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THE SPECTRAL THEORY OF SECOND ORDER TWO-POINT DIFFERENTIAL OPERATORS IV. THE ASSOCIATED PROJECTIONS AND THE SUBSPACE $\mathcal{S}_\infty(L)$

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ABSTRACT. This paper is the final part in a four-part series on the spectral theory of a two-point differential operator L in $L^2[0,1]$, where L is determined by a formal differential operator $l = -D^2 + q$ and by independent boundary values B_1, B_2. For the family of projections $\{Q_{0k}\}_{k=1}^n \cup \{Q'_k\}_{k=k_0}^\infty \cup \{Q''_k\}_{k=k_0}^\infty$ which map $L^2[0,1]$ onto the generalized eigenspaces of L, it is determined whether or not the family of all finite sums of these projections is uniformly bounded in norm. Equivalently, for the subspace $\mathcal{S}_\infty(L)$ consisting of all $u \in L^2[0,1]$ with $u = \sum_{k=1}^n Q_{0k}u + \sum_{k=k_0}^\infty Q'_k u + \sum_{k=k_0}^\infty Q''_k u$, it is determined whether or not $\mathcal{S}_\infty(L) = \overline{\mathcal{S}_\infty(L)} = L^2[0,1]$. It is necessary to modify the projections and $\mathcal{S}_\infty(L)$ in the multiple eigenvalue case.

1. Introduction. In this paper we conclude our four-part series on the spectral theory of a linear second order two-point differential operator L in the complex Hilbert space $L^2[0,1]$. Let L be the differential operator in $L^2[0,1]$ defined by

$$\mathcal{D}(L) = \{u \in H^2[0,1] \mid B_i(u) = 0,\ i = 1,2\},$$
$$Lu = lu,$$

where

$$l = -\left(\frac{d}{dt}\right)^2 + q(t)\left(\frac{d}{dt}\right)^0$$

is a second order formal differential operator on the interval $[0,1]$ with $q \in C[0,1]$, B_1, B_2 are linearly independent boundary values given by

$$B_1(u) = a_1 u'(0) + b_1 u'(1) + a_0 u(0) + b_0 u(1),$$
$$B_2(u) = c_1 u'(0) + d_1 u'(1) + c_0 u(0) + d_0 u(1),$$

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and $H^2[0,1]$ denotes the Sobolev space consisting of all functions $u \in C^1[0,1]$ with u' absolutely continuous on $[0,1]$ and $u'' \in L^2[0,1]$. From the boundary coefficient matrix,

$$A = \begin{bmatrix} a_1 & b_1 & a_0 & b_0 \\ c_1 & d_1 & c_0 & d_0 \end{bmatrix},$$

we form the six determinants A_{ij}, $1 \le i < j \le 4$, where A_{ij} is the determinant of the 2×2 submatrix of A obtained by retaining the ith and jth columns. We also write L in the form

$$L = T + S, \tag{1.1}$$

where T is the differential operator given by

$$\mathcal{D}(T) = \{u \in H^2[0,1] \mid B_i(u) = 0,\ i = 1,2\},$$
$$Tu = -u'',$$

and S is the multiplication operator given by $\mathcal{D}(S) = L^2[0,1]$, $Su = qu$.

In Part I [**14**] L and T are classified as belonging to one of five cases, Cases 1–5, by imposing conditions on the boundary parameters A_{ij}. For Cases 1–4 the spectrum $\sigma(L)$ is a countably infinite subset of $\mathbf{C}$, the eigenvalues of L satisfy certain apriori estimates, and the generalized eigenfunctions of L are complete in $L^2[0,1]$. See Theorems 4.1, 5.1, 6.1 and 7.1 in Part I. Case 5 contains many degenerate cases; some partial results are given for it in Section 8 of Part I.

The characteristic determinant $\Delta(\rho)$ of L is constructed in Part II [**15**], and its crucial asymptotic formula (4.7) is derived on a half plane $\operatorname{Im}\rho \ge -d$. A complex number $\lambda = \rho^2$ with $\operatorname{Im}\rho \ge -d$ and $|\rho| > 2e^{2d}||q||_\infty$ is an eigenvalue of L if and only if ρ is a zero of Δ, in which case the algebraic multiplicity of λ is equal to the order of the zero at ρ. See Theorems 4.1 and 5.1 in Part II.

In Part III [**16**] the eigenvalues are calculated for Cases 1–4 using Δ. The eigenvalues of L can be written as two infinite sequences $\{\lambda_k'\}_{k=k_0}^\infty$, $\{\lambda_k''\}_{k=k_0}^\infty$ for an appropriate positive integer k_0, plus a finite number of additional points $\{\lambda_{0k}\}_{k=1}^n$ about which little is known. The λ_k', λ_k'' satisfy asymptotic formulas in which the rates of convergence vary with the case and with the smoothness of q. In Cases 1, 2A, 3A, 3B and 4 the λ_{0k}, λ'_k, λ''_k are all distinct, and the corresponding algebraic multiplicities and ascents are

$$\nu(\lambda'_k) = m(\lambda'_k) = 1, \qquad \nu(\lambda''_k) = m(\lambda''_k) = 1 \tag{1.2}$$

for $k = k_0, k_0 + 1, \dots$. Case 2B contains all the multiple eigenvalue cases, where we may have $\lambda'_k \neq \lambda''_k$ for some k with

$$\nu(\lambda'_k) = m(\lambda'_k) = 1 \quad \text{and} \quad \nu(\lambda''_k) = m(\lambda''_k) = 1, \tag{1.3}$$

and $\lambda'_k = \lambda''_k$ for other k with

$$\nu(\lambda'_k) = 2 \quad \text{and} \quad m(\lambda'_k) = 1 \quad \text{or} \quad m(\lambda'_k) = 2. \tag{1.4}$$

See Theorems 2.2, 2.3, 2.4, 2.6 and 3.2 in Part III.

Here in Part IV our goal is to study the projections which map $L^2[0,1]$ onto the generalized eigenspaces of L and to develop the basic properties of the associated subspaces $\mathcal{S}_\infty(L)$ and $\mathcal{M}_\infty(L)$. Let us proceed to introduce these quantities for L belonging to Cases 1–4. For each $\lambda \in \mathbf{C}$ let $L_\lambda = \lambda I - L$, let $m(\lambda)$ denote the ascent of the operator L_λ, and let $L(\lambda) = (L_\lambda)^{m(\lambda)}$. Then for each λ belonging to $\sigma(L)$, the null space $\mathcal{N}(L(\lambda))$ is the generalized eigenspace corresponding to the eigenvalue λ, its dimension $\nu(\lambda)$ is the algebraic multiplicity of λ, and to the topological direct sum decomposition

$$L^2[0,1] = \mathcal{N}(L(\lambda)) \oplus \mathcal{R}(L(\lambda))$$

there corresponds the canonical projection Q which maps $L^2[0,1]$ onto $\mathcal{N}(L(\lambda))$ along $\mathcal{R}(L(\lambda))$.

Assume L belongs to Case 1, Case 2A, Case 3A, Case 3B or Case 4. Let $\mathcal{Q}$ be the family of projections

$$\mathcal{Q} = \{Q_{0k}\}_{k=1}^{n} \bigcup \{Q'_k\}_{k=k_0}^{\infty} \bigcup \{Q''_k\}_{k=k_0}^{\infty},$$

where the projection Q_{0k} maps $L^2[0,1]$ onto $\mathcal{N}(L(\lambda_{0k}))$ along $\mathcal{R}(L(\lambda_{0k}))$ for $k = 1, \dots, n$; the projection Q'_k maps $L^2[0,1]$ onto $\mathcal{N}(L(\lambda'_k))$ along $\mathcal{R}(L(\lambda'_k))$ for $k = k_0, k_0 + 1, \dots$; and the projection Q''_k maps $L^2[0,1]$ onto $\mathcal{N}(L(\lambda''_k))$ along $\mathcal{R}(L(\lambda''_k))$ for $k = k_0, k_0 + 1, \dots$. The family $\mathcal{Q}$ is called the *family of projections associated with* L or the *spectral family of* L. It is well known that

$$(1.5)\qquad \begin{aligned} &Q_{0k}Q_{0j} = \delta_{kj}Q_{0k}, &\quad Q'_kQ'_j = \delta_{kj}Q'_k, &\quad Q''_kQ''_j = \delta_{kj}Q''_k,\\ &Q_{0k}Q'_j = 0, &\quad Q_{0k}Q''_j = 0, &\quad Q'_kQ''_j = 0 \end{aligned}$$

for all k, j (see (3.3) in [**6**]). In terms of these projections we introduce the subspaces

$$\mathcal{S}_\infty(L) = \left\{ u \in L^2[0,1] \,\middle|\, u = \sum_{k=1}^{n} Q_{0k}u + \sum_{k=k_0}^{\infty} Q'_ku + \sum_{k=k_0}^{\infty} Q''_ku \right\}$$

and

$$\begin{aligned} \mathcal{M}_\infty(L) = \{u \in L^2[0,1] \mid Q_{0k}u = 0,\ k = 1, \dots, n;\\ Q'_ku = Q''_ku = 0,\ k = k_0, k_0+1, \dots\}. \end{aligned}$$

Implicit in the definition of $\mathcal{S}_\infty(L)$ is the fact that the two series $\sum_{k=k_0}^{\infty} Q'_ku$ and $\sum_{k=k_0}^{\infty} Q''_ku$ are convergent for each $u \in \mathcal{S}_\infty(L)$. Clearly $\mathcal{S}_\infty(L)$ contains all the generalized eigenfunctions of L, and from Part I,

$$(1.6)\qquad \overline{\mathcal{S}_\infty(L)} = L^2[0,1] \quad\text{and}\quad \mathcal{M}_\infty(L) = \{0\}.$$

Also, as a simple consequence of (1.5) $\mathcal{S}_\infty(L)$ is precisely the set of all functions $u \in L^2[0,1]$ which can be expanded in an infinite series of generalized eigenfunctions of L.

In Section 3 we prove that the family of all finite sums of the projections in $\mathcal{Q}$ is uniformly bounded in norm and $\mathcal{S}_\infty(L) = \overline{\mathcal{S}_\infty(L)}$ for Cases 1, 2A and 3A, and then in sharp contrast, we show in Sections 5 and 6 that the projections in $\mathcal{Q}$ are unbounded in norm and $\mathcal{S}_\infty(L) \neq \overline{\mathcal{S}_\infty(L)}$ for Cases 3B and 4. These results are based on special integral representations of the projections in $\mathcal{Q}$. Indeed, in Part III we constructed two sequences of circles $\{\Gamma'_k\}_{k=k_0}^{\infty}$, $\{\Gamma''_k\}_{k=k_0}^{\infty}$ in the ρ-plane having centers μ'_k, μ''_k and fixed radius δ. For each $k \geq k_0$ the characteristic determinant Δ has a unique zero ρ'_k inside Γ'_k and a unique zero ρ''_k inside Γ''_k, with ρ'_k, ρ''_k zeros of order 1 and $\lambda'_k = (\rho'_k)^2$, $\lambda''_k = (\rho''_k)^2$. In addition, for each point ρ on Γ'_k or Γ''_k, $\Delta(\rho) \neq 0$, so the point $\lambda = \rho^2$ belongs to the resolvent set $\rho(L)$ and the resolvent

$R_\lambda(L) = (\lambda I - L)^{-1}$ exists. Under the mapping $\lambda = \rho^2$ the circles Γ_k', Γ_k'' are mapped one-to-one onto smooth simple closed curves Λ_k', Λ_k'' in the λ-plane for $k \geq k_0$. From the simple geometry of the Γ_k', Γ_k'', it is obvious that Λ_k', Λ_k'' contain λ_k', λ_k'', respectively, in their interiors and no other points of $\sigma(L)$. Therefore,

$$
\begin{aligned}
Q_k' &= \frac{1}{2\pi i}\int_{\Lambda_k'} R_\lambda(L)\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma_k'} 2\rho R_{\rho^2}(L)\,d\rho,\\
Q_k'' &= \frac{1}{2\pi i}\int_{\Lambda_k''} R_\lambda(L)\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma_k''} 2\rho R_{\rho^2}(L)\,d\rho
\end{aligned}
\tag{1.7}
$$

for $k = k_0, k_0 + 1, \ldots$.

Next, assume L belongs to Case 2B, so there is the possibility of multiple eigenvalues. For this case the *family of projections associated with* L or the *spectral family* of L is given by

$$\mathcal{Q} = \{Q_{0k}\}_{k=1}^{n} \bigcup \{Q_k\}_{k=k_0}^{\infty},$$

where Q_{0k} is the projection of $L^2[0,1]$ onto $\mathcal{N}(L(\lambda_{0k}))$ along $\mathcal{R}(L(\lambda_{0k}))$ and Q_k is the projection defined as follows: if $\lambda_k' \neq \lambda_k''$, then $Q_k := Q_k' + Q_k''$ where Q_k', Q_k'' are the projections of $L^2[0,1]$ onto $\mathcal{N}(L(\lambda_k'))$, $\mathcal{N}(L(\lambda_k''))$ along $\mathcal{R}(L(\lambda_k'))$, $\mathcal{R}(L(\lambda_k''))$, respectively, and hence, Q_k is the projection of $L^2[0,1]$ onto

$$\mathcal{N}(L(\lambda_k')) \oplus \mathcal{N}(L(\lambda_k'')) \quad \text{along} \quad \mathcal{R}(L(\lambda_k')) \cap \mathcal{R}(L(\lambda_k''));$$

if $\lambda_k' = \lambda_k''$, then $Q_k := Q_k' = Q_k''$ where $Q_k' = Q_k''$ is the projection of $L^2[0,1]$ onto $\mathcal{N}(L(\lambda_k'))$ along $\mathcal{R}(L(\lambda_k'))$. For these projections we have

$$Q_{0k}Q_{0j} = \delta_{kj}Q_{0k}, \qquad Q_kQ_j = \delta_{kj}Q_k, \qquad Q_{0k}Q_j = 0 \tag{1.8}$$

for all k and j, and we can form the subspaces

$$\mathcal{S}_\infty(L) = \left\{ u \in L^2[0,1] \,\middle|\, u = \sum_{k=1}^{n} Q_{0k}u + \sum_{k=k_0}^{\infty} Q_k u \right\}$$

and

$$
\begin{aligned}
\mathcal{M}_\infty(L) = \{u \in L^2[0,1] \mid Q_{0k}u = 0,\ k = 1, \ldots, n;&\\
Q_k u = 0,\ k = k_0, k_0+1, \ldots\}.&
\end{aligned}
$$

As in the previous cases it is easy to show that

$$\overline{S_\infty(L)} = L^2[0,1] \quad \text{and} \quad \mathcal{M}_\infty(L) = \{0\}. \tag{1.9}$$

In our treatment of Case 2B in Part III, we constructed a sequence of circles $\{\Gamma_k\}_{k=k_0}^{\infty}$ with centers μ_k and fixed radius δ such that Δ has two zeros ρ_k' and ρ_k'' inside Γ_k with $\lambda_k' = (\rho_k')^2$ and $\lambda_k'' = (\rho_k'')^2$, where either $\rho_k' \neq \rho_k''$ and $\lambda_k' \neq \lambda_k''$ with ρ_k' and ρ_k'' both zeros of order 1, or $\rho_k' = \rho_k''$ and $\lambda_k' = \lambda_k''$ with ρ_k' a zero of order 2. Also, for each point ρ on Γ_k, $\Delta(\rho) \neq 0$ and $\lambda = \rho^2 \in \rho(L)$. The mapping $\lambda = \rho^2$ maps the circle Γ_k in a one-to-one manner onto a smooth simple closed curve Λ_k in the λ-plane; Λ_k contains the eigenvalues λ_k', λ_k'' in its interior and no other points of $\sigma(L)$, and hence,

$$Q_k = \frac{1}{2\pi i} \int_{\Lambda_k} R_\lambda(L)\, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_k} 2\rho R_{\rho^2}(L)\, d\rho \tag{1.10}$$

for $k = k_0, k_0 + 1, \dots$. In Section 4 we will show that the family of all finite sums of the projections in $\mathcal{Q}$ is uniformly bounded in norm and $\mathcal{S}_\infty(L) = \overline{\mathcal{S}_\infty(L)}$. These results are valid in spite of the fact that it is possible to have

$$||Q_k'|| \to \infty \quad \text{and} \quad ||Q_k''|| \to \infty \quad \text{as} \quad k \to \infty.$$

This unusual phenomenon has been studied in [**13**] for the special case $q(t) \equiv 0$.

In Section 2 we have collected all the basic results which are used in the sequel to treat the various cases: Theorem 2.1 relates the resolvents $R_\lambda(L)$ and $R_\lambda(T)$, Theorem 2.2 develops uniform bounds on the projections associated with L in terms of uniform bounds on the corresponding projections associated with T, and Theorems 2.3 and 2.4 establish the equivalence of the projections being uniformly bounded and $\mathcal{S}_\infty(L)$ being closed. Then in Sections 3–6 we proceed on a case-by-case basis to study the family of projections $\mathcal{Q}$ associated with L and the subspaces $\mathcal{S}_\infty(L)$ and $\mathcal{M}_\infty(L)$. The general strategy used in each case is as follows:

(i) Introduce the circles Γ_k', Γ_k'' and Γ_k for $k = k_0, k_0+1, \dots$, thereby establishing the geometry.

(ii) Determine the decay rate of $||R_{\rho^2}(T)||$ for ρ on Γ'_k, Γ''_k and Γ_k. For Case 3B it is also necessary to find the decay rate of $||R_{\rho^2}(T)SR_{\rho^2}(T)||$.

(iii) Utilize Theorem 2.1 to estimate the norms of $Q'_k - P'_k$, $Q''_k - P''_k$ and $Q_k - P_k$, where P'_k, P''_k and P_k are the corresponding projections for T.

(iv) Apply Theorem 2.2 to derive the uniform boundedness of the projections associated with L for Cases 1, 2A, 2B and 3A, or show the projections to be unbounded for Cases 3B and 4.

(v) Use Theorems 2.3 and 2.4 to prove that $\mathcal{S}_\infty(L)$ is closed in Cases 1, 2A, 2B and 3A, and is not closed in Cases 3B and 4.

2. Mathematical preliminaries. All of the above results for the differential operator L are also valid for the differential operator T, and in fact, we can use the *same* circles $\Gamma'_k, \Gamma''_k, \Gamma_k$ and the *same* smooth simple closed curves $\Lambda'_k, \Lambda''_k, \Lambda_k$ for both L and T (see Part III). Let us briefly summarize the results for T that are needed in the sequel.

First, assume that T belongs to Case 1, Case 2A, Case 3A, Case 3B or Case 4. Then for $k \geq k_0$ the characteristic determinant $\tilde{\Delta}$ of T has a unique zero $\tilde{\rho}'_k$ inside Γ'_k and a unique zero $\tilde{\rho}''_k$ inside Γ''_k, with $\tilde{\rho}'_k$ and $\tilde{\rho}''_k$ both zeros of order 1. The two sequences

$$\tilde{\lambda}'_k = (\tilde{\rho}'_k)^2, \qquad \tilde{\lambda}''_k = (\tilde{\rho}''_k)^2, \qquad k = k_0, k_0 + 1, \dots ,$$

consist of distinct eigenvalues of T with $\nu(\tilde{\lambda}'_k) = m(\tilde{\lambda}'_k) = 1$, $\nu(\tilde{\lambda}''_k) = m(\tilde{\lambda}''_k) = 1$, and account for all but a finite number of the points in $\sigma(T)$. For T we introduce the projections P'_k, P''_k, $k = k_0, k_0 + 1, \dots$, which map $L^2[0,1]$ onto $\mathcal{N}(T(\tilde{\lambda}'_k)), \mathcal{N}(T(\tilde{\lambda}''_k))$ along $\mathcal{R}(T(\tilde{\lambda}'_k))$, $\mathcal{R}(T(\tilde{\lambda}''_k))$, respectively. The P'_k, P''_k satisfy

$$P'_k P'_j = \delta_{kj} P'_k, \qquad P''_k P''_j = \delta_{kj} P''_k, \qquad P'_k P''_j = 0 \tag{2.1}$$

for all k and j, and have integral representations identical to those in (1.7) with L replaced by T, and hence,

$$\begin{aligned} Q'_k - P'_k &= \frac{1}{2\pi i}\int_{\Gamma'_k} 2\rho[R_{\rho^2}(L) - R_{\rho^2}(T)]\,d\rho, \\ Q''_k - P''_k &= \frac{1}{2\pi i}\int_{\Gamma''_k} 2\rho[R_{\rho^2}(L) - R_{\rho^2}(T)]\,d\rho \end{aligned} \tag{2.2}$$

for $k = k_0, k_0 + 1, \ldots$.

Second, suppose T belongs to Case 2B where multiple eigenvalues are possible. Then $\tilde{\Delta}$ has two zeros $\tilde{\rho}'_k$ and $\tilde{\rho}''_k$ inside each circle Γ_k for $k \geq k_0$, where either $\tilde{\rho}'_k \neq \tilde{\rho}''_k$ with $\tilde{\rho}'_k$ and $\tilde{\rho}''_k$ both zeros of order 1 or $\tilde{\rho}'_k = \tilde{\rho}''_k$ with $\tilde{\rho}'_k$ a zero of order 2. The two sequences

$$\tilde{\lambda}'_k = (\tilde{\rho}'_k)^2, \qquad \tilde{\lambda}''_k = (\tilde{\rho}''_k)^2, \qquad k = k_0, k_0 + 1, \ldots,$$

are made up of eigenvalues of T, accounting for all but a finite number of the eigenvalues. In [**11**] Case 2B is subdivided into Cases V–VIII. For T belonging to Case VII or Case VIII, we have $\tilde{\rho}'_k \neq \tilde{\rho}''_k$ and $\tilde{\lambda}'_k \neq \tilde{\lambda}''_k$ for all $k \geq k_0$, with $\nu(\tilde{\lambda}'_k) = m(\tilde{\lambda}'_k) = 1$ and $\nu(\tilde{\lambda}''_k) = m(\tilde{\lambda}''_k) = 1$; here we introduce the projections P_k, $k = k_0, k_0 + 1, \ldots$, where P_k maps $L^2[0,1]$ onto

$$\mathcal{N}(T(\tilde{\lambda}'_k)) \oplus \mathcal{N}(T(\tilde{\lambda}''_k)) \quad \text{along} \quad \mathcal{R}(T(\tilde{\lambda}'_k)) \cap \mathcal{R}(T(\tilde{\lambda}''_k)).$$

On the other hand, for T belonging to Case V or Case VI, $\tilde{\rho}'_k = \tilde{\rho}''_k$ and $\tilde{\lambda}'_k = \tilde{\lambda}''_k$ for all $k \geq k_0$, with $\nu(\tilde{\lambda}'_k) = 2$, $m(\tilde{\lambda}'_k) = 1$ in Case V, and $m(\tilde{\lambda}'_k) = 2$ in Case VI, and in this true multiple eigenvalue case we introduce the projections P_k, $k = k_0, k_0 + 1, \ldots$, where P_k maps $L^2[0,1]$ onto $\mathcal{N}(T(\tilde{\lambda}'_k))$ along $\mathcal{R}(T(\tilde{\lambda}'_k))$. Throughout Case 2B the P_k satisfy

$$P_k P_j = \delta_{kj} P_k \tag{2.3}$$

for all k and j, the integral representation (1.10) is valid for each P_k with L replaced by T, and

$$Q_k - P_k = \frac{1}{2\pi i} \int_{\Gamma_k} 2\rho [R_{\rho^2}(L) - R_{\rho^2}(T)]\, d\rho \tag{2.4}$$

for $k = k_0, k_0 + 1, \ldots$.

The basic properties of the projections P'_k, P''_k, P_k for T have already been developed in [**11**] and [**13**]. Using (2.1)–(2.4), we will show that the projections Q'_k, Q''_k, Q_k for L are perturbations of the projections P'_k, P''_k, P_k for T, and hence, they have analogous properties. The key to estimating the integrands in (2.2) and (2.4) is provided by the following theorem, which is a modification of Theorem 3.1 in Part I.

Theorem 2.1. *If $\lambda \in \rho(T) \cap \rho(L)$, then*

$$R_\lambda(L) - R_\lambda(T) = R_\lambda(T)SR_\lambda(L) = R_\lambda(T)SR_\lambda(T)[I + SR_\lambda(L)]. \tag{2.5}$$

In addition, if $||R_\lambda(T)|| \le (1/2)||S||^{-1}$, then $||R_\lambda(L)|| \le 2||R_\lambda(T)||$ and

$$||R_\lambda(L) - R_\lambda(T)|| \le 2||S||\,||R_\lambda(T)||^2. \tag{2.6}$$

Alternately, if $||R_\lambda(T)SR_\lambda(T)|| \le (1/2)||S||^{-1}$, then $||R_\lambda(L)|| \le 2||R_\lambda(T)|| + ||S||^{-1}$ and

$$||R_\lambda(L) - R_\lambda(T)|| \le 2(1 + ||S||\,||R_\lambda(T)||)||R_\lambda(T)SR_\lambda(T)||. \tag{2.7}$$

Proof. Clearly $L_\lambda = T_\lambda - S$. Multiplying this result by $R_\lambda(L)$ on the right and by $R_\lambda(T)$ on the left, we immediately obtain the first part of (2.5). The second part is a simple application of the first part.

If $||R_\lambda(T)|| \le (1/2)||S||^{-1}$, then from the first part of (2.5),

$$\begin{aligned}||R_\lambda(L)|| &\le ||R_\lambda(T)|| + ||R_\lambda(T)||\,||S||\,||R_\lambda(L)|| \\ &\le ||R_\lambda(T)|| + \frac{1}{2}||R_\lambda(L)||\end{aligned}$$

or $||R_\lambda(L)|| \le 2||R_\lambda(T)||$, and hence,

$$||R_\lambda(L) - R_\lambda(T)|| \le ||R_\lambda(T)||\,||S||\,||R_\lambda(L)|| \le 2||S||\,||R_\lambda(T)||^2.$$

On the other hand, if $||R_\lambda(T)SR_\lambda(T)|| \le (1/2)||S||^{-1}$, then from the second part of (2.5),

$$||R_\lambda(L)|| \le ||R_\lambda(T)|| + \frac{1}{2}||S||^{-1}[1 + ||S||\,||R_\lambda(L)||]$$

or $||R_\lambda(L)|| \le 2||R_\lambda(T)|| + ||S||^{-1}$, which yields

$$\begin{aligned}||R_\lambda(L) - R_\lambda(T)|| &\le ||R_\lambda(T)SR_\lambda(T)||[1 + ||S||(2||R_\lambda(T)|| + ||S||^{-1})] \\ &= 2(1 + ||S||\,||R_\lambda(T)||)||R_\lambda(T)SR_\lambda(T)||. \qquad \square\end{aligned}$$

To effectively use (2.6) or (2.7), we must be able to control the norms of $R_\lambda(T)$ or $R_\lambda(T)SR_\lambda(T)$ for points $\lambda = \rho^2$ with ρ on the circles $\Gamma_k', \Gamma_k'', \Gamma_k$. One useful result in this direction is given by equation (3.7) in Part I:

$$\begin{aligned} ||R_\lambda(T)|| \le \frac{1}{2|\rho|\,|\tilde{\Delta}(\rho)|} \cdot \frac{e^{|b|}}{|b|}\{6|A_{12}|\,|\rho|^2 + 4|A_{14}+A_{23}|\,|\rho| \\ + 2|A_{14}-A_{23}|\,|\rho| + 4|A_{13}|\,|\rho| + 4|A_{24}|\,|\rho| + 6|A_{34}|\} \end{aligned} \tag{2.8}$$

for all $\lambda = \rho^2 \in \rho(T)$ with $\rho = a + ib$ and $b \neq 0$. Equation (2.8) will be used in treating Case 4.

In handling Cases 1–3 we will need a variation of (2.8) which allows $b = 0$. Indeed, take any point $\lambda = \rho^2 \in \rho(T)$ with $\rho = a + ib \neq 0$. Then by (3.1)–(3.3) in Part I the Green's function for T_λ is given by

$$\tilde{G}(t,s;\lambda) = \frac{\tilde{F}(t,s;\rho)}{i\rho\tilde{\Delta}(\rho)}$$

for $t \neq s$ in $[0,1]$, where

$$\begin{aligned} |\tilde{F}(t,s;\rho)| \le e^{|b|}\{2|A_{12}|\,|\rho|^2 + |A_{14}+A_{23}|\,|\rho| \\ + |A_{14}-A_{23}|\,|\rho| + 2|A_{13}|\,|\rho| + 2|A_{24}|\,|\rho| + 2|A_{34}|\} \end{aligned}$$

for $t \neq s$ in $[0,1]$. Therefore, we conclude that

$$\begin{aligned} ||R_\lambda(T)|| \le \frac{e^{|b|}}{|\rho|\,|\tilde{\Delta}(\rho)|}\{2|A_{12}|\,|\rho|^2 + |A_{14}+A_{23}|\,|\rho| \\ + |A_{14}-A_{23}|\,|\rho| + 2|A_{13}|\,|\rho| + 2|A_{24}|\,|\rho| + 2|A_{34}|\} \end{aligned} \tag{2.9}$$

for all $\lambda = \rho^2 \in \rho(T)$ with $\rho = a+ib \neq 0$. In Case 3B it will be necessary to supplement (2.9) with estimates for the norm of $R_\lambda(T)SR_\lambda(T)$ (see Section 5).

Our principal perturbation theorem for the projections is given in [**13**, Theorem 3.1].

Theorem 2.2. *Let $\{P_k\}_{k=1}^\infty$ and $\{Q_k\}_{k=1}^\infty$ be sequences of projections on a Hilbert space H. Assume that*

(i) $P_kP_j = \delta_{kj}P_k$ *for* $k, j = 1, 2, \dots$.

(ii) *The family of all finite sums of the* P_k *is uniformly bounded in norm by a constant* $M > 0$.

(iii) $\sum_{k=1}^{\infty} ||Q_k - P_k||^2 < \infty$.

Then the family of all finite sums of the Q_k *is uniformly bounded in norm by the constant*

$$N = M + 4M^2 \bigg[\sum_{k=1}^{\infty} ||Q_k - P_k||^2\bigg]^{1/2} + \sum_{k=1}^{\infty} ||Q_k - P_k||^2.$$

The equivalence of the projections Q'_k, Q''_k, Q_k being uniformly bounded and the subspace $\mathcal{S}_\infty(L)$ being closed is established in the next two theorems. These theorems are variations of Lemma 3.4 and Theorem 3.6 in [**6**]. The second proof is similar to the first, but simpler, and is omitted.

Theorem 2.3. *Let the differential operator* L *belong to Case* 1*, Case* 2A*, Case* 3A*, Case* 3B *or Case* 4*. Then there exists a constant* $N > 0$ *such that* $||\sum_{k=k_0}^{m} Q'_k|| \le N$ *and* $||\sum_{k=k_0}^{m} Q''_k|| \le N$ *for* $m = k_0, k_0 + 1, \dots$ *if and only if*

$$\mathcal{S}_\infty(L) = \overline{\mathcal{S}_\infty(L)} = L^2[0, 1].$$

Proof. First, assume that there exists an $N > 0$ such that $||\sum_{k=k_0}^{m} Q'_k|| \le N$ and $||\sum_{k=k_0}^{m} Q''_k|| \le N$ for all $m \ge k_0$. Take any function $u \in L^2[0, 1] = \overline{\mathcal{S}_\infty(L)}$. We assert that the series $\sum_{k=k_0}^{\infty} Q'_k u$ is convergent. Indeed, for any $\varepsilon > 0$ we can choose $z \in \mathcal{S}_\infty(L)$ such that $||u - z|| \le \varepsilon/(3N)$, and then for this z we select an integer $n_0 \ge k_0$ such that $||\sum_{k=p}^{q} Q'_k z|| \le \varepsilon/3$ for all $q \ge p \ge n_0$. It follows that

$$\begin{aligned}\bigg\|\sum_{k=p}^{q} Q'_k u\bigg\| &\le \bigg\|\bigg(\sum_{k=k_0}^{q} Q'_k - \sum_{k=k_0}^{p-1} Q'_k\bigg)(u - z)\bigg\| + \bigg\|\sum_{k=p}^{q} Q'_k z\bigg\| \\ &\le 2N \cdot \frac{\varepsilon}{3N} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for all $q \geq p \geq n_0$. This establishes the assertion, and the same argument shows that the series $\sum_{k=k_0}^{\infty} Q_k'' u$ is convergent.

Now for any $z \in \mathcal{S}_\infty(L)$, we have

$$\begin{aligned}\left\| u - \sum_{k=1}^{n} Q_{0k} u - \sum_{k=k_0}^{\infty} Q_k' u - \sum_{k=k_0}^{\infty} Q_k'' u \right\| & \\ \leq ||u - z|| + \left\| \sum_{k=1}^{n} Q_{0k}(u - z) \right\| & \\ + \left\| \sum_{k=k_0}^{\infty} Q_k'(u - z) \right\| + \left\| \sum_{k=k_0}^{\infty} Q_k''(u - z) \right\| & \\ \leq \left[1 + \left\| \sum_{k=1}^{n} Q_{0k} \right\| + 2N \right] ||u - z||. &\end{aligned}$$

Since the quantity $||u - z||$ can be made arbitrarily small, we conclude that

$$u = \sum_{k=1}^{n} Q_{0k} u + \sum_{k=k_0}^{\infty} Q_k' u + \sum_{k=k_0}^{\infty} Q_k'' u \in \mathcal{S}_\infty(L).$$

Second, assume $\mathcal{S}_\infty(L) = L^2[0,1]$. Then for each $u \in L^2[0,1]$ the series $\sum_{k=k_0}^{\infty} Q_k' u$ and $\sum_{k=k_0}^{\infty} Q_k'' u$ are convergent, and

$$\left\| \sum_{k=k_0}^{\infty} Q_k' u \right\| = \lim_{m\to\infty} \left\| \sum_{k=k_0}^{m} Q_k' u \right\|,$$
$$\left\| \sum_{k=k_0}^{\infty} Q_k'' u \right\| = \lim_{m\to\infty} \left\| \sum_{k=k_0}^{m} Q_k'' u \right\|,$$

and hence,

$$\sup_{m \geq k_0} \left\| \sum_{k=k_0}^{m} Q_k' u \right\| < \infty, \qquad \sup_{m \geq k_0} \left\| \sum_{k=k_0}^{m} Q_k'' u \right\| < \infty.$$

By the Principle of Uniform Boundedness,

$$\sup_{m \geq k_0} \left\| \sum_{k=k_0}^{m} Q_k' \right\| < \infty, \qquad \sup_{m \geq k_0} \left\| \sum_{k=k_0}^{m} Q_k'' \right\| < \infty. \qquad \square$$

Theorem 2.4. *Let the differential operator L belong to Case* 2B. *Then there exists a constant $N > 0$ such that $\|\sum_{k=k_0}^m Q_k\| \le N$ for $m = k_0, k_0+1, \dots$ if and only if*

$$\mathcal{S}_\infty(L) = \overline{\mathcal{S}_\infty(L)} = L^2[0,1].$$

3. The projections and $\mathcal{S}_\infty(L)$ for Cases 1, 2A and 3A. Suppose the differential operators L and T belong to Case 1, Case 2A or Case 3A, so the A_{ij} satisfy:

Case 1. $A_{12} \neq 0$.

Case 2A. $A_{12} = 0$, $A_{14} + A_{23} \neq 0$, $A_{14} + A_{23} \neq \mp(A_{13} + A_{24})$.

Case 3A. $A_{12} = 0$, $A_{14}+A_{23} = 0$, $A_{34} \neq 0$, $A_{13}+A_{24} = 0$, $A_{13} = A_{24}$.

The conditions in Case 3A are equivalent to $A_{12} = A_{13} = A_{14} = A_{23} = A_{24} = 0$, $A_{34} \neq 0$, which correspond to Dirichlet boundary conditions.

Proceeding as in Part III (Section 2), let $\xi_0 = 1$ and $\eta_0 = -1$ for Case 1 and Case 3A, and let ξ_0 and $\eta_0 = 1/\xi_0$ be the roots of the quadratic polynomial

$$Q(z) = i(A_{14} + A_{23})z^2 + 2i(A_{13} + A_{24})z + i(A_{14} + A_{23})$$

for Case 2A, where $\xi_0 \neq \eta_0$ and $|\xi_0| \le 1$. Fix a real number d with $0 \le -\ln|\xi_0| < d$, and form the horizontal strip

$$\Omega = \{\rho = a + ib \in \mathbf{C} \mid |b| \le d\}.$$

Then the circles Γ_k', Γ_k'' are given by

$$\Gamma_k' = \{\rho \in \mathbf{C} \mid |\rho - \mu_k'| = \delta\}, \qquad \Gamma_k'' = \{\rho \in \mathbf{C} \mid |\rho - \mu_k''| = \delta\}$$

for $k = k_0, k_0+1, \dots$, where the centers are

$$\mu_k' = 2k\pi, \qquad \mu_k'' = (2k+1)\pi$$

for Case 1 and Case 3A, and

$$\mu_k' = (2k\pi + \operatorname{Arg}\xi_0) - i\ln|\xi_0|,$$
$$\mu_k'' = (2k\pi + \operatorname{Arg}\eta_0) + i\ln|\xi_0|$$

for Case 2A, the radii are equal to a constant δ with $0 < \delta \le \pi/4$ (see Part III for the additional geometric conditions satisfied by δ in Case 2A), and the positive integer k_0 has been chosen sufficiently large. The Γ_k', Γ_k'' lie in the interior of Ω, and for an appropriate constant m_0 equations (2.30) and (2.13) in Part III give

$$|\tilde{\Delta}(\rho)| \ge \frac{m_0}{2}e^{-d}|\rho|^p, \qquad |\Delta(\rho)| \ge \frac{m_0}{2}e^{-d}|\rho|^p \tag{3.1}$$

for all ρ on the circles Γ_k', Γ_k'' for $k \ge k_0$, where the integer p is equal to 2, 1 and 0 for Cases 1, 2A and 3A, respectively. It follows that if ρ lies on one of the circles Γ_k', Γ_k'', $k \ge k_0$, then $\lambda = \rho^2 \in \rho(T) \cap \rho(L)$.

Next, the estimate (2.9) immediately yields

$$||R_{\rho^2}(T)|| \le \frac{\gamma_1}{|\rho|} \tag{3.2}$$

for all ρ on Γ_k', Γ_k'' for $k \ge k_0$. Choose an integer $n_0 \ge k_0$ such that

$$\frac{\gamma_1}{|a|} \le \frac{1}{2}||S||^{-1} \tag{3.3}$$

for all $a \in \mathbf{R}$ with $a \ge z_0 := 2n_0\pi - \pi - \delta$. Then for $k \ge n_0$ and for any point $\rho = a + ib$ on Γ_k', Γ_k'', we have

$$|\rho| \ge |a| = a \ge 2k\pi - \pi - \delta \ge z_0,$$

and hence, $||R_{\rho^2}(T)|| \le (1/2)||S||^{-1}$. Thus, from (2.2), (2.6) and (3.2), the projections Q_k', P_k' satisfy

$$\begin{aligned} ||Q_k' - P_k'|| &\le \frac{1}{2\pi} \cdot \frac{4\gamma_1^2||S||}{2k\pi - \pi - \delta} \cdot 2\pi\delta \\ &= \frac{4\delta\gamma_1^2||S||}{2k\pi - \pi - \delta} \end{aligned}$$

for all $k \ge n_0$, with a similar estimate for the Q_k'', P_k''. We conclude that

$$||Q_k' - P_k'|| \le \frac{\gamma}{k}, \qquad ||Q_k'' - P_k''|| \le \frac{\gamma}{k} \tag{3.4}$$

for $k = k_0, k_0 + 1, \ldots$.

Finally, utilizing Theorems 1.1 and 2.1 of [**11**] in Case 1, Theorems 3.1 and 4.1 of [**11**] in Case 2A, and Theorem 10.1 of [**11**] in Case 3A, there exists a constant $M > 0$ such that

$$\left\| \sum_{k\in K} P_k' \right\| \le M, \qquad \left\| \sum_{k\in K} P_k'' \right\| \le M \tag{3.5}$$

for all finite subsets K of $\{k_0, k_0+1, \dots\}$. Applying Theorem 2.2 together with (3.4), there exists a constant $N > 0$ such that

$$\left\| \sum_{k\in K} Q_k' \right\| \le N, \qquad \left\| \sum_{k\in K} Q_k'' \right\| \le N \tag{3.6}$$

for all finite subsets K of $\{k_0, k_0+1, \dots\}$. From (3.6) it is immediate that the family of all finite sums of the projections in $\mathcal{Q}$ is uniformly bounded in norm, and by Theorem 2.3

$$\mathcal{S}_\infty(L) = \overline{\mathcal{S}_\infty(L)} = L^2[0,1]. \tag{3.7}$$

The above results are summarized in the following theorem which, together with Theorems 2.2, 2.3 and 2.4 of Part III, comprise our spectral theory for L belonging to Case 1, Case 2A or Case 3A.

Theorem 3.1. *Let the differential operator L belong to Case* 1*, Case* 2A *or Case* 3A*, let $\mathcal{Q}$ be the family of projections associated with L, and let $\mathcal{S}_\infty(L)$ and $\mathcal{M}_\infty(L)$ be the corresponding subspaces defined in terms of $\mathcal{Q}$. Then the family of all finite sums of the projections in $\mathcal{Q}$ is uniformly bounded in norm, and*

$$\mathcal{S}_\infty(L) = \overline{\mathcal{S}_\infty(L)} = L^2[0,1] \quad \textit{and} \quad \mathcal{M}_\infty(L) = \{0\}.$$

4. The projections and $S_\infty(L)$ for Case 2B. Assume that the differential operators L and T belong to Case 2B, where the boundary parameters satisfy

$$A_{12} = 0, \qquad A_{14} + A_{23} \ne 0, \qquad A_{14} + A_{23} = \mp(A_{13} + A_{24}).$$

Then the quadratic polynomial

$$Q(z) = i(A_{14} + A_{23})(z \mp 1)^2$$

has the double root $\xi_0 = \eta_0 = \pm 1$, and there is a possibility of multiple eigenvalues. Following Part III (Section 2), we take any constant $d > 0$ and form the horizontal strip

$$\Omega = \{\rho = a + ib \in \mathbf{C} \mid |b| \leq d\}.$$

For this case the circles

$$\Gamma_k = \{\rho \in \mathbf{C} \mid |\rho - \mu_k| = \delta\}, \qquad k = k_0, k_0 + 1, \dots ,$$

have centers

$$\mu_k = 2k\pi + \operatorname{Arg} \xi_0$$

and constant radii δ satisfying $0 < \delta \leq \pi/4$. The Γ_k are situated in the interior of Ω, and by equations (2.30) and (2.13) in Part III,

$$(4.1) \qquad |\tilde{\Delta}(\rho)| \geq \frac{m_0}{2} e^{-d} |\rho|, \qquad |\Delta(\rho)| \geq \frac{m_0}{2} e^{-d} |\rho|$$

for all ρ on Γ_k for $k \geq k_0$.

The analysis of the projections closely follows the analysis of the previous section. Indeed, for the resolvent (2.9) yields the decay rate

$$(4.2) \qquad ||R_{\rho^2}(T)|| \leq \frac{\gamma_1}{|\rho|}$$

for all ρ on Γ_k for $k \geq k_0$, and combining this with (2.4) and (2.6), we obtain the key estimate

$$(4.3) \qquad ||Q_k - P_k|| \leq \frac{\gamma}{k}, \qquad k = k_0, k_0 + 1, \dots .$$

But by earlier work (see Theorems 5.1, 6.1 and 7.1 in [**11**] and Theorem 4.1 in [**13**]) there exists a constant $M > 0$ such that

$$(4.4) \qquad \left\| \sum_{k \in K} P_k \right\| \leq M$$

for all finite subsets K of $\{k_0, k_0 + 1, \dots\}$, and applying Theorem 2.2 once more, there exists a constant $N > 0$ such that

$$(4.5) \qquad \left\| \sum_{k \in K} Q_k \right\| \leq N$$

for all finite subsets K of $\{k_0, k_0+1, \dots\}$. We conclude that the family of all finite sums of the projections in $\mathcal{Q}$ is uniformly bounded in norm, and by Theorem 2.4,

$$\mathcal{S}_\infty(L) = \overline{\mathcal{S}_\infty(L)} = L^2[0,1]. \tag{4.6}$$

The main results of this section are collected in the following theorem. Combined with Theorem 2.6 of Part III, they make up the spectral theory for the multiple eigenvalue case.

Theorem 4.1. *Let the differential operator L belong to Case* 2B*, let $\mathcal{Q}$ be the family of projections associated with L, and let $\mathcal{S}_\infty(L)$ and $\mathcal{M}_\infty(L)$ be the corresponding subspaces defined in terms of $\mathcal{Q}$. Then the family of all finite sums of the projections in $\mathcal{Q}$ is uniformly bounded in norm, and*

$$\mathcal{S}_\infty(L) = \overline{\mathcal{S}_\infty(L)} = L^2[0,1] \quad \textit{and} \quad \mathcal{M}_\infty(L) = \{0\}.$$

5. The projections and $\mathcal{S}_\infty(L)$ for Case 3B. Unlike the other cases, for Case 3B the norm of $R_{\rho^2}(T)$ does not go to 0 as $k \to \infty$ for ρ on the circles Γ'_k, Γ''_k, and our earlier methods must be modified. Assume the differential operators L and T belong to Case 3B:

$$\begin{gathered} A_{12} = 0, \qquad A_{14} + A_{23} = 0, \qquad A_{34} \neq 0, \\ A_{13} + A_{24} = 0, \qquad A_{13} \neq A_{24}. \end{gathered} \tag{5.1}$$

It is well known (see Theorem 2.1 in [**10**]) that

$$A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0,$$

which together with (5.1) yields $A_{13}^2 - A_{14}^2 = 0$ or $A_{14} = \mp A_{13}$. Thus, for this case

$$\begin{cases} A_{12} = 0, \qquad A_{13} \neq 0, \qquad A_{34} \neq 0, \\ A_{14} = \mp A_{13}, \qquad A_{23} = \pm A_{13}, \qquad A_{24} = -A_{13}, \end{cases} \tag{5.2}$$

and everything can be expressed in terms of the two parameters A_{13} and A_{34}.

As in Part III (Section 2) choose a real number d with $d \geq 1$ and

$$\frac{\beta_0}{d} \leq \frac{1}{2}||S||^{-1},$$

where $\beta_0 = (2/|A_{34}|)\{12|A_{13}| + 6|A_{34}|\}$, and form the horizontal strip

$$\Omega = \{\rho = a + ib \in \mathbf{C} \mid |b| \leq d\}.$$

For Case 3B the circles Γ_k', Γ_k'', the centers μ_k', μ_k'' and the constant radii δ are the same as in Case 1 and Case 3A. The Γ_k', Γ_k'' lie in the interior of Ω, and by equations (2.30) and (2.13) of Part III,

$$|\tilde{\Delta}(\rho)| \geq \frac{m_0}{2}e^{-d}, \qquad |\Delta(\rho)| \geq \frac{m_0}{2}e^{-d} \tag{5.3}$$

for all ρ on the circles Γ_k', Γ_k'' for $k \geq k_0$. Combining (5.3) with (2.9), we obtain the estimate

$$||R_{\rho^2}(T)|| \leq \frac{e^d|\rho|}{(m_0/2)e^{-d}|\rho|}\{6|A_{13}| + 2|A_{34}|\} := \gamma_1 \tag{5.4}$$

for all ρ on Γ_k', Γ_k'' for $k \geq k_0$.

Next, we show that the decay rate (5.4) is the best possible for Case 3B. From Part I (see (1.2) and (3.1)–(3.3)) the characteristic determinant of T is given by

$$\tilde{\Delta}(\rho) = -A_{34}e^{i\rho} + A_{34}e^{-i\rho}$$

for $\rho \in \mathbf{C}$, and for $\lambda = \rho^2 \neq 0$ in $\rho(T)$ the Green's function for T_λ can be written as

$$\tilde{G}(t,s;\lambda) = \frac{\tilde{H}(t,s;\rho)}{\tilde{\Delta}(\rho)} + \frac{\tilde{J}(t,s;\rho)}{i\rho\tilde{\Delta}(\rho)}$$

for $t \neq s$ in $[0,1]$, where

$$\begin{aligned}\tilde{H}(t,s;\rho) = {} & \pm A_{13}e^{i\rho(1-t-s)} \mp A_{13}e^{-i\rho(1-t-s)} \\ & + A_{13}e^{i\rho(t-s)} - A_{13}e^{-i\rho(t-s)}\end{aligned}$$

for $t \neq s$ in $[0,1]$,

$$\begin{aligned}\tilde{J}(t,s;\rho) = {} & \frac{1}{2}A_{34}e^{i\rho(1+t-s)} + \frac{1}{2}A_{34}e^{-i\rho(1+t-s)} \\ & - \frac{1}{2}A_{34}e^{i\rho(1-t-s)} - \frac{1}{2}A_{34}e^{-i\rho(1-t-s)}\end{aligned}$$

for $0 \le t < s \le 1$, and

$$\tilde{J}(t,s;\rho) = \frac{1}{2}A_{34}e^{i\rho(1-t+s)} + \frac{1}{2}A_{34}e^{-i\rho(1-t+s)} - \frac{1}{2}A_{34}e^{i\rho(1-t-s)} - \frac{1}{2}A_{34}e^{-i\rho(1-t-s)}$$

for $0 \le s < t \le 1$. Clearly $|\tilde{\Delta}(\rho)| \le 2|A_{34}|e^d$ for $\rho \in \Omega$, and $|\tilde{H}(t,s;\rho)| \le 4|A_{13}|e^d$, $|\tilde{J}(t,s;\rho)| \le 2|A_{34}|e^d$ for $t \ne s$ in $[0,1]$ and for $\rho \in \Omega$.

Take any point $\rho = a+ib$ on one of the circles Γ'_k, Γ''_k for $k \ge k_0$, and consider the function

$$u_\rho(t) = e^{i\rho t}, \qquad 0 \le t \le 1.$$

Clearly $|u_\rho(t)| = e^{-bt} \le e^d$, $||u_\rho|| \le e^d$, and

$$||R_{\rho^2}(T)|| \ge \frac{1}{||u_\rho||}||R_{\rho^2}(T)u_\rho|| \ge e^{-d}||R_{\rho^2}(T)u_\rho||, \tag{5.5}$$

where

$$\begin{aligned} R_{\rho^2}(T)u_\rho(t) &= \frac{1}{\tilde{\Delta}(\rho)}\int_0^1 \tilde{H}(t,s;\rho)e^{i\rho s}\,ds \\ &\quad + \frac{1}{i\rho\tilde{\Delta}(\rho)}\int_0^1 \tilde{J}(t,s;\rho)e^{i\rho s}\,ds \\ &= \frac{A_{13}}{\tilde{\Delta}(\rho)}v_\rho(t) + w_\rho(t) \end{aligned} \tag{5.6}$$

with $v_\rho(t) = \pm e^{i\rho(1-t)} + e^{i\rho t}$ and $||w_\rho|| \le \gamma_0/|\rho| \le \gamma_0/|a|$. Now

$$\begin{aligned} |v_\rho(t)|^2 &= e^{-2b(1-t)} \pm 2e^{-b}\cos a(1-2t) + e^{-2bt} \\ &\ge 2e^{-2d} \pm 2e^{-b}\cos a(1-2t), \\ ||v_\rho||^2 &\ge 2e^{-2d} \pm 2e^{-b}\frac{\sin a}{a} \ge 2e^{-2d} - \frac{2e^d}{|a|}, \end{aligned}$$

and hence, by (5.5) and (5.6),

$$\begin{aligned} ||R_{\rho^2}(T)|| &\ge e^{-d}\left\{\frac{|A_{13}|}{|\tilde{\Delta}(\rho)|}||v_\rho|| - ||w_\rho||\right\} \\ &\ge e^{-d}\left\{\frac{|A_{13}|}{2|A_{34}|e^d}\left[2e^{-2d} - \frac{2e^d}{|a|}\right]^{1/2} - \frac{\gamma_0}{|a|}\right\}. \end{aligned}$$

It follows that there exists an integer $k_1 \geq k_0$ such that

$$||R_{\rho^2}(T)|| \geq e^{-d} \cdot \frac{|A_{13}|}{2|A_{34}|e^d} \cdot e^{-d} := \gamma_2 \tag{5.7}$$

for all points ρ on Γ'_k, Γ''_k for $k \geq k_1$.

In view of (5.7), it is no longer possible to force the condition $||R_{\rho^2}(T)|| \leq (1/2)||S||^{-1}$ on the circles Γ'_k, Γ''_k and use (2.6) in Theorem 2.1 to estimate the projections. However, we can still obtain the alternate condition $||R_{\rho^2}(T)SR_{\rho^2}(T)|| \leq (1/2)||S||^{-1}$ for ρ on the Γ'_k, Γ''_k with k sufficiently large, thereby permitting us to use (2.7) in Theorem 2.1 to estimate the projections. Let us proceed to develop these ideas.

Take any integer $k \geq k_0$ and any point ρ on Γ'_k, Γ''_k. Then the operator $R_{\rho^2}(T)SR_{\rho^2}(T)$ is an integral operator on $L^2[0,1]$ with L^2-kernel $\tilde{K}(t,s;\rho^2)$ given by

$$\tilde{K}(t,s;\rho^2) = \int_0^1 \tilde{G}(t,\xi;\rho^2)q(\xi)\tilde{G}(\xi,s;\rho^2)\,d\xi$$

for $t, s \in [0,1]$, which upon simplification becomes

$$\begin{aligned}
\tilde{K}(t,s;\rho^2) = {} & \frac{[A_{13}]^2}{[\tilde{\Delta}(\rho)]^2} \int_0^1 e^{2i\rho\xi} q(\xi)\,d\xi \\
& \{e^{-i\rho(2-t-s)} \mp e^{-i\rho(1-t+s)} \pm e^{-i\rho(1+t-s)} - e^{-i\rho(t+s)}\} \\
& + \frac{[A_{13}]^2}{[\tilde{\Delta}(\rho)]^2} \int_0^1 e^{-2i\rho\xi} q(\xi)\,d\xi \\
& \{e^{i\rho(2-t-s)} \mp e^{i\rho(1-t+s)} \pm e^{i\rho(1+t-s)} - e^{i\rho(t+s)}\} \\
& + \tilde{\theta}(t,s;\rho^2)
\end{aligned} \tag{5.8}$$

for $t, s \in [0,1]$, where $\tilde{\theta}(\cdot,\cdot;\rho^2)$ is bounded and measurable on $[0,1] \times [0,1]$ with $||\tilde{\theta}(\cdot,\cdot;\rho^2)||_\infty \leq \gamma_3/|\rho|$. Earlier we showed that (see (2.4) in Part III)

$$\left| \int_0^1 e^{2i\rho\xi} q(\xi)\,d\xi \right| \leq e^{2d}||q - \tilde{q}||_\infty + \frac{e^{2d}}{|\rho|}[||\tilde{q}||_\infty + ||\tilde{q}'||_\infty], \tag{5.9}$$

where $\tilde{q}$ is an arbitrary function in $C^1[0,1]$, and replacing ρ by $-\rho$, we see that this estimate is also valid for the integral $\int_0^1 e^{-2i\rho\xi} q(\xi)\,d\xi$. Therefore, combining (5.3) and (5.9) with (5.8), we conclude that

$$|\tilde{K}(t,s;\rho^2)| \le \gamma_4 \left\{ ||q-\tilde{q}||_\infty + \frac{1}{|\rho|}[||\tilde{q}||_\infty + ||\tilde{q}'||_\infty + 1] \right\}$$

for $t, s \in [0,1]$, and

$$||R_{\rho^2}(T)SR_{\rho^2}(T)|| \le \gamma_4 \left\{ ||q-\tilde{q}||_\infty + \frac{1}{|\rho|}[||\tilde{q}||_\infty + ||\tilde{q}'||_\infty + 1] \right\} \tag{5.10}$$

for all ρ on Γ_k', Γ_k'' for $k \ge k_0$.

Finally, we turn to the projections Q_k', Q_k'', $k \ge k_0$, and P_k', P_k'', $k \ge k_0$, associated with L and T, respectively. In (11.12) of [**11**] we showed that

$$||P_k'|| \ge \frac{2|A_{13}|}{|A_{34}|}(2k\pi), \qquad ||P_k''|| \ge \frac{2|A_{13}|}{|A_{34}|}(2k+1)\pi \tag{5.11}$$

for all $k \ge k_0$. Choose a function $\hat{q} \in C^1[0,1]$ satisfying

$$\begin{aligned} \gamma_4 ||q-\hat{q}||_\infty &\le \frac{1}{4}||S||^{-1}, \\ \gamma_4 ||q-\hat{q}||_\infty &\le \frac{|A_{13}|}{16\delta(1+\gamma_1||S||)|A_{34}|}, \end{aligned} \tag{5.12}$$

and for this fixed $\hat{q}$ choose an integer $n_0 \ge k_0$ such that

$$\begin{aligned} \frac{\gamma_4}{2k\pi-\delta}[||\hat{q}||_\infty + ||\hat{q}'||_\infty + 1] &\le \frac{1}{4}||S||^{-1}, \\ \frac{\gamma_4}{2k\pi-\delta}[||\hat{q}||_\infty + ||\hat{q}'||_\infty + 1] &\le \frac{|A_{13}|}{16\delta(1+\gamma_1||S||)|A_{34}|} \end{aligned} \tag{5.13}$$

for all $k \ge n_0$. Then for $k \ge n_0$ and for ρ on Γ_k', from (5.10), (5.12) and (5.13), we obtain

$$||R_{\rho^2}(T)SR_{\rho^2}(T)|| \le \frac{1}{2}||S||^{-1}, \tag{5.14}$$

and by (2.7), (5.4), (5.10), (5.12) and (5.13), we get

$$||R_{\rho^2}(L) - R_{\rho^2}(T)|| \leq \frac{2(1+\gamma_1||S||)|A_{13}|}{8\delta(1+\gamma_1||S||)|A_{34}|} = \frac{|A_{13}|}{4\delta|A_{34}|}, \tag{5.15}$$

and hence, by (2.2),

$$||Q'_k - P'_k|| \leq \frac{1}{2\pi} \cdot 2(2k\pi + \delta) \cdot \frac{|A_{13}|}{4\delta|A_{34}|} \cdot 2\pi\delta \leq \frac{|A_{13}|}{|A_{34}|}(2k\pi) \tag{5.16}$$

for all $k \geq n_0$. It follows from (5.11) and (5.16) that

$$||Q'_k|| \geq ||P'_k|| - ||Q'_k - P'_k|| \geq \frac{|A_{13}|}{|A_{34}|}(2k\pi) \tag{5.17}$$

for $k = n_0, n_0 + 1, \ldots$, and similarly,

$$||Q''_k|| \geq \frac{|A_{13}|}{|A_{34}|}(2k+1)\pi \tag{5.18}$$

for $k = n_0, n_0 + 1, \ldots$. Thus, the norms of the Q'_k, Q''_k are growing at the same rate as the norms of the P'_k, P''_k, and as an application of Theorem 2.3, we have

$$\mathcal{S}_\infty(L) \neq \overline{\mathcal{S}_\infty(L)} = L^2[0,1]. \tag{5.19}$$

We summarize the above results in the following theorem. Together with Theorem 2.4 of Part III, they make up our spectral theory for Case 3B.

Theorem 5.1. *Let the differential operator L belong to Case* 3B*, let $\mathcal{Q}$ be the family of projections associated with L, and let $\mathcal{S}_\infty(L)$ and $\mathcal{M}_\infty(L)$ be the corresponding subspaces defined in terms of $\mathcal{Q}$. Then the projections in $\mathcal{Q}$ are not uniformly bounded in norm,*

$$||Q'_k|| \longrightarrow \infty \quad \textit{and} \quad ||Q''_k|| \longrightarrow \infty \quad \textit{as} \quad k \to \infty,$$

and

$$\mathcal{S}_\infty(L) \neq \overline{\mathcal{S}_\infty(L)} = L^2[0,1] \quad \textit{and} \quad \mathcal{M}_\infty(L) = \{0\}.$$

6. The projections and $\mathcal{S}_\infty(L)$ for Case 4. Suppose the differential operators L and T belong to Case 4, the logarithmic case, where

$$A_{12} = 0, \qquad A_{14} + A_{23} = 0, \qquad A_{34} \neq 0, \qquad A_{13} + A_{24} \neq 0.$$

In setting up this case we follow Part III (Section 3). Let $\mu = -2i(A_{13} + A_{24})/A_{34}$; choose constants α and β with $0 < \alpha \leq 1/2$, $\beta \geq 2$, and

$$\frac{6\beta_0}{|A_{34}|\beta} \leq \frac{1}{2}||S||^{-1},$$

where $\beta_0 = 2|A_{14}| + 2|A_{13}| + 2|A_{24}| + 2|A_{34}|$, and set $\xi = [1 + (|\mu|^2/\alpha^2)]^{1/2}$ and $\eta = [(\beta^2/|\mu|^2) + 1]^{1/2}$; and introduce the logarithmic strip

$$\Omega = \left\{ \rho = a + ib \in \mathbf{C} \;\middle|\; |a| \geq \frac{\alpha}{|\mu|} \text{ and } \ln \frac{|\mu|\,|a|}{\beta} \leq |b| \leq \ln \frac{|\mu|\,|a|}{\alpha} \right\}.$$

The circles

$$\Gamma_k' = \{\rho \in \mathbf{C} \mid |\rho - \mu_k'| = \delta\}, \qquad \Gamma_k'' = \{\rho \in \mathbf{C} \mid |\rho - \mu_k''| = \delta\},$$

$k = k_0, k_0 + 1, \ldots$, now have their centers at the points

$$\mu_k' = (2k\pi - \operatorname{Arg}\mu) + i \ln|\mu|(2k\pi - \operatorname{Arg}\mu),$$
$$\mu_k'' = -[(2k+1)\pi + \operatorname{Arg}\mu] + i \ln|\mu|[(2k+1)\pi + \operatorname{Arg}\mu],$$

and have constant radii δ satisfying $0 < \delta \leq \pi/4$ and $0 < \delta < (\ln 2)/(|\mu| + 1)$. From equations (3.31), (3.22) and (3.27) in Part III, the characteristic determinants satisfy

$$(6.1) \qquad |\tilde{\Delta}(\rho)| \geq \frac{\alpha m_0}{4\xi}|A_{34}|e^{|b|}, \qquad |\Delta(\rho)| \geq \frac{\alpha m_0}{4\xi}|A_{34}|e^{|b|}$$

for all points $\rho = a + ib$ on Γ_k', Γ_k'' for $k \geq k_0$.

Take any point $\rho = a + ib$ on one of the circles Γ_k', Γ_k'' for $k \geq k_0$. It follows from (6.1) that $\lambda = \rho^2$ belongs to $\rho(T) \cap \rho(L)$,

$$|\rho| \geq 2k\pi - \pi - \delta \geq k$$

and

$$|\rho| \leq (2k\pi + 2\pi) + \ln|\mu|(2k\pi + 2\pi) + \delta \leq (1 + |\mu|)(4k\pi)$$

because $k \geq k_0 \geq 2$ and $|\mu|(2k\pi + 2\pi) > \beta \geq 2$ by (3.11) in Part III, and

$$\frac{1}{2}\ln|\rho| \leq |b|$$

by (3.5) in Part III. Thus, by (2.8) and (6.1)

$$||R_{\rho^2}(T)|| \leq \frac{\gamma_1}{|b|} \leq \frac{2\gamma_1}{\ln|\rho|} \leq \frac{2\gamma_1}{\ln k}. \tag{6.2}$$

Next, for the projections P_k', P_k'', it has been shown that there exists a constant $\gamma_0 > 0$ such that

$$||P_k'|| \geq \gamma_0 \frac{k}{\ln k}, \qquad ||P_k''|| \geq \gamma_0 \frac{k}{\ln k} \tag{6.3}$$

for all $k \geq k_0$ (see (9.25) in [**11**]). Select an integer $n_0 \geq k_0$ such that

$$\frac{2\gamma_1}{\ln k} \leq \frac{1}{2}||S||^{-1}, \qquad \frac{16\delta(1+|\mu|)\pi||S||(2\gamma_1)^2}{\ln k} \leq \frac{1}{2}\gamma_0 \tag{6.4}$$

for all $k \geq n_0$. Then from (6.2) and (6.4), $||R_{\rho^2}(T)|| \leq (1/2)||S||^{-1}$ for ρ on Γ_k', Γ_k'' for $k \geq n_0$, and for the projections Q_k', P_k' equations (2.2), (2.6), (6.2) and (6.4) yield

$$\begin{aligned} ||Q_k' - P_k'|| &\leq \frac{1}{2\pi} \cdot 2(1+|\mu|)(4k\pi) \cdot 2||S|| \cdot \left[\frac{2\gamma_1}{\ln k}\right]^2 \cdot 2\pi\delta \\ &\leq \frac{1}{2}\gamma_0 \frac{k}{\ln k} \end{aligned} \tag{6.5}$$

for all $k \geq n_0$, and hence, by (6.3) and (6.5),

$$||Q_k'|| \geq ||P_k'|| - ||Q_k' - P_k'|| \geq \frac{1}{2}\gamma_0 \frac{k}{\ln k} \tag{6.6}$$

for $k = n_0, n_0 + 1, \ldots$. Similarly,

$$||Q_k''|| \geq \frac{1}{2}\gamma_0 \frac{k}{\ln k} \tag{6.7}$$

for $k = n_0, n_0+1, \dots$, and by Theorem 2.3,

$$\mathcal{S}_\infty(L) \neq \overline{\mathcal{S}_\infty(L)} = L^2[0,1]. \tag{6.8}$$

The spectral theory for Case 4 is contained in Theorem 3.2 of Part III and in the following theorem, which summarizes the results of this section.

Theorem 6.1. *Let the differential operator L belong to Case 4, let $\mathcal{Q}$ be the family of projections associated with L, and let $\mathcal{S}_\infty(L)$ and $\mathcal{M}_\infty(L)$ be the corresponding subspaces defined in terms of $\mathcal{Q}$. Then the projections in $\mathcal{Q}$ are not uniformly bounded in norm,*

$$||Q_k'|| \longrightarrow \infty \quad \textit{and} \quad ||Q_k''|| \longrightarrow \infty \quad \textit{as} \quad k \to \infty,$$

and

$$\mathcal{S}_\infty(L) \neq \overline{\mathcal{S}_\infty(L)} = L^2[0,1] \quad \textit{and} \quad \mathcal{M}_\infty(L) = \{0\}.$$

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