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QUALITATIVE ANALYSIS OF NONLINEAR SYSTEMS BY THE METHOD OF MATRIX LYAPUNOV FUNCTIONS

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ABSTRACT. This paper surveys applications of the method of matrix Lyapunov functions in the investigation of stability, asymptotic stability and instability of systems modelling real phenomena in engineering and technology. As a framework of the stability analysis systems of ordinary differential equations (ODE) under structural perturbations, the concept of matrix Lyapunov function is used.

0. What is a matrix Lyapunov function? As is well known, an auxiliary function with properties similar to those of a norm is an essential tool in Lyapunov's direct method for the qualitative theory of differential equations. At present, scalar Lyapunov functions are widely used for the solution of various problems in engineering, technology, mathematical biology, economics, etc. In the early 1970s, there appeared techniques utilizing several auxiliary functions that were actually components of a vector Lyapunov function. The idea was extensively developed when the dynamical properties of solutions of so-called "large scale systems" were investigated.

Further development of ideas by Lyapunov and Poincare resulted in the concept of a "matrix auxiliary function." Our subsequent presentation deals with this concept.

0.1. Let $(\mathbf{R}^n, ||\cdot||)$ be a real Euclidean normed space. We denote by $B(\rho) = \{x : ||x|| < \rho\}$ an open ball with radius ρ and center at the origin, and $\mathcal{D} = \mathcal{T}_0 \times B(\rho)$ is a direct Descart product, where $\mathcal{T}_0 = \{t : t_0 \leq t < \infty\}$ and $t_0 \in \mathcal{T}_\tau \subseteq \mathbf{R}$, where $\mathcal{T}_\tau = \{t : \tau \leq t < +\infty\}$,

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$\tau \in \mathbf{R}$. Moreover, we will require an open, connected, time-invariant neighborhood $\mathcal{N} \subseteq \mathbf{R}^n$ of the point $x = 0$.

We consider a system of differential equations

$$\frac{dx}{dt} = f(t, x), \qquad x(t_0) = x_0, \tag{0.1}$$

where $x \in \mathbf{R}^n$ and $f \in C[\mathcal{D}, \mathbf{R}^n]$. Suppose that the solution $x(t; t_0, x_0)$ of (0.1) is continuous for all $t \in \mathcal{T}_0$ and $(t_0, x_0) \in \operatorname{Int}(\mathcal{T}_0 \times B(\rho))$, where $x(t_0; t_0, x_0) = x_0$.

Here the definitions of stability, attraction, asymptotic stability and also instability of the zero solution of (0.1) are the usual.

0.2. Together with system (0.1) we consider a two-index system of functions (a matrix-function)

$$U(t, x) = [v_{ij}(t, x)], \qquad i, j = 1, 2, \dots, m, \ m > 1 \tag{0.2}$$

with elements $v_{ij} \in C[\mathcal{T}_\tau \times \mathbf{R}^n, \mathbf{R}]$. As is well known, the property of having a fixed sign of the auxiliary function is the most important property in the method of Lyapunov functions. Taking function (0.2) as an auxiliary function, we introduce its property of having a fixed sign for using Lyapunov's direct method, so that the following three points hold.

(a) The property of having a fixed sign of the matrix-function (0.2) is to be coordinated with the well-known concept of this property of a numerical matrix.

(b) The property of having a fixed sign of the matrix-function (0.2) is to be coordinated with the classical notion of the property of being a Lyapunov function.

(c) The property of having a fixed sign of the matrix-function (0.2) is to be natural in the framework of the direct method of Lyapunov.

Let $y \in \mathbf{R}^m$, $\eta \in \mathbf{R}^m_+$ or $\psi \in C[\mathbf{R}^n, \mathbf{R}^m]$, $\psi(0) = 0$ be given. By means of the vectors y and η, or the vector-function $\psi(x)$, we introduce the following functions:

1. $V(t, x, y) = y^T U(t, x) y$;
2. $V(t, x, \eta) = \eta^T U(t, x) \eta$;

3. $V(t, x, \psi) = \psi^T U(t, x)\psi$.

Any of the functions 1–3 can be utilized in order to define the property of having a fixed sign of the matrix-function (0.2). Hence, without loss of generality, we shall only consider function 1.

Definition 0.3. Matrix-function $U : \mathcal{T}_\tau \times \mathbf{R}^n \to \mathbf{R}^{m\times m}$ is called

(1) *positive definite* on $\mathcal{T}_\tau$ if and only if there exist a time-invariant connected neighborhood $\mathcal{N} \subseteq \mathbf{R}^n$ of the point $x = 0$, a positive definite function $W : \mathcal{N} \to \mathbf{R}_+$ and a vector $y \in \mathbf{R}^m$, $y \neq 0$, such that

(a) $U \in C[\mathcal{T}_\tau \times \mathcal{N}, \mathbf{R}^{m\times m}]$;

(b) $U(t, 0) = 0$ for all $t \in \mathcal{T}_\tau$;

(c) $V(t, x, y) \geq W(x)$ for all $(t, x \neq 0, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{N} \times \mathbf{R}^m$;

(2) *positive definite* on $\mathcal{T}_\tau \times \mathcal{L}$ if and only if conditions (a)–(c) of Definition 0.3 (1) are satisfied for $\mathcal{N} = \mathcal{L} \subseteq \mathbf{R}^n$ is an arbitrary set;

(3) *positive definite in the whole* on $\mathcal{T}_\tau$ if and only if conditions (a)–(c) of Definition 0.3 (1) are satisfied for $\mathcal{N} = \mathbf{R}^n$;

(4) *negative definite* (*in the whole*) on $\mathcal{T}_\tau \times \mathcal{N}$ if and only if $(-U)$ is positive definite (in the whole) on $\mathcal{T}_\tau \times \mathcal{N}$ (on $\mathcal{T}_\tau$).

Remark 0.4. The expression "on $\mathcal{T}_\tau$" in Definition 0.3 is omitted if and only if all its conditions are satisfied for $\tau \in \mathbf{R}$.

Definition 0.5. A matrix function $U : \mathcal{T}_\tau \times \mathbf{R}^n \to \mathbf{R}^{m\times m}$ is called

(1) *decreasing* on $\mathcal{T}_\tau$, $\tau \in \mathbf{R}$ if and only if there exist a time-invariant connected neighborhood $\mathcal{N} \subseteq \mathbf{R}^n$ of the point $x = 0$, a positive definite function $u : \mathcal{N} \to \mathbf{R}_+$ and a vector $y \in \mathbf{R}^m$ such that conditions (a) and (b) and

(d) $V(t, x, y) = y^T U(t, x) y \leq u(x) \forall\, (t, x \neq 0, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{N} \times \mathbf{R}^m$

are satisfied;

(2) *decreasing* on $\mathcal{T}_\tau \times \mathcal{L}$ if and only if all conditions of Definition 0.5 are satisfied for $\mathcal{N} = \mathcal{L}$, $\mathcal{L} \subseteq \mathbf{R}^n$;

(3) *decreasing in the whole* of $\mathcal{T}_\tau$ if and only if all conditions of Definition 0.5 are satisfied for $\mathcal{N} = \mathbf{R}^n$.

Remark 0.6. If system (0.1) is autonomous and matrix function (0.2) is autonomous, condition 1 (c) of Definition 0.3 is simplified to the following

(c′) $V(x,y) = y^T U(x) y > 0 \forall\, (x \neq 0, y \neq 0) \in \mathcal{N} \times \mathbf{R}^m$.

Remark 0.7. The definitions of positive semi-definiteness of an autonomous matrix function are formulated on the basis of Definition 0.5 in the context of condition (c').

Definition 0.8. A matrix-function $U : \mathcal{T}_\tau \times \mathbf{R}^n \to \mathbf{R}^{m\times m}$ is called:

(1) *radially unbounded* on $\mathcal{T}_\tau$, $\tau \in \mathbf{R}$ if as $||x|| \to +\infty$, $V(t,x,y) \to +\infty$ for all $(t, x \neq 0, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{N} \times \mathbf{R}^m$;

(2) *radially unbounded* if as $||x|| \to +\infty$, the condition $V(t,x,y) \to +\infty$ for all $(t, x \neq 0, y \neq 0) \in \mathcal{T}_\tau \times \mathcal{N} \times \mathbf{R}^m$, $\tau \in \mathbf{R}$ holds.

We determine the total derivative of matrix-function (0.2) along solutions of system (0.1) as follows:

$$D^+U(t,x) = [D^+v_{ij}(t,x), i,j \in [1,m]]; \tag{0.9}$$

$$D_+U(t,x) = [D_+v_{ij}(t,x), i,j \in [1,m]]; \tag{0.10}$$

where

$$D^+v_{ij}(t,x) = \limsup\{[v_{ij}(t+\theta, x+\theta f(t,x)) - v_{ij}(t,x)]\theta^{-1} : \theta \to 0^+\};$$
$$D_+v_{ij}(t,x) = \liminf\{[v_{ij}(t+\theta, x+\theta f(t,x)) - v_{ij}(t,x)]\theta^{-1} : \theta \to 0^+\}.$$

The notation $D^*U(t,x)$ means that either of expression (0.9) or (0.10) may be used.

Definition 0.11. The two-index system of functions (0.2) is called a *matrix Lyapunov function* (MLF) if and only if

(1) the matrix-function $U(t,x)$ has a fixed sign;

(2) the matrix-function $D^*U(t,x)$ is semi-definite (has a fixed sign) with sign opposite to that of function $U(t,x)$ and $D^*U(t,0) = 0$ for all $t \in \mathcal{T}_\tau$.

Definition 0.11 shows that an MLF solves the problem of stability for the zero solution of system (0.1). Definition 0.11 can be modified with reference to the various dynamical properties of the zero solution. We designate by $S(y)$ a set of MLF's, which solve the stability problem.

Definition 0.12. A matrix-function $U : \mathcal{T}_\tau \times \mathbf{R}^n \to \mathbf{R}^{m\times m}$ is called MLF of $S(y)$ type if and only if on $\mathcal{T}_\tau \times \mathcal{N}$, the conditions

(1) the matrix-function $U(t,x)$ is positive definite (decreasing);

(2) the matrix-function $D^+U(t,x)$ is nonpositive and $D^+U(t,0) = 0$ are satisfied.

We designate by $AS(y)$ a set of MLF's solving the problem of asymptotic stability.

Definition 0.13. A matrix-function $U : \mathcal{T}_\tau \times \mathbf{R}^n \to \mathbf{R}^{m\times m}$ is called MLF of $AS(y)$ type if and only if on $\mathcal{T}_\tau \times \mathcal{N}$ the conditions:

(1) the matrix-function $U(t,x)$ is positive definite (decreasing);

(2) the matrix-function $D^+U(t,x)$ is negative definite and for all t, $D^+U(t,0) = 0$ are satisfied.

We designate by $NS(y)$ a set of matrix-functions solving the problem of instability of the zero solution. We now characterize several properties of U-functions in terms of special types of comparison functions (see [**4, 14**]).

A continuous function $a : [0,h] \to \mathbf{R}_+$ (or a continuous function $a : [0,\infty) \to \mathbf{R}_+$) is said to belong to class K, i.e., $a \in K$, if $a(0) = 0$ and if a is strictly increasing on $[0,h]$ (or on $[0,\infty)$). If $a : \mathbf{R}_+ \to \mathbf{R}_+$, $a \in K$ and if $\lim_{r\to\infty} a(r) = \infty$, then a is said to belong to class KR.

Definition 0.14. A matrix-function $U : \mathcal{T}_\tau \times \mathbf{R}^n \to \mathbf{R}^{m\times m}$ is called a *matrix-function of Lyapunov-Chetayev type* of $NS(y)$ if and only if there exists a $t_0 \in \mathcal{T}_\tau$, $\Delta > 0$ such that $\mathcal{B}(\Delta) \subset \mathcal{N}$, and an open set $W \subset \mathcal{B}(\Delta)$ such that on $\mathcal{T}_\tau \times W$ the following conditions are satisfied.

(1) $0 < U(t,x) \le C$, where C is a real $m \times m$ matrix;

(2) $y^T U(t,x) y \ge a(V(t,x,y))$, where a is of class K;

(3) $0 \in \partial W$;

(4) $U(t,x) = 0$ on $\mathcal{T}_\tau \times (\partial W \cap \mathcal{B}(\Delta))$.

In terms of existence of an MLF, the stability and instability theorems of Lyapunov's direct method are formulated as follows.

Theorem 0.15. *For the zero solution of* (0.1) *to be stable* (*uniformly*) *it is necessary and sufficient that the* MLF $U : \mathcal{T}_\tau \times \mathbf{R}^n \to \mathbf{R}^{m\times m}$ *of* $S(y)$ *type exists for some positive integer* m.

Proof. For $m = 1$ the proof of Theorem 0.13 is well known (see for example Lyapunov [**5**]). If $m > 1$, then the function $V(t, x, y)$ for $y \in \mathbf{R}^m$ is scalar and the theorem is proved similarly to the $m = 1$ case. □

Theorem 0.16. *For the zero solution of* (0.1) *to be asymptotically stable* (*uniformly*) *it is necessary and sufficient that the* MLF $U : \mathcal{T}_\tau \times \mathbf{R}^n \to \mathbf{R}^{m\times m}$ *of* $AS(y)$ *type exists for some positive integer* m.

Theorem 0.17. *For the zero solution of* (0.1) *to be unstable it is sufficient that the* MF *of Lyapunov-Chetayev type of* $NS(y)$, $U : \mathcal{T}_\tau \times \mathbf{R}^n \to \mathbf{R}^{m\times m}$, *exists for positive integers* m.

We consider a simple example

$$\dot{x} = Px, \tag{0.10}$$

where $x \in R^2$,

$$P = \begin{pmatrix} -6 & 4 \\ -3 & 1 \end{pmatrix}.$$

The elements of MF are taken as

$$v_{11} = x_1^2, \qquad v_{22} = x_2^2, \qquad v_{12} = v_{21} = \alpha x_1 x_2,$$

where α is a constant. It is easy to verify that for $-1 < \alpha < -25/7$ the application of the matrix-function

$$U(x) = \begin{pmatrix} x_1^2 & \alpha x_1 x_2 \\ \alpha x_1 x_2 & x_2^2 \end{pmatrix}$$

and vector $\eta = (1, 1)^T$ allows one to solve the problem of stability of the zero solution of problem (0.10).

1. Systems described by ODE's under structural perturbations. Let the matrix $P = (p_1^T, p_2^T, \dots, p_s^T)^T \in \mathbf{R}^{s\times q}$ be characteristics of inner (or outer) perturbations. We designate the class of all admissible matrices P by

$$\mathcal{P} = \{P : P_1 \le P(t) \le P_2, t \in [-\infty, +\infty]\}, \tag{1.1}$$

where the matrices P_1 and P_2 are *a priori* defined. Note that $\mathcal{P}$ can also consist of a single element $\{0\}$.

We define a structural matrix $S_i : \mathbf{R} \to \mathbf{R}^{n_i \times N n_i}$ which contain the structural parameters $s_{ij} : [-\infty, +\infty] \to \{0, 1\}$ which take binary values as functions of t, i.e., $s_{ij} : [-\infty, +\infty] \to [0, 1]$ by the formula

$$\begin{gathered} S_i = [s_{i1} I_i; s_{i2} I_i, \dots, s_{ir-1} I_i, I_i, s_{i,r+1} I_i, \dots, s_{iN} I_i], \\ r \ne i, \qquad I_i = \operatorname{diag}\{1, 1, \dots, 1\} \in \mathbf{R}^{n_i \times n_i}. \end{gathered} \tag{1.2}$$

Here N is the number of all possible structures. It should be noted that it is sufficient but not necessary to require that the condition $s_{ij}(t) = 1$ implies that $s_{ik}(t) = 0$ for all $k \ne j$.

The matrix

$$S(t) = \operatorname{diag}(S_1, S_2, \dots, S_s),$$

describing all structural variations is called a structural matrix. The set of all possible matrices $S(t)$ is designated by $\mathcal{L}_s$:

$$\mathcal{L}_s = \{S : S = \operatorname{diag}(S_1, \dots, S_s)\}.$$

Let some real system S be described by system of ODE's (of dimension n)

$$dx/dt = f(t, x, P), \tag{1.3}$$

where $x \in \mathbf{R}^n$, $f \in \mathcal{F}$ and $\mathcal{F} = \{f^1, \dots, f^N\}$,

$$f^k \in C[\mathcal{T}_\tau \times \mathbf{R}^n \times \mathbf{R}^{s\times q}, \mathbf{R}^n] \qquad \forall\, k = 1, 2, \dots, N.$$

System (1.3) can be presented in the form

$$\frac{dx_i}{dt} = f_i(t, x^i, 0) + S_i(t) f_i^*(t, x, p_i), \tag{1.4}$$

where $x_i \in \mathbf{R}^{n_i}$, $x = (x_1^T, \dots, x_s^T)^T \in \mathbf{R}^n$, $\sum_{i=1}^s n_i = n$,

$$x^i = (0^T, \dots, 0^T, x_i^T, 0^T, \dots, o^T) \in \mathbf{R}^n.$$

The number N in the definition of the family $\mathcal{F}$ and $\mathcal{F}_i = \{f_i^1, \dots, f_i^N\}$, describes structural variation of the whole system. The function $k = k(t)$ on the set $\mathcal{N} = \{1, 2, \dots, N\}$, $k(t) \in \mathcal{N}$ for all $t \in \mathbf{R}$, describes the variation of system (1.3). It is clear that system (1.3) is structurally invariant if and only if $K(t)$ is constant, $K(t) \equiv K$, i.e., the set $\mathcal{N} = \{K\}$ consists of one element.

In this section we establish stability conditions for system (1.4) by applying an MLF.

Definition 1.5. The zero solution of system (1.3) is called

i) *asymptotically stable* (*in the whole*) on $\mathcal{P} \times \mathcal{L}_s$ if and only if it is asymptotically stable (in the whole) for any $(P, S) \in \mathcal{P} \times \mathcal{L}_s$;

ii) *uniformly asymptotically stable* (*in the whole*) on $\mathcal{P} \times \mathcal{L}_s$ if and only if it is uniformly asymptotically stable (in the whole) for any $(P, S) \in \mathcal{P} \times \mathcal{L}_s$.

Assumptions 1.6. There exist

(1) open connected neighborhoods $\mathcal{N}_{ix} \subset \mathbf{R}^{n_i}$ of states $x_i = 0$, $i = 1, \dots, s\}$;

(2) functions $\varphi_{ik} : \mathcal{N}_{ik} \to \mathbf{R}_+$, $i = 1, 2, \dots, s$, $k = 1, 2, (\varphi_{ik} \in K$-class $(KR))$;

(3) real constants $\underline{\alpha}_{ij}, \bar{\alpha}_{ij}$, $i, j = 1, \dots, s$;

(4) a matrix-function

$$U(t,x) = \begin{pmatrix} v_{11}(t, x_1) & \cdots & v_{1s}(t, x_1, x_s \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ v_{s1}(t, x_1, x_s) & \cdots & v_{ss}(t, x_s) \end{pmatrix}, \qquad v_{ij} = v_{ji} \tag{1.7}$$

the elements of which satisfy the following estimates

(a) $\underline{\alpha}_{ii}\varphi_{i1}^2(||x_i||a(t) \le v_{ii}(t, x_i) \le \bar{\alpha}_{ii}\varphi_{i2}^2(||x_i||)$, for all $(t, x_i) \in \mathbf{R}_+ \times \mathcal{N}_{ix}$, $i = 1, 2, \dots, s$;

(b) $\underline{\alpha}_{ij}\varphi_{i1}(||x_i||)\varphi_{j1}(||x_j||)a(t) \le v_{ij}(t, x_i, x_j) \le \bar{\alpha}_{ij}\varphi_{i2}(||x_i||)\varphi_{j2}(||x_j||)$ for all $(t, x_i, x_j) \in \mathbf{R}_+ \times \mathcal{N}_{ix} \times \mathcal{N}_{jx}$, where $a \in C[\mathbf{R}, \mathbf{R}_+]$, $a \ge \beta > 0$ for all $t \in \mathbf{R}_+$, $(i \ne j) \in [1, s]$.

Lemma 1.8. *If all conditions of Assumptions* 1.6 *hold, the function*

$$V(t, x) = \eta^T U(t, x)\eta, \qquad \eta \in R^s_+,\ \eta > 0 \tag{1.9}$$

satisfies the two-sided estimate for all $(t, x) \in \mathbf{R}_+ \times \mathcal{N}_x$

$$a(t)u_1^T H^T A_1 H u_1 \le V(t, x) \le u_2^T H^T A_2 H u_s, \tag{1.10}$$

where

$$\begin{aligned} u_1^T &= (\varphi_{11}(||x_1||), \dots, \varphi_{1s}(||x_s||))^T, \\ u_2^T &= (\varphi_{12}(||x_1||), \dots, \varphi_{2s}(||x_s||))^T, \\ H &= \operatorname{diag}[\eta_1, \eta_2, \dots, \eta_s], \\ A_1 &= [\underline{\alpha}_{ij}], \qquad A_2 = [\bar{\alpha}_{ij}], \qquad i, j = 1, 2, \dots, s. \end{aligned}$$

Proof. Inequalities (1.10) are obtained by a direct substitution of estimates (a) and (b) from condition (1.6) (4) in the extended expression for function $V(t, x)$. $\square$

Assumptions 1.11. There exist

(1) functions φ_i, v_{ij}; $i, j = 1, \dots, s$ satisfying the conditions of Assumptions 1.6 and, moreover,

(a) functions $v_{ii}(t, x_i) \in C[\mathbf{R}_+ \times \mathcal{N}_{ix_0}, \mathbf{R}_+]$ or $v_{ii}(t, x_i) \in C[\mathbf{R}_+ \times \mathbf{R}^{n_i}, \mathbf{R}_+]$, where $\mathcal{N}_{ix_0} = \{x_i : x_i \in \mathcal{N}_{ix}, x_i \ne 0\}$;

(b) for $i \ne j$ functions $v_{ij}(t, x_i, x_j) \in C[\mathbf{R}_+ \times \mathcal{N}_{ix_0} \times \mathcal{N}_{jx_0}, \mathbf{R}]$ or $v_{ij}(t, x_i, x_j) \in C[\mathbf{R}_+ \times \mathbf{R}^{n_i} \times \mathbf{R}^{n_j}, \mathbf{R}]$;

(2) constants ρ_i, $\rho_i^*(P, S)$, $\rho_{ij}(P, S)$, $i = 1, \dots, s$; $j = 2, \dots, s$; $j > i$ such that the following conditions hold

(a) $\eta_i^2(D_t^+ v_{ii} + (D_{x_i}^+ v_{ii})^T f_i(t, x^i, 0)) \le \rho_i \varphi_i^2(||x_i||)$ for all $(t, x_i) \in \mathbf{R} \times \mathcal{N}_{ix_0}$, $i = 1, 2, \dots, s$;

(b) $\sum_{i=1}^s \eta_i^2 (D_{x_i}^+ v_{ii})^T S_i f_i^*(t, x, p_i) + \sum_{i=1}^s \sum_{j=2, j>i}^s 2\eta_i\eta_j \{D_t^+ v_{ij} + (D_{x_i}^+ v_{ij})^T f_i(t, x^i, 0) + (D_{x_j}^+ v_{ij})^T f_j(t, x^j, 0) + (D_{x_i}^+ v_{ii})^T S_i f_i^*(t, x, p_i) +$

$(D^+_{x_j} v_{ij})^T S_j f^*_j(t,x,p_j) \le \sum_{i=1}^s \rho^*_i(P,S)\varphi_i^2(\|x_i\|) + 2\sum_{i=1}^s \sum_{j=2,j>i}^s \rho_{ij}(P,S)\varphi_i(\|x_i\|)\varphi_j(\|x_j\|)$ for all $(t,x,P,S) \in \mathbf{R}_+ \times \mathcal{N}_{x_0} \times \mathcal{P} \times \mathcal{L}_s$.

Lemma 1.12. *If all conditions of Assumptions* 1.11 *are satisfied, then the total derivative of function* (1.9) *with respect to system* (1.4) *satisfies the inequality*

$$D^+V(t,x) = \eta^T D^+U(t,x)\eta \le w^T G(P,S) w \quad \forall\,(t,x,P,S) \in \mathbf{R}_+ \times \mathcal{N}_{x_0} \times \mathcal{P} \times \mathcal{L}_s, \tag{1.13}$$

where

$$\begin{aligned} w^T &= (\varphi_1(\|x_1\|), \varphi_2(\|x_2\|), \dots, \varphi_s(\|x_s\|)) \\ G(P,S) &= [\sigma_{ij}(P,S)], \qquad i,j = 1,2,\dots,s \\ \sigma_{ij}(\cdot) &= \sigma_{ji}(\cdot) = \rho_{ij}(P,S) \qquad \forall\, i \ne j, \\ \sigma_{ii}(\cdot) &= \rho_i + \rho^*_i(P,S). \end{aligned}$$

Proof. Inequality (1.13) results directly by substituting inequalities (2)(a) and (2)(b) from Assumptions 1.11 into the expression for the function $D^+V(t,x)$. Let $\zeta \in \mathbf{R}_+$, $\zeta > 0$ be given. The set $V_\zeta(t)$ is called a maximal connected neighborhood of point $x = 0$ for $t \in \mathbf{R}$ and can be obtained by using the function $V, V : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}_+$ so that the condition $x \in V_\zeta(t)$ implies $V(t,x) < \zeta$. The definition of maximal connected neighborhood of the point $x_i = 0$ is introduced similarly, with the help of functions $v_{ii}(t,x_i)$ constructed for independent subsystems

$$\frac{dx_i}{dt} = f_i(t, x^i, 0), \qquad i = 1,2,\dots,s. \qquad \square$$

Theorem 1.15. *Let the perturbed equations of motion* (1.4) *be such that all conditions of Assumptions* 1.6 *and* 1.11 *are satisfied*; *except for the upper estimate of the function* v_{ij}, $i,j = 1,2,\dots,n$ *in Assumptions* 1.6*, and assume*

(a) *there exist positive numbers* ξ_i (*or* $\xi_i = +\infty$) *such that the set* $V_{i\zeta}(t)$ *is asymptotically contracted for any* $\zeta \in [0,\xi_i]$ *and every* $i = 1,2,\dots,s$;

(b) *the matrix* $A = H^T A_1 H$ *is positive definite*;

(c) *there exists a negative definite matrix* $G^* \in \mathbf{R}^{s\times s}$ *such that for matrix* $G(P,S)$ *from the* (1.13) *estimate,*

$$G(P,S) \le G^* \qquad \forall\,(P,S) \in \mathcal{P}\times\mathcal{L}_s$$

is valid.

Then the equilibrium state $x = 0$ *of* (1.4) *is structurally asymptotically stable on* $\mathcal{P}\times\mathcal{L}^s$.

If all hypotheses of Theorem 1.15 are satisfied for $\mathcal{N}_{ix} = \mathbf{R}^{n_i}$ for radially unbounded function φ_{ki} and for $\xi_i = +\infty$ when every $i = 1, 2, \dots, s$, then the equilibrium state $x = 0$ of (1.4) is structurally asymptotically stable in the whole on $\mathcal{P}\times\mathcal{L}_s$.

Proof. Let the conditions of Assumptions 1.6 and hypothesis (b) of Theorem 1.14 be satisfied, and let the function $V(t,x)$ be positive definite. Hypothesis (a) of Theorem 1.14 ensures the asymptotic contraction of the set $V_\zeta(t)$ constructed based on the function $V(t,x)$. Assumptions 1.11, Lemma 1.12 and hypothesis (c) of Theorem 1.15 imply that $D^+V(t,x)$ is a negative definite function for every $(P,S) \in \mathcal{P}\times\mathcal{L}_s$. These conditions are known [**4**] to be sufficient to provide structural asymptotic stability of the state $x = 0$ of (1.4).

In the case that $\mathcal{N}_{ix} = \mathbf{R}^{n_i}$ and φ_{ik} are radially unbounded and $\xi_i = +\infty$, then function $V(t,x)$ is positive definite in the whole and radially unbounded. Together with other hypotheses of Theorem 1.15, this ensures the structural asymptotic stability in the whole on $\mathcal{P}\times\mathcal{L}_s$.

Let the large scale system (S) of Lurie system type be decomposed into S subsystems

$$\begin{aligned}\frac{dx_i}{dt} &= \sum_{l=1}^{s} S_{il}^{(1)} A_{il} x_l + \sum_{l=1}^{s} S_{il}^{(2)} q_{il} f_{il}(\sigma_{il}),\\ \sigma_{il} &= c_{il}^T x, \qquad i = 1, 2, \dots, s,\end{aligned} \tag{1.17}$$

where $\sigma_{il}^{-1} f_{il}(\sigma_{il}) \in [0, k_{il}] \subseteq \mathbf{R}_+$, A_{il} are constant matrices, $x_i \in \mathbf{R}^{n_i}$, $\sum_{i=1}^{s} n_i = n$, $x \in \mathbf{R}^n$, k_{il} are constants. Every matrix and every vector in system (1.17) is of the corresponding dimension and, moreover, matrices $S_{il}^{(1)}$ and $S_{il}^{(2)}$ are diagonal. With the help of structural matrices

$$S_i = \begin{bmatrix} S_{il}^{(1)} & S_{i2}^{(1)} & \cdots & S_{i,i-1}^{(1)} & I & S_{i,i+1}^{(1)} & \cdots & S_{is}^{(1)} \\ S_{i1}^{(2)} & S_{i2}^{(2)} & \cdots & S_{i,i-1}^{(2)} & 0 & S_{i,i+1}^{(2)} & \cdots & S_{is}^{(2)} \end{bmatrix};$$
$$S = \operatorname{diag}(S_1, S_2, \dots, S_s),$$

the structural set is defined as follows

$$\mathcal{L}_s = \{S : 0 \le S_{il}^{(k)} \le I, S_{ii}^{(1)} = I, i, l = 1, 2, \dots, s, k = 1, 2\}$$

where I is a unit matrix of corresponding dimension. The set $\mathcal{P}$ for the system is the zero set, i.e., $\mathcal{P} = \{0\}$.

Substituting for the vector x by the vector x^i we subdivide system (1.17) into independent subsystems of the form

$$\frac{dx_i}{dt} = A_{ii}x_i + S_{ii}^{(2)} q_{ii} f_{ii}(\tilde{\sigma}_{ii}), \tag{1.18}$$

where $\tilde{\sigma}_{ii} c_{ii}^T x^i$, $x^i = (0, \dots, 0, x_i, 0, \dots, 0)^T \in \mathbf{R}^n$, $i \in [1, s]$. We introduce the following notations:

$$\begin{aligned} f_i(x^i) &= A_{ii}x_i + S_{ii}^{(2)} q_{ii} f_{ii}(\tilde{\sigma}_{ii}), \qquad \tilde{\sigma}_{ii} = c_{ii}^T x^i; \\ f_i^*(x, S) &= \sum_{l,1,l\neq i}^{s} S_{il}^{(1)} A_{il} x_l + \sum_{\substack{l=1 \\ l\neq i}}^{s} S_{il}^{(2)} q_{il} f_{il}(\sigma_{il}) \\ &\quad + S_{ii}^{(2)} q_{ii} [f_{ii}(\sigma_{ii}) + f_{ii}(\tilde{\sigma})], \\ \sigma_{il} &= c_{il}^T x, \qquad i = 1, 2, \dots, s. \end{aligned} \tag{1.19}$$

In view of (1.19), system (1.16) can be written as

$$\frac{dx_i}{dt} = f_i(x^i) + f_i^*(x, S), \qquad i = 1, 2, \dots, s, \tag{1.20}$$

where $f_i(0) = 0$ and $f_i^*(0, s) = 0$ for all $i = 1, 2, \dots, s$. Stability of the equilibrium state $x = 0$ of (1.20) is investigated by means of the matrix Lyapunov function

$$U(x) = [v_{ij}(x_i, x_j)], \qquad v_{ij} = v_{ji}, \tag{1.21}$$

the elements v_{ij} of which are defined as

$$v_{ij}(x_i, x_j) = x_i^T P_{ij} x_j, \qquad i, j \in [1, s], \tag{1.22}$$

where the P_{ii} are symmetric positive definite matrices, and P_{ij}, $i \neq j$, are constant matrices.

Functions (1.22) satisfy the following estimates [**1**]:

(1.23)
$$\begin{gathered}\lambda_m(P_{ii})||x_i||^2 \leq v_{ii}(x_i) \leq \lambda_M(P_{ii})||x_i||^2;\\ -\lambda_M^{1/2}(P_{ij}P_{ij}^T)||x_i||\,||x_j|| \leq v_{ij}(x_i, x_j) \leq \lambda_M^{1/2}(P_{ij}P_{ij}^T)||x_i||\,||x_j||\\ \forall\,(x_i, x_j) \in \mathbf{R}^{n_i} \times \mathbf{R}^{n_j} \quad (i \neq j), \quad i, j = 1, 2, \dots, s.\end{gathered}$$

Here $\lambda_m(P_{ii})$ are minimal and $\lambda_M(P_{ii})$ are maximal eigenvalues of the matrices P_{ii}, $\lambda_M(P_{ij}P_{ij}^T)$ are maximal eigenvalues of the matrices $P_{ij}P_{ij}^T$ and $(i \neq j) \in [1, s]$.

If estimates (1.23) are valid for the function

$$V(x) = \eta^T U(x)\eta, \qquad \eta \in \mathbf{R}_+^s,\ \eta > 0, \tag{1.24}$$

the two-sided inequality

$$u^T A u \leq V(x) \leq u^T B u \qquad \forall\, x \in \mathbf{R}^n, \tag{1.25}$$

similar to inequality (1.10), holds, where

$$\begin{aligned}&u^T = (||x_1||, ||x_2||, \dots, ||x_s||), \qquad A = H^T A_1 H, \qquad B = H^T A_2 H,\\ &\underline{\alpha}_{ii} = \lambda_m(P_{ii}), \qquad \bar{\alpha}_{ii} = \lambda_M(P_{ii}),\\ &\underline{\alpha}_{ij} = \underline{\alpha}_{ji} = -\lambda_M^{1/2}(P_{ij}P_{ij}^T), \qquad \bar{\alpha}_{ij} = \bar{\alpha}_{ji} = \lambda_M^{1/2}(P_{ij}P_{ij}^T). \qquad \square\end{aligned}$$

Lemma 1.26. *If for system* (1.17) *the matrix function* (1.21) *with elements* (1.22) *is constructed, then for the Dini derivatives of functions* (1.22) *along solutions of system* (1.17) *the estimates*

(a) $\eta^2(D^+_{x_i}v_{ii})^T f_i(x^i) \leq \rho_i^{(1)}(S)||x_i||^2$ for all $x_i \in \mathcal{N}_{ix_n}$, $i = 1, 2, \dots, s$;

(b) $\sum_{i=1}^s \eta_i^2 (D^+_{x_i}v_{ii})^T f_i^*(x, S) + 2\sum_{i=1}^s \sum_{j=2, j>i}^s \eta_i\eta_j\{(D^+_{x_i}v_{ij})^T(f_i(x^i) + f_i^*(x, S)) + (D^+_{x_j}v_{ij})^T(f_j(x^j) + f_j^*(x, S)) \leq \sum_{i=1}^s \rho_i^{(2)}(S)||x_i||^2 + 2\sum_{i=1}^s \sum_{j=2, j>i}^s \rho_{ij}(S)||x_i||\,||x_j||$ for all $(x_i, x_j) \in \mathcal{N}_{ix_0} \times \mathcal{N}_{jx_0} \times \mathcal{L}_s$

hold. Here $\rho_i^{(k)}(S)$, $k = 1, 2$; $i \in [1, s]$ are maximal eigenvalues of matrices

$$\begin{aligned}
&\eta_i^2[P_{ii}A_{ii} + A_{ii}^T P_{ii} + P_{ii}S_{ii}^{(2)}q_{ii}k_{ii}^*(c_{ii}^i)^T + (S_{ii}^{(2)}q_{ii}k_{ii}^*(c_{ii}^i)^T)^T P_{ii}];\\
&\sum_{l=1}^{i-1}\eta_i\eta_l\{[(S_{li}^{(1)}A_{li})^T + (S_{li}^{(2)}q_{li}k_{li}^*(c_{li}^i)^T)^T]P_{li}\\
&\qquad\quad + P_{li}^T[S_{li}^{(1)}A_{li} + S_{li}^{(2)}q_{li}k_{li}^*(c_{li}^i)^T]\}\\
&\quad + \sum_{l=i+1}^{s}\eta_i\eta_l\{P_{il}[S_{li}^{(1)}A_{li} + S_{li}^{(2)}q_{li}k_{li}^*(c_{li}^i)^T]\\
&\qquad\qquad + [(S_{li}^{(1)}A_{li})^T + (S_{li}^{(2)}q_{li}k_{li}^*(c_{li}^i)^T)^T]P_{li}^T\}\\
&\quad + \eta_i^2[P_{ii}S_{ii}^{(2)}q_{ii}k_{ii}^*(c_{ii}^i)^T + (S_{ii}^{(2)}q_{ii}k_{ii}^{**}(c_{ii}^i)^T)^T P_{ii}],
\end{aligned}$$

respectively, and $\rho_{ij}(S)$, $i < j$, $i = 1, 2, \dots, s$; $j = 2, \dots, s$ are norms of the matrices

$$\begin{aligned}
&\sum_{l=1}^{j-1}\eta_j\eta_l[(S_{li}^{(1)}A_{li})^T + (S_{li}^{(2)}q_{li}k_{li}^*(c_{li}^i)^T)^T]P_{lj}\\
&+ \sum_{l=j+1}^{s}\eta_j\eta_l[(S_{li}^{(1)}A_{li})^T + (S_{li}^{(2)}q_{li}k_{li}^*(c_{li}^i)^T)^T]P_{jl}^T\\
&+ \sum_{l=1}^{i-1}\eta_i\eta_l P_{li}^T[S_{lj}^{(1)}A_{lj} + S_{lj}^{(2)}q_{lj}k_{lj}^*(c_{lj}^j)^T]\\
&+ \sum_{l=i+1}^{s}\eta_i\eta_j P_{il}[S_{lj}^{(1)}A_{lj} + S_{lj}^{(2)}q_{lj}k_{lj}^*(c_{lj}^T)]\\
&+ \frac{1}{2}\eta_i^2\{P_{ii}(S_{ij}^{(1)}A_{ij}) + (S_{ij}^{(1)}A_{ij})^T P_{ii} + P_{ii}(S_{ij}^{(2)}q_{ij}k_{ij}^*(c_{ij}^j)^T)\\
&+ (S_{ij}^{(2)}q_{ij}k_{ij}^*(c_{ij}^j)^T)^T P_{ii} + P_{ii}(S_{ii}^{(2)}q_{ii}k_{ii}^*(c_{ii}^j)^T) + (S_{ii}^{(2)}q_{ii}k_{ii}^*(c_{ii}^j)^T)^T P_{ii}\}\\
&+ \frac{1}{2}\eta_j^2\{P_{ji}(S_{ji}^{(1)}A_{ji}) + (S_{ji}^{(1)}A_{ji})^T P_{ji} + P_{jj}(S_{ji}^{(2)}q_{ji}k_{jk}^*(c_{ji}^i)^T)\\
&+ (S_{ji}^{(2)}q_{ji}k_{ji}^*(c_{ji}^i)^T)^T P_{jj} + P_{jj}(S_{jj}^{(2)}q_{jj}k_{jj}^*(c_{jj}^i)^T)\\
&+ (S_{jj}^{(2)}q_{jj}k_{jj}^*(c_{jj}^i)^T)^T P_{jj}\},
\end{aligned}$$

respectively. Here

$$k_{ij}^* = \begin{cases} k_{ij} & \text{for } \sigma_{ij}(S_{ij}^{(k)}q_{ij})^T P_{ij}x_j > 0, \\ & i,j = 1,2,\dots,s;\ k = 1,2; \\ 0 & \text{in other cases} \end{cases}$$
$$k_{ij}^{**} = \begin{cases} k_{ii} & \text{for } \sigma_{ii}(S_{ii}^{(2)}q_{ii})^T P_{ii}x_i > 0,\ i = 1,2,\dots,s; \\ -k_{ii} & \text{for } \sigma_{ii}(S_{ii}^{(2)}q_{ii})^T P_{ii}x_i < 0,\ i = 1,2,\dots,s \end{cases}$$

$c_{ij}^i \in R^{n_k}$ is the k^{th} component of vector c_{ij}.

Lemma 1.26 is proved by the immediate transformations of expression $\eta^T D^+U(x)\eta$.

The application of Lemma 1.12 to system (1.20) when the estimates from Lemma 1.26 hold, enables us to get for $D^+V(x)$ the inequality

$$D^+V(x) \le u^T C(S^*)u \qquad \forall\,(x,S) \in \mathcal{N}_{x_0} \times \mathcal{L}_s, \tag{1.27}$$

where $C(S^*) = [c_{ij}(S^*)]$ and

$$c_{ii}(S^*) = \rho_i^{(1)}(S^*) + \rho_i^{(2)}(S^*),$$
$$c_{ij} = c_{ji} = \rho_{ij}(S^*) \qquad (i \ne j) \in [1,s].$$

Here $S^* \in \mathcal{L}_s$ is a constant $s \times s$ matrix such that

$$\rho_i^{(k)}(S) \le \rho_i^{(k)}(S^*); \qquad \rho_{ij}(S) \le \rho_{ij}(S^*), \qquad k = 1,2.$$

Estimate (1.27) is proved by the immediate transformation of expression $D^+V(x)$ in view of inequalities (a) and (b) from Lemma 1.26.

Theorem 1.28. *Let equations* (1.17) *be such that there exists matrix-function* (1.21) *with elements* (1.22) *satisfying estimates* (1.23), *and for the Dini derivative of function* (1.24), *estimate* (1.27) *holds. If*

(a) *matrix A in equality* (1.25) *is positive definite*;

(b) *matrix C in inequality* (1.27) *is negative definite. Then the equilibrium state $x = 0$ of system* (1.17) *is uniformly asymptotically stable on $\mathcal{L}_s$.*

If the hypotheses of Theorem 1.28 *are satisfied for* $\mathcal{N}_{ix} = \mathbf{R}^{n_i}$, *then the equilibrium state* $x = 0$ *of system* (1.7) *is uniformly asymptotically stable in the whole on* $\mathcal{L}_s$.

Proof. It is easily seen that if all hypotheses of Theorem 1.28 are satisfied, all hypotheses of Theorem 1.15 are also satisfied, and thus Theorem 1.28 is a corollary of Theorem 1.15. □

Example 1.29. Let system (1.17) be a system of fourth order of Lurie type decomposed into two interconnected second order systems defined by the following vectors and matrices:

$$A_{11} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}; \qquad A_{12} = \begin{bmatrix} -5 & 0 \\ -1 & -5 \end{bmatrix}; \qquad A_{21} = \begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix};$$
$$A_{22} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}; \qquad q_{1,l} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}; \qquad q_{2,l} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix};$$

$$\begin{aligned} &c_{1,l}^T = (0.1; 0; 0.1; 0); && c_{2l}^T = (0.1; 0; 0; 0.1); \\ &k_{il} = 1, && i, l = 1, 2; \\ &S_{il}^{(1)} = I; && S_{ij}^{(2)} = s_{ij} I; i, j = 1, 2; \qquad I = \operatorname{diag}(1, 1). \end{aligned}$$

Here $s_{ij} : [-\infty, +\infty] \to [0, 1]$ is a structural parameter. The structural set is defined as

$$\mathcal{L}_s = \{S : 0 \le s_{ij}^{(k)} \le I, S_{ij}^{(1)} = I, i, j, k = 1, 2\}.$$

For the elements v_{ij} of matrix-function (1.21) taken in the form

$$\begin{aligned} v_{ii}(x_i) &= x_i^T I x_i, \qquad i = 1, 2 \\ v_{12}(x_1, x_2) &= v_{21}(x_1, x_2) = 0,1 x_1^T I x_2, \end{aligned} \tag{1.30}$$

the estimates

$$\begin{aligned} v_{ii}(x_i) &\ge ||x_i||^2, \qquad i = 1, 2 \\ v_{12}(x_1, x_2) &\ge -0,1 ||x_1|| \, ||x_2|| \end{aligned}$$

hold. Let $\eta^T = (1, 1)$. Then matrix $\tilde{A}$, corresponding to matrix A in estimate (1.25), is:

$$\tilde{A} = \begin{bmatrix} 1 & -0.1 \\ -0.1 & 1 \end{bmatrix}.$$

If the elements of matrix-function (1.21) are taken in the form of (1.30), then

(1) for $k_i^* = 0$, $\rho_1^{(1)}(S) = -6$; $\rho_2^{(1)}(S) = 0.2$;

$$\rho_{12}^{(1)}(S) = 1,1 + 0.02 s_{11}; \qquad \rho_{21}^{(2)}(S) = -0.9 + 0.02 s_{22},$$
$$\rho_{12}(S) = 0.29;$$

(2) for $k_i^* = k = 1$; $\rho_1^{(1)}(S) = -6 + 0.02 s_{11}$;

$$\rho_2^{(1)}(S) = 0.2 + 0.01 s_{22}; \qquad \rho_1^{(2)}(S) = 1.1 + 0.02 s_{11} + 0.001 s_{21};$$
$$\rho_2^{(2)}(S) = -0.9 + 0.0001 s_{12} + 0.02 s_{22};$$
$$\rho_{12}(S) = 0.29 + 0.011 s_{11} + 0.01 s_{12} + 0.005 s_{21} + 0.007 s_{22}.$$

Matrix $\widetilde{C}$ corresponding to matrix C in estimate (1.27) has the form

$$\widetilde{C} = \begin{cases} \begin{bmatrix} -4.28 & 0.29 \\ 0.29 & -0.68 \end{bmatrix} & \text{for } k_i^* = 0; \\ \begin{bmatrix} -4.859 & 0.323 \\ 0.323 & -0.669 \end{bmatrix} & \text{for } k_i^* = k_i = 1 \end{cases}$$

and is negative definite.

Thus, all hypotheses of Theorem 1.28 are satisfied and the equilibrium state $x = 0$ of (1.17) with vectors and matrices from (1.29) is structurally asymptotically stable in the whole on $\mathcal{L}_s$.

2. Conclusions. The method of qualitative analysis of nonlinear systems based on matrix Lyapunov functions embraces the advantages of both the method of scalar Lyapunov functions and that of vector Lyapunov functions. First, it is simple in its applications (given the presence of an appropriate function). Second, it can be effectively applied to large scale dynamical systems. By the same token, the method

(i) enlarges the classes of functions suitable for the construction of an appropriate Lyapunov function;

(ii) allows a more precise accounting of the interconnections between subsystems in large scale systems and extends the assumptions on the dynamical properties of the subsystems;

(iii) does not require the construction of quasimonotone comparison systems as vector functions do;

(iv) simplifies the testing of conditions leading to the property of having a fixed sign of special matrices in nonnegative cones in stability (instability problems).

3. Comments. The famous dissertation paper by A.M. Lyapunov [**5**] containing the basic method of qualitative analysis of nonlinear systems is very close to ideas of H. Poincare [**15**]. There are many monographs developing this or that Lyapunov's method of functions. Results obtained for large-scale systems are presented in books by L.T. Grujić [**3**], A.A. Martynyuk [**6**], A.N. Michel and R.K. Miller [**14**], D.D. Siljak [**16**], L.T. Grujić, A.A. Martynyuk, M. Ribbens-Pavella [**4**], etc. The concept of matrix Lyapunov functions was proposed by A.A. Martynyuk (September, 1976, Yablonna, Poland). Papers [**1,2, 7**–**10**] were the first to develop the idea of auxiliary matrix functions. In our paper we present some results from these and some generalizations (see A.A. Martynyuk, V.G. Miladzhanov [**11, 12**]). In the description of structural perturbations we follow L.T. Grujić, A.A. Martynyuk and M. Ribbens-Pavella [**4**].

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