses a unique solution in the sense of Carathéodory.*

REFERENCES

1. D. Idczak, *Doctoral dissertation*, Institute of Math., Łódź Univ., 1991.

2. S. Łojasiewicz, *An introduction to the theory of real functions*, J. Wiley & Sons, Chichester, 1988.

3. J. Mawhin, *Problèmes de Dirichlet variationnels non-linéaires*, L'Université de Montréal, 1987.

4. J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, 1989.

5. S. Saks, *Theory of the integral*, Warsaw, 1937.

6. M. Struwe, *Variational methods*, Springer-Verlag, 1990.

7. M.B. Suryanarayana, *A Sobolev space and Darboux problem*, Pacific J. Math. **69** (1979), 535–550.

8. S. Walczak, *Absolutely continuous functions of several variables and their application to differential equations*, Bull. Polish Acad. Sci. Math. **35** (1987), 733–744.

9. ———, *On the differentiability of absolutely continuous functions of several variables*, ibid. **36** (1988), 513–520.

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