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DOMAINS OF TRIGONOMETRIC TRANSFORMS

GILBERT MURAZ AND P. SZEPTYCKI

ABSTRACT. It is shown that the maximal solid domains of trigonometric transforms coincide with appropriate amalgam spaces.

1. Introduction. To formulate our objectives in this note, we need to recall the definition and some properties of the extended (maximal solid) domain of an integral operator. For details, we refer to [**2**] and [**7**].

Let X, Y be σ-finite measure spaces with the measures denoted by dx, dy, and let $L^\circ(X)$, $L^\circ(Y)$ denote, as usual, the spaces of measurable finite a.e. (complex valued) functions on X and Y, respectively. L° is an F-space (vector metric complete) with the topology of convergence in measure on all subsets of finite measure. This topology on $L^\circ(x)$ may be defined, for instance, by any of the F-norms of the form $u \to \rho_x(u) := \int_x \Phi(|u(x)|)\phi(x)\,dx$, where Φ is a positive increasing continuous subadditive bounded function from $\mathbf{R}_+$ into $(0,1)$ with $\Phi(0) = 0$ and $\phi > 0$ with $\int_\phi < \infty$. The F-norms ρ_Y are defined similarly. It is convenient to assume that some such F-norms ρ_X, ρ_Y have been fixed throughout this paper.

For a kernel k, a measurable function on $X \times Y$, the corresponding integral operator

$$K : D_K \subset L^\circ(Y) \to L^\circ(X)$$

is defined by:

(1.1)

$$Ku(x) = \int_Y k(x,y)u(y)\,dy,$$

$$D_K = \left\{ u \in L^\circ(Y); K_a u(x) = \int_Y |k(x,y)||u(y)|\,dy < \infty \text{ a.e.} \right\}.$$

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D_K is referred to as the *proper domain* of K. It is an F-space when equipped with the F-norm $\rho_K(u) := \rho_Y(u) + \rho_X(K_a u)$.

A subset of L° is solid if together with every function u it contains the order interval $\{v \in L^\circ; |v| \leq |u| \text{ a.e.}\}$. A topological vector subspace of L° is solid if the topology is given by a base of neighborhoods of 0 which are solid or, in the case of metric space, by a solid (monotone) F-norm.

Clearly D_K is a solid subspace of $L^\circ(Y)$.

On $L^\circ(Y)$ we consider the weakest solid additive group topology in which the operator $K : D_K \subset L^\circ(Y) \to L^\circ(X)$ is continuous. It can be shown that this topology can be defined by the (group) norm

$$\tilde{\rho}_K := \rho_Y(u) + \sup\{\rho_X(Kv);\ v \in D_K,\ |v| \leq |u|\}.$$

The *extended domain* $\tilde{D}_K$ of K is then defined as the closure of D_K in this topology–it turns out to be a solid F-space. The extension by continuity of K to $\tilde{D}_K$ is denoted by $\tilde{K}$.

We shall need the following characterization of $\tilde{D}_K$, [**7**]:

(1.2) $u \in \tilde{D}_K$ if and only if for every sequence $u_n \in D_K$, u_n with disjoint supports, such that $|u_n| \leq |u|$ a.e. we have $\sum |Ku_n|^2 < \infty$ almost everywhere.

The interest of $\tilde{D}_K$ lies in the following maximality property.

K can be extended by continuity to a topological vector subspace L of $L^\circ(Y)$, provided:

(i) $L \cap D_K$ is dense in L,

(ii) $K : L \cap D_K \to L^\circ(X)$ is continuous.

Theorem 1.1 [**2**]. *If L is any solid topological vector subspace of $L^\circ(Y)$ satisfying* (i) *and* (ii), *then $L \subset \tilde{D}_K$ and the extension* (*by continuity*) *of K to L is the restriction of $\tilde{K}$ to L.*

By the definition, $\tilde{D}_K$ itself satisfies (i) and (ii). In other words,

(1.3) $\tilde{D}_K$ is the maximal solid subspace of $L^\circ(Y)$ to which K can be extended by continuity.

We notice that if $k = k_1 + ik_2$, where k_1 and k_2 are real, and if K_1 and K_2 denote the corresponding integral operators, then

$$D_K = D_{K_1} \cap D_{K_2} \quad \text{and} \quad \tilde{D}_K = \tilde{D}_{K_1} \cap \tilde{D}_{K_2}. \tag{1.4}$$

The first identity follows readily from (1.1) and the second from (1.2).

In this paper we are interested in the trigonometric transforms related to the Fourier transform.

Let $X = Y = \mathbf{R}$, and let

$$f(x,y) = e^{-ixy} = \cos xy - i \sin xy = c(x,y) - is(x,y).$$

We write the corresponding integral operator as $F = C - iS$ and, in the usual way, we identify C and S with operators on the positive half-axis, $x, y \geq 0$.

It is obvious that the proper domain of F is L^1.

To describe the extended domain of F, we recall the definition of the amalgam space $l^p(L^q)$; for details we refer to [**4, 3**]. If $1 \leq p,\ q < \infty$, then

$$l^p(L^q(\mathbf{R})) = \left\{ u \in L^q_{\text{loc}}(\mathbf{R});\ \sum_{n=-\infty}^{\infty} (||u||_{L^q(n,n+1)})^p = (||u||_{l^p(L^q)})^p < \infty \right\}. \tag{1.5}$$

There is an obvious modification of this definition if p or q is ∞.

We recall that in (1.5) the intervals $(n, n+1)$ can be replaced by the intervals $(na, (n+1)a)$, where $a > 0$ is arbitrary; the resulting spaces are the same and the norms obtained in this way are equivalent.

The space $l^p(L^q(\mathbf{R}_+))$ and the amalgam spaces involving L^q with weight are defined similarly.

The domains of F can now be described as follows:

$$D_F = L^1(\mathbf{R}), \qquad \tilde{D}_F = l^2(L^1). \tag{1.6}$$

The first statement is a repetition of what was said above, for the second we refer to [**8, 3**]. The last reference contains the relevant discussion of the Fourier transform on a locally compact group.

In particular, the Fourier transform is continuous from $l^2(L^1)$ into L° (actually the range $F(l^2(L^1))$ is contained in the amalgam $l^\infty(L^2)$).

By (1.4), the proper domains D_C and D_S contain L^1; however, it is easily seen that the domain D_S contains functions which are not integrable at 0 and therefore it cannot be equal to L^1.

It is plausible, but not obvious, that $D_C = L^1$.

Also, it is plausible, but not obvious, that $\tilde{D}_C$ coincides with $l^2(L^1)$.

It follows from (1.4) that $\tilde{D}_C$ and $\tilde{D}_S$ contain $l^2(L^1)$. Since $\tilde{D}_S$ contains D_S, it is clear from the preceding remark that $\tilde{D}_S$ is strictly larger than $l^2(L^1)$.

In the first part of the paper we prove the assertions above which we have qualified as plausible, but not obvious, and we describe $\tilde{D}_S$.

Several concrete and general examples, see, e.g., [**9**], show that the size of $\tilde{D}_K$ depends on the rate of oscillations of the kernel k; for instance, if k is of fixed sign, or if K is a convolution operator with an almost periodic kernel, then $\tilde{D}_K = D_K$ and K cannot be extended by continuity to solid spaces beyond D_K. However, certain operators with kernels of modulus one can be extended to "arbitrarily large" spaces, provided the kernels oscillate "sufficiently fast" about 0. In examples of this kind, regularity of k seems to have played a role, at least in the proofs. To get an idea if this regularity assumption is essential, we consider, as an example, the operator denoted by $T = \mathrm{SIGN}\,(S)$, with the kernel $k(x,y) = \mathrm{sign}\,(\sin xy)$. Clearly, this kernel oscillates at the "same" rate as s but is discontinuous.

We show that T is continuous from L^2 to L^2 and, consequently, its extended domain contains $L^1 + L^2$. We also show that this extended domain is contained in $l^2(L^1)$. We are unable, however, to arrive at the desired conclusion that $\tilde{D}_T = l^2(L^1)$. It is possible that this conclusion may be false, but we do not have a counterexample.

Concerning notations: whenever a measure space in consideration is understood, we omit it from symbols such as L°, L^1, $l^2(L^1)$, etc. By $u|E$ we mean the restriction of a function u to a set E and 1_E denotes the characteristic function of E. By a set we always mean a measurable set and by a function we mean a measurable function.

2. Domains of C and of S. We will have several occasions to use the following proposition.

Proposition 2.1 (see [**6**]). *Let $E \subset [0,1]$ be a set of positive measure, and denote by N_x, $x \in \mathbf{R}$, the set $\{n \in \{0, \pm 1, \pm 2, \dots\}; nx \in E(\text{mod } 1)$. Suppose that $a_n > 0$, $n = 0, \pm 1, \pm 2, \dots$ is a sequence such that $\sum a_n = \infty$. Then $\sum_{n \in N_x} a_n = \infty$ for almost every $x \in R$.*

The usefulness of this proposition can be seen in the proof of the following result:

Proposition 2.2. *Let $\kappa : \mathbf{R}_+ \to \mathbf{R}$ be a bounded periodic function, satisfying one of the following two conditions.*

(i) *There exist $b > 0$ and $\varepsilon > 0$ such that $(\kappa(t)) > \varepsilon$ for all $t \in (0, b)$.*

(ii) *There exists a $b > 0$ such that for every $a \in (0, b)$ there is an $\varepsilon > 0$ such that $\kappa(t) \geq \varepsilon$ for all $t \in (a, b)$.*

Consider the integral operator with the kernel $k(x, y) = \kappa(xy)$, $x, y \in \mathbf{R}_+$. In the first case (i) we have $D_K = L^1$ and $\tilde{D}_K \subset l^2(L^1)$. In the second case (ii) we have $D_K|(1, \infty) = L^1(1, \infty)$ and $\tilde{D}_K|(1, \infty) \subset l^2(L^1(1, \infty))$.

Remark. Clearly, what matters in the above proposition is that, near 0, κ is of constant sign and that it is bounded away from zero; the size of the period is irrelevant for the result.

Proof. We assume for simplicity that the period of κ is 1, and we deal first with the case (i). If $u \in D_K$, $u \geq 0$, then

$$\int_0^\infty |\kappa(xy)| u(y)\, dy = \sum_{n \geq 0} \int_0^1 |\kappa(xy + nx)| u(y + n)\, dy < \infty \quad \text{for almost all } x. \tag{2.1}$$

We now let $E = (0, b/2)$. Then for $x \in E$, $n \in N_x$ and $y \in (0, 1)$, we have $\kappa(xy + nx) \geq \varepsilon$ and (2.1) implies that $\sum_{n \in N_x} \int_0^1 u(y + n)\, dy < \infty$ for almost all x. We use now Proposition 2.1 to conclude that

$\sum \int_0^1 u(y+n)\,dy < \infty$ and that $u \in L^1$. To prove the statement about $\tilde{D}_K$, we first verify that $\tilde{D}_K \subset L^1_{\mathrm{loc}}$. To this effect, for $c > 0$ we consider the integral operator $K_{b,c}$ with the kernel $k(x,y)$, as before, restricted to $x \in (0, b/c)$ and $y \in (0,c)$. This is an operator with positive kernel bounded away from zero, and general properties of integral operators [**2**] imply that its extended domain coincides with its proper domain which is $L^1(0,c)$. Also, by the same argument, we have $\tilde{D}_K|(0,c) \subset \tilde{D}_{K_{b,c}}$ which proves that $\tilde{D}_K \subset L^1_{\mathrm{loc}}$.

Suppose now that $u \in \tilde{D}_K$, $u \geq 0$, and let $u_n = 1_{(n,n+1)}u$. Then, as in the preceding paragraph, $u_n \in D_K$ for $n = 0, 1, \ldots,$ and we can apply (1.2) to conclude that:

(2.2)
$$\sum_{n\geq 0}\left(\int_n^{n+1} \kappa(xy)u(y)\,dy\right)^2 = \sum_{n\geq 0}\left(\int_0^1 \kappa(xy+nx)u(y+n)\,dy\right)^2 < \infty.$$

We now use Proposition 2.1 to conclude from (2.2) that

$$\sum_{n\geq 0}\left(\int_n^{n+1} u(y)\,dy\right)^2 < \infty$$

in the same manner as we concluded from (2.1) that if u is in D_K, then $u \in L^1$.

In the case (ii), we use the same argument as above, with the following modifications. To verify the statement about D_K, we take $a = b/8$ in the condition (ii), choose ε accordingly, let $E = (0, b/8)$, and observe that $\kappa(xy + x + nx) \geq \varepsilon$ if $x \in (b/8, b/4)$, $y \in (0,1)$ and $n \in N_x$. The same choice is appropriate for the proof of the inclusion $\tilde{D}_K \subset l^2(L^1(1,\infty))$ provided we verify first that $\tilde{D}_K \subset L^1_{\mathrm{loc}}([1,\infty))$. To see this, we notice that for every $c > 1$ there is an $\varepsilon > 0$ such that $\kappa(xy) \geq \varepsilon$ for all $y \in [1,c]$ and $x \in (b/(2c), b/c)$. □

We now apply Proposition 2.2 with suitable choices of κ in order to obtain the corresponding information about the domains of the operators C and T introduced in Section 1.

Proposition 2.3. (i) *If* $K = C$ *or if* $K = T$, *then* $D_K = L^1$ *and* $\tilde{D}_K \subset l^2(L^1)$.

(ii) *For* $K = S$, $D_S = L^1(w)$ *where* w *is the weight,* $w = \min(y, 1)$ *for* $y \geq 0$, *and* $\tilde{D}_S \subset l^2(L^1(w))$. *In particular, it follows from* (1.4) *and from* (1.6) *that* $\tilde{D}_C = l^2(L^1)$ *and* $\tilde{D}_S = l^2(L^1(w))$.

Proof. In the part (i) the choice of κ is obvious and the conclusion follows from part (i) of Proposition 2.2.

In (ii) the information about D_S and about $\tilde{D}_S$ for $y > 1$ follows from (ii) in Proposition 2.2. Restricting the kernel $s(x, y)$ to $y \in (0, 1)$ and $x \in (0, 1/4)$, we rewrite the corresponding operator as

$$x \int_0^1 \frac{\sin(xy)}{xy} yu(y)\, dy$$

and notice that kernel $(\sin xy)/(xy)$ is positive and bounded away from zero, hence its proper and extended domains are both L^1.

Since the cosine transform C is continuous from $l^2(L^1)$ into L° and $\tilde{D}_C$ is contained in $l^2(L^1)$, it follows from Theorem 1.1 that the two spaces are equal. For the same reason, $l^2(L^1) \subset \tilde{D}_S$; to see that the same inclusion also holds for the larger space $l^2(L^1(w))$, we write this space as the direct sum of $L^1(0, 1; w) = L^1(0, 1; y)$ and $l^2(L^1(1, \infty))$. The first summand is contained in D_S, therefore in $\tilde{D}_S$, and the second, with obvious identification, is contained in $l^2(L^1)$. This completes the description of extended domains of C and of S. □

3. Continuity properties of the operator T**.** In this section we show that the operator $T = \mathrm{SIGN}\,(\mathrm{S})$ considered on a suitable dense subset is continuous from L^2 to L^2, in particular, its extended domain contains $L^1 + L^2$ and therefore is larger than its proper domain which is L^1.

We begin with the following lemma which in a more general form can be found in [**1**].

Lemma 3.1. *Let* $\kappa(t) \geq 0$, $t > 0$, *be such that* $\int_0^\infty t^{-3/2}\kappa(t)\, dt = c < \infty$, *and consider the integral operator* K *with the kernel* $k(x, y) = (1/x)\kappa(x/y)$, $x, y > 0$. *Then* K *is a bounded operator in* $L^2(0, \infty)$ *and* $||K|| \leq c$.

For the sake of completeness we include a simple proof. The hypothesis on κ implies that, with $\phi(x) = x^{-1/2}$,

$$\begin{aligned} K^*\phi(y) &= \int_0^\infty x^{-1/2} k(x,y)\,dx \\ &= y^{-1/2} \int_0^\infty \kappa(t) t^{-3/2}\,dt = c\phi(y) \end{aligned}$$

and

$$K\phi(x) = \int_0^\infty y^{-1/2} k(x,y)\,dy = x^{-1/2} \int_0^\infty \kappa(t) t^{-3/2}\,dt = c\phi(x).$$

It follows that K satisfies the well-known Schur criterion of boundedness in L^2; see, for instance, [**5**] for a more general result, and the argument is complete. □

We now consider the operator, T,

$$\begin{aligned} Tu(x) &= \int_0^\infty \operatorname{sign}[\sin(2\pi xy)] u(y)\,dy \\ &= \sum_{n=0}^\infty (-1)^n \int_{n/x}^{(n+1)/x} u(z)\,dz. \end{aligned} \tag{3.1}$$

Proposition 3.2. *If u is a characteristic function of an interval, then $Tu \in L^2$ and Tu can be written in the form*

$$Tu(x) = \int_0^{1/x} u(z)\,dz - \sum_{n=1}^\infty \int_{(2n-1)/x}^{2n/x} [u(z) - u(z+1/x)]\,dz,$$

where the last sum is convergent in L^2. We also have $Tu(x) = \lim_{r\to 1} T_r u(x)$, where

$$T_r u(x) = \sum_{n=0}^\infty r^n \int_{2n/x}^{(2n+1)/x} [u(z) - u(z+1/x)]\,dz,$$

with the sum is understood as a limit in L^2.

Proof. If $u(z) = 0$ for all $z > b$, then for $x \leq 1/b, Tu(x) = ||u||_{L^1}$ and therefore $Tu(x)$ is bounded. For large x the sum consists, after cancellations, of at most two nonzero terms and can be estimated by $(2/x)$; it follows that $Tu \in L^2$. The statements concerning the limits follow from the dominated convergence theorem. □

We consider next the composition of the sine transform with T; this can be represented in a form resembling somewhat the Poisson kernel.

Proposition 3.3. *The composition ST can be written in the form $STu = \lim_{r \to 1}(1-r)K_r u$, where K_r is the integral operator with the kernel*

$$k_r(x,y) = -\left[\frac{4}{x}\cos\frac{x}{y}\sin^2\left(\frac{x}{2y}\right)\right]\left[(1-r)^2 + 4r\left(\sin\frac{x}{y}\right)^2\right],$$
$$0 \leq r < 1.$$

The formula is valid for functions u which are characteristic functions of intervals (*and hence for step functions*) *and the limit is understood in the sense of L^2.*

Proof. We use Proposition 3.2 and the fact that S is a bounded operator in L^2 to write $STu = S \lim T_r u = \lim ST_r u$.

We have, after a suitable change of variables,

$$ST_r u = \sum_{n=0}^{\infty} r^n \int_0^{\infty} \sin\frac{x}{y}\left[\int_{2ny}^{(2n+1)y}\left(u(z) - u\left(z + \frac{1}{y}\right)\right)dz\right] y^{-2}\,dy.$$

Integrating by parts and noticing that $\sin(x/y)y^{-2}\,dy = (1/x)(d(\cos x/y))$, we get

$$\begin{aligned} ST_r u &= \frac{1}{x}\sum_{n=0}^{\infty} r^n \left(\int_0^{\infty}\cos\frac{x}{y}[2nu(2ny) - 2(2n+1)u((2n+1)y)\right.\\ &\qquad\qquad \left. + (2n+2)u((2n+2)y)]\,dy\right)\\ &= \frac{1}{x}\int_0^{\infty}\left\{\sum_{n=0}^{\infty} r^n\left[\cos\left(2n\frac{x}{y}\right) - 2\cos\left((2n+1)\frac{x}{y}\right)\right.\right.\\ &\qquad\qquad \left.\left. + \cos\left((2n+2)\frac{x}{y}\right)\right]\right\}u(y)\,dy. \end{aligned}$$

By a standard calculation involving a geometric series, we see that the sum in the curly parentheses equals $(1-r)(1-2\cos t+\cos 2t)/((1-r)^2+2r(1-\cos 2t))$, where $t=x/y$ (here we use the assumption that $r<1$), which yields the conclusion. □

We now study boundedness properties of the operators K_r. Since S^2 is a scalar multiple of the identity, and since T, at least by the nature of the oscillations of its kernel, resembles S, it is expected that ST should inherit some of the properties of the identity.

Proposition 3.4. *K_r are bounded operators from L^2 into L^2;*

$$||K_r||_{L^2\to L^2}\le \text{const}\,(1-r)^{-1},\qquad 0\le r<1.$$

Proof. We use Lemma 3.1 with $\kappa(t)=(|\cos t|\sin^2(t/2))/((1-r)^2+4r\sin^2 t)$. The integral in question can be written in the form

$$\int_0^\infty \kappa(t)t^{-3/2}\,dt=\int_A \kappa(t)t^{-3/2}\,dt+\int_B \kappa(t)t^{-3/2}\,dt,$$

where for some fixed α, $0<\alpha<1$, $A=\{t>\pi/2; |\sin t|>\alpha\}$ and $B=\mathbf{R}_+\backslash A$. Then the first integral (over A) can be estimated by $\text{const}/[(1-r)^2+2\alpha r]$, the second can be estimated by $\text{const}\,(1-r)^{-1}$ and the proof is complete. □

Theorem 3.5. *The operator T is continuous from L^2 to L^2. The extended domain of T contains L^1+L^2, hence is strictly larger than its proper domain (which coincides with L^1), and is contained in $l^2(L^1)$.*

Proof. Since S is a scalar multiple of a unitary operator in L^2, in particular S^2 is a scalar multiple of the identity, the theorem follows from Proposition 3.4 by composing S (from the left) with the continuous operator $K_r: L^2\to L^2$. The inclusion $\tilde{D}_T$ in $l^2(L^1)$ was established in Proposition 2.3. □

Remark. We were unable to obtain Theorem 3.5 directly, without the intermediary of the composition ST. Also, as already indicated in the introduction, we were unable to establish our outstanding conjecture that the extended domain of T is $l^2(L^1)$.

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Institut Fourier, BP 74, St. Martin D'hères, Cedex, France

University of Kansas, Department of Mathematics, Lawrence, KS 66045