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WEAK SEQUENTIAL COMPLETENESS OF β-DUALS

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1. Introduction. The property of weak sequential completeness in sequence spaces has been considered by many authors and has been used to prove results in summability theory and functional analysis (see [**2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 15**]). In this paper we present a generalization of the following theorem of D. Noll (see below for relevant definitions):

Theorem 1.1 [**9**, Theorem 6]. *If E is a sequence space containing Φ that has the weak gliding hump property, then E^β is $\sigma(E^\beta, E)$-sequentially complete.*

We show, in Theorem 3.5, that if E is a sequence space containing Φ that has the signed weak gliding hump property (Definition 3.4), then E^β is $\sigma(E^\beta, E)$-sequentially complete. The sequence space of bounded series, bs, is shown to have the signed weak gliding hump property. It is known that bs fails the weak gliding hump property (see [**9, 5**]).

2. Preliminaries. A *sequence space* is a vector space of sequences, which can be scalar ($\mathbf{R}$ or $\mathbf{C}$) or vector-valued. In this paper all vector spaces are over $\mathbf{R}$, largely for convenience.

A real-valued sequence space E is called a *K-space* if the inclusion map $E \to \omega$ (the space of all sequences) is continuous, when ω is given the product topology ($\omega = \prod_{i=1}^{\infty}(\mathbf{R})_i$). A K-space with a Frechet (complete, metrizable and locally convex) topology is called an *FK-space*; if the topology is a Banach topology, then E is called a *BK-space*.

The α-, β- and γ-duals of a sequence space E are defined to be

$$E^\alpha = \left\{(y_i) : \sum_{i=1}^{\infty} |x, y_i| < \infty \text{ for all } (x_i) \in E\right\},$$

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$$E^\beta = \left\{ (y_i) : \sum_{i=1}^{\infty} x_i y_i \text{ converges for all } (x_i) \in E \right\},$$

and

$$E^\gamma = \left\{ (y_i) : \sup_n \left| \sum_{i=1}^{n} x_i y_i \right| < \infty \text{ for all } (x_i) \in E \right\}.$$

For a fairly complete list of sequence spaces and their duals, see [**8**, p. 68].

Let $\Phi = \operatorname{span}\{e^i : i \in \mathbf{N}\}$, where e^i denotes the sequence with 1 in the ith position and zeros elsewhere. If $E \supseteq \Phi$, then E and E^α or E^β are in duality with respect to the bilinear form $x \cdot y = \sum_{i=1}^{\infty} x_i y_i$, where $x = (x_i) \in E$, $y = (y_i) \in E^\alpha$ or E^β.

We can define weak topologies (topologies of pointwise convergence) $\sigma(E, E^\alpha)$ and $\sigma(E, E^\beta)$ on E, and $\sigma(E^\alpha, E)$ and $\sigma(E^\beta, E)$ on E^α and E^β, respectively. E^β is *weakly sequentially complete* if every $\sigma(E^\beta, E)$-Cauchy sequence converges to an element of E^β. Similar definitions hold for E and E^α.

3. Main results. The primary tool used in proving Theorem 3.5 is the following result, which is a useful generalization of the basic matrix theorem of Antosik and Mikusinski, which has been used to prove many fundamental results in functional analysis and measure theory (see [**1, 13**]). The theorem is stated and proved in a very general setting, that of Abelian topological groups.

Lemma 3.1. *Let X be an Abelian topological group and $x_{ij} \in X$ for $i, j \in \mathbf{N}$. If $\lim_i x_{ij} = 0$ for all j and $\lim_j x_{ij} = 0$ for all i, and if (U_k) is a sequence of neighborhoods of 0 in X, then there exists an increasing sequence of positive integers (p_i) such that $x_{p_i p_j}, x_{p_j p_i} \in U$ for $j > i$.*

Proof. See [**13**, Lemma 1]. □

We have the following generalization of the basic matrix theorem of Antosik and Mikusinski.

Theorem 3.2. *Let X be an Abelian topological group and $x_{ij} \in X$ for all $i, j \in \mathbf{N}$. Suppose*

(i) $\lim_i x_{ij} = x_j$ *exists for all j and*

(ii) *for each increasing sequence of positive integers (m_j) there is a subsequence (n_j) and a choice of signs $s_j \in \{-1, 1\}$ such that $(\sum_{j=1}^{\infty} s_j x_{in_j})_{i=1}^{\infty}$ is Cauchy.*

Then $\lim_i x_{ij} = x_j$ *uniformly for $j \in \mathbf{N}$. In particular,*

$$\lim_i \lim_j x_{ij} = \lim_j \lim_i x_{ij} = 0 \quad \textit{and} \quad \lim_i x_{ii} = 0.$$

Proof. (The proof is essentially that given in [**13**, Theorem 2].)

If the conclusion fails, there is a closed, symmetric neighborhood U_0 of 0 and increasing sequences of positive integers (m_k) and (n_k) such that $x_{m_k n_k} - x_{n_k} \notin U_0$ for all k. Pick a closed, symmetric neighborhood U_1 of 0 such that $U_1 + U_1 \subseteq U_0$ and set $i_1 = m_1$, $j_1 = n_1$. Since

$$x_{i_1 j_1} - x_{j_1} = (x_{i_1 j_1} - x_{i j_1}) + (x_{i j_1} - x_{j_1}),$$

there exists i_0 such that $x_{i_1 j_1} - x_{i j_1} \notin U_1$ for $i \geq i_0$. Choose k_0 such that $m_{k_0} > \max\{i_1, i_0\}$, $n_{k_0} > j_1$ and set $i_2 = m_{k_0}$, $j_2 = n_{k_0}$.

Then $x_{i_1 j_1} - x_{i_2 j_1} \notin U_1$ and $x_{i_2 j_2} - x_{j_2} \notin U_0$. Proceeding in this manner produces increasing sequences (i_k) and (j_k) such that $x_{i_k j_k} - x_{j_k} \notin U_0$ and $x_{i_k j_k} - x_{i_{k+1} j_k} \notin U_1$. For convenience, set $z_{kl} = x_{i_k j_l} - x_{i_{k+1} j_l}$, so $z_{kk} \notin U_1$.

Choose a sequence of closed, symmetric neighborhoods of 0, (U_n), such that $U_n + U_n \subseteq U_{n-1}$ for $n \geq 1$. Note that

$$U_3 + U_4 + \cdots + U_m = \sum_{j=3}^{m} U_j \subseteq U_2 \quad \text{for each } m \geq 3.$$

By (i), $\lim_k z_{kl} = 0$ for each l and by (ii), $\lim_l z_{kl} = 0$ for each k so by the lemma there is an increasing sequence of positive integers (p_k) such that $z_{p_k p_l}, z_{p_l p_k} \in U_{k+2}$ for $k > l$. By (ii) there is a subsequence (q_k) of (p_k) and a choice of signs s_k such that $(\sum_{k=1}^{\infty} s_k x_{i q_k})_{i=1}^{\infty}$ is Cauchy, so

$$\lim_k \sum_{l=1}^{\infty} s_l z_{q_k q_l} = 0.$$

Thus, there exists a k_0 such that

$$\sum_{l=1}^{\infty} s_l z_{q_{k_0} q_l} \in U_2.$$

Then for $m > k_0$,

$$\begin{aligned}\sum_{\substack{l=1\\ l\neq k_0}}^{m} s_l z_{q_{k_0} q_l} &= \sum_{l=1}^{k_0-1} s_l z_{q_{k_0} q_l} + \sum_{l=k_0+1}^{m} s_l z_{q_{k_0} q_l}\\ &\in \sum_{l=1}^{k_0-1} U_{k_0+2} + \sum_{l=k_0+1}^{m} U_{l+2}\\ &\subseteq \sum_{l=3}^{m+2} U_l \subseteq U_2,\end{aligned}$$

so

$$z_{k_0} = \sum_{\substack{l=1\\ l\neq k_0}}^{\infty} s_l z_{q_{k_0} q_l} \in U_2.$$

Thus,

$$s_{k_0} z_{q_{k_0} q_{k_0}} = \sum_{l=1}^{\infty} s_l z_{q_{k_0} q_l} - z_{k_0} \in U_2 + U_2 \subseteq U_1.$$

Since U_1 is symmetric, $z_{q_{k_0} q_{k_0}} \in U_1$ as well, which is a contradiction. □

Definition 3.3. A matrix which satisfies the hypotheses of Theorem 3.2 will be referred to as a *signed $\mathcal{K}$-matrix*.

A *$\mathcal{K}$-matrix* as originally introduced by Antosik and Mikusinski satisfies condition (i) of Theorem 3.2 and condition (ii) without the choice of signs s_j.

Theorem 3.2 is used to prove Theorem 3.5, which is the main result concerning weakly sequentially complete β-duals. First, a definition which generalizes Noll's definition of the weak gliding hump property (WGHP).

Definition 3.4. Let E be a sequence space containing Φ. E has the *signed weak gliding hump property* (signed WGHP) if, given any $x \in E$ and any disjoint sequence $(I_n) \subset I_0$ (the set of all finite subintervals of $\mathbf{N}$), there exists a subsequence (I_{n_k}) and a choice of signs $(s_k) \in \{-1,1\}^{\mathbf{N}}$ such that the coordinatewise sum $\sum_k s_k C_{I_{n_k}} x \in E$ (C_A denotes the characteristic function of A).

As suggested by the name, the difference between the signed WGHP and Noll's WGHP is that the "humps" in Definition 3.4 are multiplied by ± 1. Many sequence spaces, both scalar- and vector-valued, satisfy the signed WGHP, in particular the space bs.

Theorem 3.5. *Assume that E is a sequence space containing Φ with the signed* WGHP*. Then E^β is $\sigma(E^\beta, E)$ sequentially complete.*

Proof. (The proof is a modification of the proof of Theorem 7 in [**12**].)

Let (y^k) be a $\sigma(E^\beta, E)$ Cauchy sequence. Denote by y the sequence defined by $y_j = \lim_k y^k \cdot e^j$, that is, the coordinatewise limit of (y^k). We need to show that $\lim_n \sum_{j=1}^n y_j x_j = \lim_k y^k \cdot x$ for all $x \in E$. Obviously, this will imply that $y \in E^\beta$ and complete the proof. $\square$

If the desired conclusion is not true, there exist an increasing sequence of integers (n_l), $x \in E$, $\varepsilon > 0$, such that

$$\left| \sum_{j=1}^{n_l} y_j x_j - \lim_k \sum_{j=1}^{\infty} y_j^k x_j \right| > \varepsilon \quad \text{for all } l.$$

Manipulating the lefthand side yields

$$\begin{aligned} \left| \sum_{j=1}^{n_l} y_j x_j - \lim_k \sum_{j=1}^{\infty} y_j^k x_j \right| &= \left| \sum_{j=1}^{n_l} y_j x_j - \lim_k \left(\sum_{j=1}^{n_l} y_j^k x_j + \sum_{j=n_l+1}^{\infty} y_j^k x_j \right) \right| \\ &= \left| \lim_k \left(\sum_{j=1}^{n_l} (y_j x_j - y_j^k x_j) - \sum_{j=n_l+1}^{\infty} y_j^k x_j \right) \right|. \end{aligned}$$

Since $\lim_k y_j^k x_j = y_j x_j$, $\lim_k \sum_{j=1}^{n_l} (y_j x_j - y_j^k x_j) = 0$. So

$$\left| \lim_k \left(- \sum_{j=n_l+1}^{\infty} y_j^k x_j \right) \right| = \left| \lim_k \sum_{j=n_l+1}^{\infty} y_j^k x_j \right| > \varepsilon \quad \text{for all } l.$$

Choose k_1 such that $|\sum_{j=n_1+1}^{\infty} y_j^{k_1} x_j| > \varepsilon$. Since the series is convergent, there exists $m_1 > n_1 + 1$ such that $|\sum_{j=m_1}^{\infty} y_j^{k_1} x_j \Big| < \varepsilon/2$. Therefore, $|\sum_{j=n_1+1}^{m_1} y_j^{k_1} x_j| > \varepsilon/2$, by the triangle inequality.

Let $I_1 = \{n_1 + 1, \dots , m_1\}$. Now choose $k_2 > k_1$ and an integer $n_2 > m_1$ (so named for notational ease) such that $|\sum_{j=n_2+1}^{\infty} y_j^{k_2} x_j| > \varepsilon$. As above, we can find $m_2 > n_2 + 1$ such that $|\sum_{j=n_2+1}^{m_2} y_j^{k_2} x_j| > \varepsilon/2$. Let $I_2 = \{n_2+1, \dots , m_2\}$. Proceeding inductively produces a sequence $I_j = \{n_j + 1, \dots , m_j\}$. Note that

$$(*) \qquad |y^{k_i} \cdot C_{I_i} x| > \varepsilon/2 \quad \text{for all } i.$$

Consider the matrix $(y^{k_i} \cdot C_{I_j} x) = M = (M_{ij})$. We show that M is a signed $\mathcal{K}$-matrix. The columns of M converge to $y \cdot C_{I_j} x$. By the signed WGHP, for every subsequence (p_j) there exists a further subsequence (q_j) and a choice of signs (s_j) such that the coordinatewise sum

$$\tilde{x} = \sum_{j=1}^{\infty} s_j C_{I_{q_j}} x \in E.$$

Hence,

$$\lim_i \sum_{j=1}^{\infty} y^{k_i} \cdot s_j C_{I_{q_j}} x = \lim_i y^{k_i} \cdot \tilde{x}$$

converges by hypothesis.

So M is a signed $\mathcal{K}$-matrix and $M_{ii} \to 0$, contradicting $(*)$. Therefore, $y \cdot x = \lim_i y^i \cdot x$, and so E^β is $\sigma(E^\beta, E)$-sequentially complete. □

The sequence space

$$bs = \left\{ (x_i) : \sup_n \left| \sum_{i=1}^{n} x_i \right| < \infty \right\}$$

was the motivation in defining the signed WGHP. To see that bs fails the WGHP, consider $x = (1, -1, 1, -1, \dots)$. Clearly, $x \in bs$ but for $I_n = \{2n - 1\}$, $C_{\cup n I_n} x = (1, 0, 1, 0, \dots) \notin bs$, and for no subsequence

(I_{n_k}) of (I_n) is $C_{\cup_k I_{n_k}} x \in bs$. However, bs does satisfy the signed WGHP, as we now prove.

Proposition 3.6. *bs has the signed* WGHP.

Proof. (Actually we show the stronger result that for all $x \in bs$ and increasing $(I_n) \subset I_0$, there exists a choice of signs $(s_n) \in \{-1,1\}^{\mathbf{N}}$ such that $\sum_n s_n C_{I_n} x \in bs$.)

Let $x \in bs$, and let (I_n) be increasing in I_0. Note that $|C_I \cdot x| \leq M$ for any interval $I \in I_0$ and some $M > 0$, because there exists $M/2$ such that $\sup_n |\sum_{i=1}^n x_i| < M/2$, and so

$$|C_I \cdot x| = \left| \sum_{i=1}^{\max(I)} x_i - \sum_{i=1}^{\min(I)-1} x_i \right| < M.$$

Define a choice of signs recursively:

$$s_1 = \operatorname{sgn}(C_{I_1} \cdot x)$$

and

$$s_{n+1} = \left[-\operatorname{sgn}\left(\sum_{k=1}^n s_k C_{I_k} \cdot x \right) \right] [\operatorname{sgn}(C_{I_{n+1}} \cdot x)]$$

where $\operatorname{sgn}(0) = +1$.

Let $y = \sum_k s_k C_{I_k} x$ (coordinatewise sum). We show that $y \in bs$ by showing that $|\sum_{i=1}^N y_i| \leq 2M$ for any N.

We first prove by induction that $|\sum_{i=1}^{\max(I_n)} y_i| \leq M$ for any n. For $n = 1$ the result is clear. If $|\sum_{i=1}^{\max(I_n)} y_i| \leq M$, then, by construction,

$$\left| \sum_{i=1}^{\max(I_n+1)} y_i \right| = \left| \sum_{i=1}^{\max(I_n)} y_i + \sum_{i \in I_n+1} y_i \right| \leq M,$$

since $|\sum_{i \in I_{n+1}} y_i| \leq M$ and $\sum_{i \in I_{n+1}} y_i$ is opposite in sign to $\sum_{i=1}^{\max(I_n)} y_i$.

Now for any $N > 0$ we can write

$$\left| \sum_{i=1}^N y_i \right| \leq \left| \sum_{i=1}^{P_N} y_i \right| + \left| \sum_{i=P_N+1}^{N} y_i \right|$$

where $P_N = \max\{\max(I_k) : \max(I_k) \le N\}$ and $P_N = 0$ if $N < \max(I_1)$. So

$$\left|\sum_{i=1}^{N} y_i\right| \le \left|\sum_{i=1}^{P_N} y_i\right| + \left|\sum_{i=P_N+1}^{N} y_i\right| \le 2M \quad \text{for all } N,$$

using the observation at the beginning of the proof. This proves the result. □

It should be noted that Boos and Leiger have reported to the author through private communication that there are examples of $\sigma(E^\beta, E)$-sequentially complete spaces that fail the signed WGHP. It would be highly desirable to have a "gliding hump" characterization of weakly sequentially complete sequence spaces.

Finally, we present a modest application of weak sequential completeness to show continuity of infinite matrices mapping between sequence spaces. Let $A = (a_{nk})$ be an infinite matrix. $A : E \to F$ means $Ax = (\sum_k a_{nk}x_k) = (y_n) \in F$ for all $x = (x_k) \in E$. The following theorem of Swetits will be used.

Theorem 3.7 [**14**, Theorem 2.1]. *Let E and F be spaces containing Φ such that E^β is $\sigma(E^\beta, E)$-sequentially complete, and F is $\sigma(F, F^\beta)$-sequentially complete. If A is an infinite matrix, then the following are equivalent*:

a) $A : E \to F$.

b) $A' : F^\beta \to E^\beta$.

c) $A : E^{\beta\beta} \to F$.

(Here, A' denotes the transpose of A.)

Actually, an inspection of Swetits' proof shows that a) ⇒ b) is true only with the assumption that E^β is $\sigma(E^\beta, E)$-sequentially complete. Indeed, he shows that $Ax \cdot y = A'y \cdot x$ for $x \in E$ and $y \in F^\beta$. With this observation we can prove the following:

Theorem 3.8. *Let E and F be sequence spaces containing Φ.*

Assume that E^β is $\sigma(E^\beta, E)$-sequentially complete and that $A : E \to F$. Then A is $(E, E^\beta) \to \sigma(F, F^\beta)$ continuous, that is, continuous with respect to the weak topologies on E and F.

Proof. Let x^δ be a net in E which converges to $x \in E$ in the topology $\sigma(E, E^\beta)$. We need to show that $Ax^\delta \to Ax$ in $\sigma(F, F^\beta)$. By Swetits' result, $y \cdot Ax^\delta = A'y \cdot x^\delta$ for $y \in F^\beta$, and $A'y \in E^\beta$. Therefore,

$$A'y \cdot x^\delta \to A'y \cdot x = y \cdot Ax.$$

So, $Ax^\delta \to Ax$ with respect to the topology $\sigma(F, F^\beta)$, which completes the proof. □

It is interesting to note that, by the last theorem, the nontopological signed WGHP implies a topological result, the weak continuity of A. This can be viewed as an example of "automatic" continuity. That is, continuity implied by nontopological assumptions, in this case on the domain space.

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