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KRONROD EXTENSION OF GENERALIZED GAUSS-RADAU AND GAUSS-LOBATTO FORMULAE

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ABSTRACT. Kronrod extensions of Gauss-Radau and Gauss-Lobatto formulae having end points of multiplicity 2 are studied. For the four Chebyshev measures, expansions of the respective Stieltjes polynomials in terms of appropriate Chebyshev polynomials are given whenever possible; otherwise, an efficient computational algorithm is given. Explicit formulas are derived for the weights associated with the end points.

1. Introduction. In 1964, Kronrod [**9, 10**] initiated the idea of extending Gaussian quadrature formulae. He proposed to add $n+1$ nodes to an n-point Gauss-Legendre formula and to choose the new nodes and all weights of the extended formula so that it has maximum degree of exactness. It turns out that the additional nodes are the zeros of a polynomial of degree $n+1$ (known as Stieltjes polynomial) that is orthogonal to all lower-degree polynomials with respect to the Legendre polynomial of degree n as the weight function. Work in this direction has intensified in the last ten years. The reader is referred to surveys by Gautschi [**3**] and Monegato [**12**] for a detailed discussion of recent developments in this area. Here we apply Kronrod's idea to generalized Gauss-Radau and Gauss-Lobatto formulae with double end points, recently discussed by C. Bernardi and Y. Maday [**1**], and Gautschi and Li [**6, 7**].

We assume that the weight function w associated with the integration is one of the four Chebyshev weights:

$$\begin{aligned}
w_1(t) &= (1-t^2)^{-1/2},\\
w_2(t) &= (1-t^2)^{1/2},\\
w_3(t) &= (1-t)^{-1/2}(1+t)^{1/2},\\
w_4(t) &= (1-t)^{1/2}(1+t)^{-1/2}.
\end{aligned}$$

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The Kronrod extension (see [**3, 9, 10**]) for the generalized Gauss-Radau quadrature rule has the form

$$\begin{aligned}\int_{-1}^{1} f(t)w(t)\,dt = \sigma_0 f(-1) + \sigma_0' f'(-1) + \sum_{\nu=1}^{n} \sigma_\nu f(\tau_\nu) \\ + \sum_{\mu=1}^{n+1} \sigma_\mu^\star f(\tau_\mu^\star) + R_n^{KR}(f),\end{aligned} \tag{1.1}$$

where $\tau_\nu = \tau_\nu^{(n)}$ are the zeros of $\pi_n(\,\cdot\,;(1+t)^2w(t))$, the nth-degree (monic) orthogonal polynomial relative to the weight $w^R(t) = (1+t)^2w(t)$, and $\tau_\mu^\star = \tau_{\mu,n}^\star, \sigma_0, \sigma_0', \sigma_\nu$ and $\sigma_\mu^\star$ are chosen so that formula (1.1) has maximum degree of exactness $\geq 3n+3$. It turns out that $\tau_\mu^\star$ must be the zeros of the (monic) polynomial $\pi_{n+1}^\star$ of degree $n+1$, known as Stieltjes polynomial (cf. [**12**]), orthogonal to all polynomials of lower degree with respect to the weight function

$$w^{KR}(t) = \pi_n(t; w^R(t))w^R(t).$$

Similarly, there exists an optimal extension of the Gauss-Lobatto formula with double end points associated with the weight function w. It has the form

$$\begin{aligned}\int_{-1}^{1} f(t)w(t)\,dt = \sigma_0 f(-1) + \sigma_0' f'(-1) + \sigma_{n+1} f(1) + \sigma_{n+1}' f'(1) \\ + \sum_{\nu=1}^{n} \sigma_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} \sigma_\mu^\star f(\tau_\mu^\star) + R_n^{KL}(f),\end{aligned} \tag{1.2}$$

where $\tau_\nu = \tau_\nu^{(n)}$ are the zeros of $\pi_n(\,\cdot\,;(1-t^2)^2w(t))$, and $\tau_\mu^\star = \tau_{\mu,n}^\star$ are the zeros of $\pi_{n+1}^\star$. In this case, (1.2) has degree of exactness $\geq 3n+5$ and $\pi_{n+1}^\star$ is orthogonal to all polynomials of degree $\leq n$ in the sense

$$\int_{-1}^{1} \pi_{n+1}^\star(t)p(t)\pi_n(t; w^L(t))w^L(t)\,dt = 0, \quad \text{all } p \in P_n,$$

where $w^L(t) = (1-t^2)^2w(t)$. It is well known that $\pi_{n+1}^\star$ uniquely exists for both (1.1) and (1.2). In practice, one wants all the zeros of $\pi_{n+1}^\star$ to be real and inside $[-1,1]$. These questions, and questions of interlacing of the nodes τ_ν, $\tau_\mu^\star$ and positivity of the weights, have been studied by Gautschi and Notaris [**5**]. Here, our main interest is in deriving explicit formulae for the polynomials $\pi_{n+1}^\star$, whenever possible, or else, obtaining constructive procedures for generating them. In Section 2 we study the expansions of the Stieltjes polynomials $\pi_{n+1}^\star$ in terms of appropriate Chebyshev polynomials. Inclusion and interlacing properties of the zeros of π_n and $\pi_{n+1}^\star$ are also summarized in Section 2. Finally, in Section 3 various representations of the weights in the extension formulae (1.1) and (1.2) are obtained.

2. Stieltjes polynomials. Let $w_i^R(t) = (1+t)^2 w_i(t)$ be the weight functions for the Gauss-Radau formulae with double end point at -1 and Chebyshev weight w_i. Similarly, we let $w_i^L(t) = (1-t^2)^2 w_i(t)$ denote the weight functions associated with Gauss-Lobatto formulae. The nodal polynomials $\pi_n^{R,i}(t)$ $(= \pi_n(t; w_i^R))$ and $\pi_n^{L,i}(t)$ $(= \pi_n(t; w_i^L))$ associated with the interior nodes in the generalized Gauss-Radau and Gauss-Lobatto formulae can be expressed in terms of the Chebyshev polynomials. These formulas are straightforward consequences of corresponding ones given in [**4**]. We will not present them here.

2.1. The Stieltjes polynomials. To avoid a proliferation of superscripts and subscripts needed to identify the polynomials $\pi_{n+1}^\star$, we suppress some of them whenever they are clear from the context. Explicit formulas for $\pi_{n+1}^\star$ are not available in all cases under investigation. The results for $w = w_i^R$, $i = 1, 2, 3, 4$, and $w = w_1^L$, w_2^L are given in Theorems 2.1–2.6.

Theorem 2.1. *The Stieltjes polynomial relative to the weight function w_1^R is given by*

(2.1)
$$\pi_{n+1}^\star(t) = \frac{1}{2^n}\biggl\{T_{n+1}(t) + 4\frac{n+1}{2n+3}\sum_{k=0}^{n-1}(-1)^{k+1}\left(\frac{2n+1}{2n+3}\right)^k T_{n-k}(t) + (-1)^{n+1}\frac{2(n+1)}{2n+3}\left(\frac{2n+1}{2n+3}\right)^n T_0\biggr\},$$

where T_k is the first-kind Chebyshev polynomial of degree k.

Alternatively,

$$\pi^{\star}_{n+1}(t) = \frac{1}{2^n}\bigg\{V_{n+1}(t) - \frac{2n+1}{2n+3}V_n(t) - \frac{8(n+1)}{(2n+3)^2}\sum_{k=1}^{n}\bigg(-\frac{2n+1}{2n+3}\bigg)^{k-1}V_{n-k}(t)\bigg\}, \tag{2.2}$$

where $V_n(t) = \cos((n+1/2)\theta)/\cos((1/2)\theta)$, $t = \cos\theta$ *and* V_n *is the third-kind Chebyshev polynomial of degree* n.

Proof. We first note that the Stieltjes polynomial $\pi^{\star}_{n+1}$ must satisfy the orthogonality relation

$$\int_{-1}^{1}\pi^{\star}_{n+1}(t)p(t)\pi_n^{R,1}(t)w_1^R(t)\,dt = 0, \quad \text{all } p \in \mathbf{P}_n. \tag{2.3}$$

Now we expand $\pi^{\star}_{n+1}$ in terms of T_r, i.e.,

$$2^n\pi^{\star}_{n+1}(t) = T_{n+1}(t) + \sum_{r=0}^{n}c_rT_{n-r}(t). \tag{2.4}$$

Next we let $p = T_k$, $k \le n$, in (2.3), and apply the formula

$$\pi_n^{R,1}(t) = \frac{1}{2^{n+1}(1+t)^2}\bigg\{T_{n+2} + 4\frac{n+1}{2n+1}T_{n+1} + \frac{2n+3}{2n+1}T_n\bigg\}$$

(this is a straightforward consequence of corresponding one given in [**4**]) and (2.4) to write (2.3) in the form

$$\int_{-1}^{1}\bigg(T_{n+1} + \sum_{r=0}^{n}c_rT_{n-r}\bigg)T_k\bigg(T_{n+2} + 4\frac{n+1}{2n+1}T_{n+1} + \frac{2n+3}{2n+1}T_n\bigg)w_1(t)\,dt = 0, \quad k = 0,1,2,\dots,n. \tag{2.5}$$

Using $T_kT_l = (T_{k+l} + T_{|k-l|})/2$ and the orthogonality of the T_r, we can reduce (2.5) to the following system of linear equations in the c_k, valid for $n \geq 2$,

$$\begin{aligned} 4(n+1) + (2n+3)c_0 &= 0,\\ 4(n+1) + 4(n+1)c_0 + (2n+3)c_1 &= 0,\\ (2n+1)c_{k-2} + 4(n+1)c_{k-1} + (2n+3)c_k &= 0,\\ k &= 2, 3, \dots, n-1,\\ (2n+1)c_{n-2} + 4(n+1)c_{n-1} + 2(2n+3)c_n &= 0. \end{aligned} \tag{2.6}$$

It can be checked directly, or derived from the theory of difference equations, that the solution of (2.6) is

$$c_k = (-1)^{k+1}\frac{4(n+1)}{2n+3}\left(\frac{2n+1}{2n+3}\right)^k, \quad k = 0, 1, 2, \dots, n-1,$$

and

$$c_n = (-1)^{n+1}\frac{2(n+1)}{2n+3}\left(\frac{2n+1}{2n+3}\right)^n.$$

This proves (2.1) for $n \geq 2$. For $n = 1$, (2.5) becomes the system

$$8 + 5c_0 = 0, \qquad 4 + 4c_0 + 5c_1 = 0,$$

which has the solution $c_0 = -8/5$ and $c_1 = 12/25$ in agreement with (2.1). For (2.2), one writes

$$2^n\pi^\star_{n+1}(t) = V_{n+1}(t) + \sum_{r=0}^{n} d_r V_{n-r}(t)$$

and uses the formula

$$\pi_n^{R,1}(t) = \frac{1}{2^{n+1}(1+t)}\left\{V_{n+1}(t) + \frac{2n+3}{2n+1}V_n(t)\right\}.$$

Then letting $p = V_k$ in (2.3) and applying the same procedure as before yields formula (2.2). □

In a similar, though more complicated, manner one can prove the following five theorems.

Theorem 2.2. *The Stieltjes polynomial associated with the weight function* w_2^R *has the following expansion*

$$\pi^\star_{n+1}(t) = \frac{1}{2^{n+1}}\bigg(U_{n+1}(t) + \sum_{k=0}^{n} c_{n-k}U_{n-k}(t)\bigg), \tag{2.7}$$

where

$$c_{n-k} = a_0 \cos k\theta + a_1 \sin k\theta,$$

the coefficients a_0 *and* a_1 *being given by*

$$\begin{aligned} a_0 &= -\frac{4(n+1)}{2n+5}, \\ a_1 &= \frac{2}{\sqrt{3}(2n+5)} \\ &\quad \times \bigg(\frac{\sqrt{(2n+3)(2n+5)}(4n^3+14n^2-2n-21)}{(n+3)(2n+5)} \\ &\qquad - 4(n+1)\sqrt{(n+1)(n+3)}\bigg) \end{aligned}$$

and the angle θ *satisfying the relation*

$$\cos\theta = -2\sqrt{\frac{(n+1)(n+3)}{(2n+3)(2n+5)}}, \qquad 0 < \theta < \pi.$$

Theorem 2.3. *For the weight function* w_3^R*, the Stieltjes polynomial admits the form*

$$\pi^\star_{n+1}(t) = \frac{1}{2^{n+1}}\bigg(V_{n+1}(t) + \sum_{k=0}^{n} c_{n-k}V_{n-k}(t)\bigg), \tag{2.8}$$

where

$$c_{n-k} = a_0 \cos k\theta + a_1 \sin k\theta,$$

the coefficients a_0 *and* a_1 *being given by*

$$\begin{aligned} a_0 &= -\frac{2n+1}{n+2}, \\ a_1 &= \frac{1}{\sqrt{3}(n+2)}\biggl(\frac{2(n+1)^{3/2}(4n^2+4n-17)}{(n+2)^{1/2}(2n+5)} \\ &\qquad - (2n+1)\sqrt{(2n+5)(2n+1)}\biggr) \end{aligned}$$

and the angle θ *being determined by the relation*

$$\cos\theta = -\frac{1}{2}\sqrt{\frac{(2n+5)(2n+1)}{(n+1)(n+2)}}.$$

Theorem 2.4. *For the weight function* w_4^R*, the Stieltjes polynomial can be expanded either as*

$$\begin{aligned} \pi_{n+1}^{\star}(t) = \frac{1}{2^n}\biggl\{T_{n+1}(t) &+ \sum_{k=1}^{n}\biggl(-\frac{n+1}{n+2}\biggr)^{n-k+1}T_k(t) \\ &+ \frac{1}{2}\biggl(-\frac{n+1}{n+2}\biggr)^{n+1}T_0\biggr\}, \end{aligned} \tag{2.9}$$

or as

$$\begin{aligned} \pi_{n+1}^{\star}(t) = \frac{1}{2^{n+1}}\biggl\{U_{n+1}(t) &- \frac{n+1}{n+2}U_n(t) \\ &- \frac{2n+3}{(n+2)^2}\sum_{k=1}^{n}\biggl(-\frac{n+1}{n+2}\biggr)^{k-1}U_{n-k}(t)\biggr\}. \end{aligned} \tag{2.10}$$

Theorem 2.5. *For the weight function* $w = w_1^L$*, the Stieltjes polynomial is given by*

$$\begin{aligned} \pi_{n+1}^{\star}(t) = \frac{1}{2^n}\biggl\{T_{n+1}(t) &+ \sum_{k=1}^{(n-1)/2}\biggl(\frac{n+1}{n+3}\biggr)^{k}T_{n-2k+1}(t) \\ &+ \frac{1}{2}\biggl(\frac{n+1}{n+3}\biggr)^{(n+1)/2}T_0(t)\biggr\} \end{aligned} \tag{2.11}$$

when n is odd, and by

$$\pi^\star_{n+1}(t) = \frac{1}{2^n}\left\{T_{n+1}(t) + \sum_{k=1}^{n/2}\left(\frac{n+1}{n+3}\right)^k T_{n-2k+1}(t)\right\} \tag{2.12}$$

when n is even.

Theorem 2.6. *For the weight function $w = w_2^L$, the Stieltjes polynomial can be expressed as*

$$\begin{aligned}\pi^\star_{n+1}(t) = {}& \frac{1}{2^{n+1}}\left\{U_{n+1}(t) + \sum_{k=1}^{[n/2]} c_{n-2k+1}U_{n-2k+1}(t)\right\}\\ &+ \frac{1+(-1)^{n-1}}{4}c_0U_0(t),\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}c_{n-2k+1} = {}& \left[a_0\left(1+\sqrt{\frac{2n+6}{(n+5)(n+2)}}\right)^k\right.\\ &\left.+a_1\left(1-\sqrt{\frac{2n+6}{(n+5)(n+2)}}\right)^k\right]\left(\frac{n+2}{n+4}\right)^k,\end{aligned}$$

$$a_0 = \frac{n\sqrt{(2n+6)(n+5)} - 4\sqrt{n+2}}{2\sqrt{n+2}(2n+6+\sqrt{(2n+6)(n+5)(n+2)})},$$

$$a_1 = \frac{4\sqrt{n+2} + n\sqrt{(2n+6)(n+5)}}{2\sqrt{n+2}(\sqrt{(2n+6)(n+5)(n+2)} - 2n - 6)}.$$

In the theorems stated above, two expansions for $\pi^\star_{n+1}$ are given where possible. Since T_k, U_k and V_k are easy to generate, expansions of $\pi^\star_{n+1}$ in terms of these polynomials allow for efficient ways of finding the zeros of Stieltjes polynomials by Newton's method. In particular, it is easy to evaluate $\pi^\star_{n+1}(t)$ and $(\pi^\star_{n+1})'(t)$ for any t in $[-1,1]$.

Unfortunately, these formulas are not suitable for establishing the interlacing property for the zeros of π_n and $\pi^\star_{n+1}$.

There is essentially one more case to be considered, namely $w = w_3^L$; the case $w = w_4^L$ can be settled by noting

$$\pi^\star_{n+1}(t; w_4^L) = (-1)^{n+1}\pi^\star_{n+1}(-t; w_3^L).$$

Again, we expand $\pi^\star_{n+1}(t) = \pi^\star_{n+1}(t; w_3^L)$ as

$$\pi^\star_{n+1}(t) = \frac{1}{2^{n+1}}\left(U_{n+1} + \sum_{k=0}^{n} c_k U_{n-k}\right)$$

and use the techniques employed before. The result is no longer an explicit expression for the coefficients c_k, but the following recursive procedure:

Initialization

$$c_{-1} = 1, \qquad c_0 = -\frac{n+1}{n+3}, \qquad c_1 = \frac{n^2-5}{(n+3)^2}$$

For $k = 2, 3, \ldots, n$ do

$$c_k = \frac{n+1}{n+3}\left(\frac{n+2}{n+4}c_{k-3} + c_{k-2} - c_{k-1}\right)$$

2.2. Inclusion and interlacing. We are now interested in the location of the zeros of $\pi^\star_{n+1}$ and their relation to the zeros of π_n. Specifically, if all the zeros of $\pi^\star_{n+1}$ are inside the interval $[-1, 1]$, for any n, then we say that *inclusion* holds. The *interlacing* property can be stated as

$$\tau_1^\star < \tau_1 < \tau_2^\star < \cdots < \tau_n^\star < \tau_n < \tau_{n+1}^\star \quad \text{for each } n,$$

where the zeros τ_ν and $\tau_\mu^\star$ are assumed to be arranged in increasing order. The two questions of inclusion and interlacing are discussed by Gautschi and Notaris [**5**] for Gauss-Kronrod quadratures relative to the Jacobi weight functions. In Table 2.1 we list all the known results for the eight special Jacobi weight functions studied in Section 2.

TABLE 2.1. Inclusion and interlacing properties.

Weight	Inclusion	Interlacing
w_1^R	true for even n	true
w_2^R	true	true
w_3^R	true for even n	true
w_4^R	true $(*)$	true $(*)$
w_1^L	true $(*)$	true $(*)$
w_2^L	true $(*)$	true
w_3^L	true	true
w_4^L	true	true

In the above table, $(*)$ indicates that the result has been proved theoretically. Otherwise, it is verified numerically only for $n \leq 40$. The case where $w = w_1^L$ is a special case of Gegenbauer weight $w(t) = (1-t^2)^{\mu-1/2}$ with $\mu = 2$. For Gegenbauer weight functions, it is well known from a result by Szegö [**14**] that interlacing holds if $0 < \mu \leq 2$. Monegato [**12**] proved that inclusion and interlacing hold for the weight function w_4^R. The inclusion property for the weight function w_1^R is shown by Rabinowitz [**13**]. The inclusion property of the zeros of $\pi_{n+1}^\star$ in the case of the weight functions w_4^R, w_1^L and w_2^L can be shown by using Theorems 2.4–2.6.

3. Formulas for the quadrature weights. It can be shown by applying the theory of interpolation that the weights in (1.1) admit the following representations:

(3.1)

$$\sigma_0 = \int_{-1}^{1} \frac{1 - \left(\dfrac{\pi_n'(-1)}{\pi_n(-1)} + \dfrac{(\pi_{n+1}^\star)'(-1)}{\pi_{n+1}^\star(-1)}\right)(1+t)}{\pi_n(-1)\pi_{n+1}^\star(-1)}\, \pi_n(t)\pi_{n+1}^\star(t)\, dt,$$

$$\sigma_0' = \int_{1}^{1} \frac{(1+t)\pi_n(t)\pi_{n+1}^\star(t)}{\pi_n(-1)\pi_{n+1}^\star(-1)} w(t)\, dt,$$

$$\sigma_\nu = \int_{-1}^{1} \frac{(1+t)^2 \pi_n(t) \pi^\star_{n+1}(t)}{(1+\tau_\nu)^2 \pi'_n(\tau_\nu) \pi^\star_{n+1}(\tau_\nu)(t-\tau_\nu)} w(t)\, dt, \qquad \nu = 1, 2, \dots, n,$$

$$\sigma^\star_\mu = \int_{-1}^{1} \frac{(1+t)^2 \pi_n(t) \pi^\star_{n+1}(t)}{(1+\tau^\star_\mu)^2 \pi_n(\tau^\star_\mu)(\pi^\star_{n+1})'(\tau^\star_\mu)(t-\tau^\star_\mu)} w(t)\, dt,$$
$$\mu = 1, 2, \dots, n+1.$$

Similar, though more complicated, formulae can be obtained for the weights in (1.2), but we omit them here.

By using techniques similar to those used by Monegato [**11**] in the proof of his Theorems 1 and 2, the formulas in (3.1) can be simplified to

(3.2)
$$\sigma_0 = \kappa^R_0 - \frac{\pi'_n(-1)/\pi_n(-1) + (\pi^\star_{n+1})'(-1)/\pi^\star_{n+1}(-1)}{\pi_n(-1)\pi^\star_{n+1}(-1)} \parallel \pi_n \parallel^2_{w^R},$$
$$\sigma'_0 = \kappa^R_1 + \frac{\parallel \pi_n \parallel^2_{w^R}}{\pi_n(-1)\pi^\star_{n+1}(-1)},$$
$$\sigma_\nu = \lambda^R_\nu + \frac{\parallel \pi_n \parallel^2_{w^R}}{(1+\tau_\nu)^2 \pi'_n(\tau_\nu)\pi^\star_{n+1}(\tau_\nu)}, \qquad \nu = 1, 2, \dots, n,$$
$$\sigma^\star_\mu = \frac{\parallel \pi_n \parallel^2_{w^R}}{(1+\tau^\star_\mu)^2 \pi_n(\tau^\star_\mu)(\pi^\star_{n+1})'(\tau^\star_\mu)}, \qquad \mu = 1, 2, \dots, n+1,$$

where κ^R_0, κ^R_1 and λ^R_ν are the weights in the Gauss-Radau quadrature rule with double endpoints (see [**6**]) and $\parallel \pi_n \parallel^2_{w^R} = \int_{-1}^{1} \pi_n^2(t) w^R(t)\, dt$. The weights λ^R_ν are given by

$$\text{(3.3)} \qquad \lambda^R_\nu = -\frac{\parallel \pi_n \parallel^2_{w^R}}{(1+\tau_\nu)^2 \pi_{n+1}(\tau_\nu)\pi'_n(\tau_\nu)}, \qquad \nu = 1, 2, \dots, n.$$

For the weights in the Kronrod extension of the Gauss-Lobatto rule with double end points (cf. (1.2)), we are able to develop the following relations that may be useful in computation:

(3.4)
$$\sigma_0 = \kappa^L_0 + \frac{1 - \pi'_n(-1)/\pi_n(-1) - (\pi^\star_{n+1})'(-1)/\pi^\star_{n+1}(-1)}{4\pi_n(-1)\pi^\star_{n+1}(-1)} \parallel \pi_n \parallel^2_{w^L},$$
$$\sigma'_0 = \kappa^L_1 + \frac{\parallel \pi_n \parallel^2_{w^L}}{4\pi_n(-1)\pi^\star_{n+1}(-1)},$$

$$\sigma_{n+1} = \mu_0^L - \frac{1 + \pi_n'(1)/\pi_n(1) + (\pi_{n+1}^\star)'(1)/\pi_{n+1}^\star(1)}{4\pi_n(1)\pi_{n+1}^\star(1)} \parallel \pi_n \parallel_{w^L}^2,$$

$$\sigma_{n+1}' = \mu_1^L - \frac{\parallel \pi_n \parallel_{w^L}^2}{4\pi_n(1)\pi_{n+1}^\star(1)},$$

$$\sigma_\nu = \lambda_\nu^L + \frac{\parallel \pi_n \parallel_{w^L}^2}{(1-\tau_\nu^2)^2\pi_n'(\tau_\nu)\pi_{n+1}^\star(\tau_\nu)}, \qquad \nu = 1, 2, \dots, n,$$

$$\sigma_\mu^\star = \frac{\parallel \pi_n \parallel_{w^L}^2}{(1-(\tau_\mu^\star)^2)^2\pi_n(\tau_\mu^\star)(\pi_{n+1}^\star)'(\tau_\mu^\star)}, \qquad \mu = 1, 2, \dots, n+1.$$

Here, κ_0^L, κ_1^L, μ_0^L, μ_1^L and λ_ν^L are the weights in Gauss-Lobatto quadrature with double end points. The weights κ_0^L, κ_1^L, μ_0^L and μ_1^L relative to the four Chebyshev weight functions were studied in Gautschi and Li [**6**]. It is not difficult to see that

$$\text{(3.5)} \qquad \lambda_\nu^L = -\frac{\parallel \pi_n \parallel_{w^L}^2}{(1-\tau_\nu^2)^2\pi_{n+1}(\tau_\nu)\pi_n'(\tau_\nu)}, \qquad \nu = 1, 2, \dots, n.$$

The positivity of the weights σ_ν has been studied by Gautschi and Notaris (see [**5**]). If the interlacing property holds for the zeros of π_n and $\pi_{n+1}^\star$, then the weights $\sigma_\mu^\star$ are positive (Monegato [**11**]). In this section our main interest is to obtain explicit formulae for the weights in the boundary terms of (1.1) and (1.2) and to exam their signs, in the case of the four Chebyshev weight functions. Some alternative procedures for the computation of the internal nodes are proposed in [**2, 8**]. In the following we state the respective results and sketch their proofs. Only three weight functions will be considered; for the remaining five (except w_3^R which follows from the result for w_4^R) we were unable to obtain results because of the complexity of the computations.

Theorem 3.1. *For $w(t) = w_1(t)$, the weights σ_0 and σ_0' in the* *quadrature formula* (1.1) *are given by*

$$\sigma_0 = \frac{3\pi}{(2n+1)(2n+3)^3\left[1 - \frac{4(n+1)^2}{(2n+3)^2}\left(\frac{2n+1}{2n+3}\right)^n\right]}$$

$$\left\{ 2(n+1)(2n^2+4n+1)\left(\frac{2n+1}{2n+3}\right)^n \right.$$

(3.6)

$$+ \frac{(2n+3)^2 - 8(n+1)^2\left(\frac{2n+1}{2n+3}\right)^n}{2(n+1)}$$

$$\left. \cdot \left[\frac{n(n+2)}{5} + \frac{(n+1)^2\left(1 - 2\left(\frac{2n+1}{2n+3}\right)^{n+1}\right)}{1 - \frac{4(n+1)^2}{(2n+3)^2}\left(\frac{2n+1}{2n+3}\right)^n}\right]\right\},$$

(3.7)

$$\sigma_0' = \frac{3\pi((2n+3)^{n+2} - 8(n+1)^2(2n+1)^n)}{2(n+1)(2n+1)(2n+3)((2n+3)^{n+2} - 4(n+1)^2(2n+1)^n)}.$$

Moreover, they are both positive.

To prove this theorem, it suffices to observe that

$$\| \pi_n \|^2 = \int_{-1}^{1} \pi_n^2(t)(1+t)^2(1-t^2)^{-1/2}\,dt = \frac{1}{2^{2n+1}}\frac{2n+3}{2n+1}\pi,$$

(3.8)

$$\pi_n(-1) = \frac{1}{2^{n+1}}\frac{2(-1)^n(n+1)(2n+3)}{3},$$

$$\pi_{n+1}^\star(-1) = \frac{1}{2^n}(-1)^{n+1}\left(2n+3 - 4\frac{(n+1)^2}{2n+3}\left(\frac{2n+1}{2n+3}\right)^n\right),$$

$$\pi_n'(-1) = \frac{1}{2^{n+1}}\frac{(-1)^{n+1}}{15}2n(n+1)(n+2)(2n+3),$$

$$(\pi_{n+1}^\star)'(-1) = \frac{1}{2^n}(-1)^n(n+1)^2\left(2n+3-2(2n+1)\left(\frac{2n+1}{2n+3}\right)^n\right),$$
$$\kappa_0^R = \frac{3\pi}{5}\frac{6n^2+12n+5}{(n+1)(2n+1)(2n+3)},$$
$$\kappa_1^R = \frac{3\pi}{(n+1)(2n+1)(2n+3)}.$$

The second, fourth and last two relations in (3.8) have been shown in [**6**]. The third and fifth formula follows from Theorem 2.1 by elementary but tedious calculations. The positivity of σ_0 and σ_0' follows in a straightforward manner from (3.6) and (3.7).

Theorem 3.2. *For $w(t) = w_4(t)$, the weights σ_0 and σ_0' in the quadrature formula* (1.1) *are given by*

(3.9)
$$\sigma_0 = \frac{3\pi}{(n+1)(n+2)(2n+3)}\left\{\frac{2(3n^2+9n+5)}{5} - \frac{1}{2\left(1-\frac{2n^2+5n+3}{2(n+2)^2}\left(\frac{n+1}{n+2}\right)^n\right)}\left[\frac{n(n+3)}{5}+(n+1)(n+2)\frac{1-\frac{2n^2+5n+3}{(n+2)^2}\left(\frac{n+1}{n+2}\right)^n}{1-\frac{2n^2+5n+3}{2(n+2)^2}\left(\frac{n+1}{n+2}\right)^n}\right]\right\},$$

and

(3.10)
$$\sigma_0' = \frac{3\pi}{(n+1)(n+2)}\left(\frac{1}{n+3} - \frac{1}{2(2n+3)\left(1-\frac{2n^2+5n+3}{2(n+2)^2}\left(\frac{n+1}{n+2}\right)^n\right)}\right).$$

Moreover, both σ_0 and σ_0' are positive.

The proof of this theorem utilizes the following equalities:

$$
\begin{aligned}
\pi_n(-1) &= \frac{1}{2^{n+1}}(-1)^n \frac{(n+2)(2n+3)}{3}, \\
\pi_n'(-1) &= \frac{1}{2^{n+1}}(-1)^{n+1} \frac{n(n+2)(n+3)(2n+3)}{15}, \\
\pi_{n+1}^\star(-1) &= \frac{1}{2^n}(-1)^{n+1}\left(n+2 - \frac{2n^2+5n+3}{2(n+2)}\left(\frac{n+1}{n+2}^2\right)^n\right), \\
(\pi_{n+1}^\star)'(-1) &= \frac{1}{2^n}(-1)^n(n+1)(n+2)^2 \\
&\qquad \left(1 - \frac{2n^2+5n+3}{(n+2)^2}\left(\frac{n+1}{n+2}^2\right)^n\right), \\
\| \pi_n \|_{w_4^R}^2 &= \frac{1}{2^{2n+2}} \frac{n+2}{n+1}\pi, \\
\kappa_0^R &= \frac{6\pi(3n^2+9n+5)}{5(n+1)(n+2)(2n+3)}, \\
\kappa_1^R &= \frac{3\pi}{(n+1)(n+2)(2n+3)}.
\end{aligned} \tag{3.11}
$$

The first two and the last two are given in [**6**]; the fifth can be computed directly, while the remaining two follow from Theorem 2.4. Again, the positivity follows readily from (3.9) and (3.10).

Theorem 3.3. *For the first-kind Chebyshev weight function, we have the following explicit formulas for the weights* σ_0, σ_0', σ_{n+1} *and* σ_{n+1}' *in the Lobatto-type formula* (1.2):

If n *is even, then*

$$
\begin{aligned}
\sigma_{n+1} &= \sigma_0 \\
&= \frac{3\pi(1-2c_n)[6n^2+24n+20-(11n^2+44n+40)c_n]}{40(n+1)(n+2)(n+3)(1-c_n)^2}, \\
\sigma_{n+1}' &= \sigma_0' \\
&= \frac{3\pi(1-2c_n)}{8(n+1)(n+2)(n+3)(1-c_n)};
\end{aligned} \tag{3.12}
$$

if n is odd, then

$$
\begin{aligned}
\sigma_{n+1} = \sigma_0 = {} & \frac{3\pi}{40(n+1)(n+2)[n+3-(n+2)d_n]^2} \\
& \left\{6n^2+24n+20 - \frac{d_n}{n+3}\left(23n^3+128n^2+219n+120\right)\right. \\
& \left. + 2\left(11n^3+61n^2+103n+55\right)\frac{n+2}{(n+3)^2}d_n^2\right\}, \\
\sigma'_{n+1} = \sigma'_0 = {} & \frac{3\pi(1-(2(n+2)/(n+3))d_n)}{8(n+1)(n+2)(n+3-(n+2)d_n)},
\end{aligned}
\tag{3.13}
$$

where

$$c_n = \left(\frac{n+1}{n+3}\right)^{(n/2)+1}, \qquad d_n = \left(\frac{n+1}{n+3}\right)^{(n+1)/2}.$$

Moreover, the weights σ_0, σ'_0, σ_{n+1} and σ'_{n+1} are all positive.

To prove Theorem 3.3, one needs to show, and then use, the following results:

$$
\begin{aligned}
\pi_n(-1) &= \frac{1}{2^{n+2}}(-1)^n\frac{2(n+2)(n+3)}{3}, \\
\pi'_n(-1) &= \frac{1}{2^{n+2}}(-1)^{n+1}\frac{2n(n+2)(n+3)(n+4)}{15}, \\
\| \pi_n \|^2_{w_1^L} &= \frac{1}{2^{2n+3}}\frac{n+3}{n+1}\pi, \\
\kappa_0^L &= \frac{3\pi(3n^2+12n+10)}{10(n+1)(n+2)(n+3)}, \\
\kappa_1^L &= \frac{3\pi}{4(n+1)(n+2)(n+3)};
\end{aligned}
\tag{3.14}
$$

$$
\pi^\star_{n+1}(-1) = \begin{cases} -\dfrac{1}{2^{n+1}}(n+3)\left(1-\left(\dfrac{n+1}{n+3}\right)^{(n/2)+1}\right), & n \text{ even}, \\ \dfrac{1}{2^{n+1}}\left(n+3-(n+2)\left(\dfrac{n+1}{n+3}\right)^{(n+1)/2}\right), & n \text{ odd}; \end{cases}
\tag{3.15}
$$

(3.16) $(\pi^{\star}_{n+1})'(-1)$

$$= \begin{cases} \dfrac{1}{2^{n+1}}\,(n+1)\left[(n+3)^2-(2n^2+8n+7)\left(\dfrac{n+1}{n+3}\right)^{n/2}\right], & n \text{ even},\\[2ex] -\dfrac{1}{2^{n+1}}\,(n+1)\,(n+3)\left[n+3-2(n+2)\left(\dfrac{n+1}{n+3}\right)^{(n+1)/2}\right], & n \text{ odd}.\end{cases}$$

Again, the equations in (3.14) are from [**6**]. The formulas (3.15) and (3.16) follow from Theorem 2.5. The positivity result follows from (3.12) by an easy calculation, and from (3.13) by a more complicated, though elementary, calculation.

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REFERENCES

1. C. Bernardi and Y. Maday, *Some spectral approximations of one-dimensional fourth-order problems*, *Progress in approximation theory* (P. Nevai and A. Pinkus, eds.), Academic Press, New York, 1991.

2. F. Caliò, W. Gautschi and E. Marchetti, *On computing Gauss-Kronrod quadrature formulae*, Math. Comp. **47** (1986), 639–650.

3. W. Gautschi, *Gauss-Kronrod quadrature—A survey*, in *Numerical methods and approximation theory* III (G.V. Milovanović, ed.), Faculty of Electronic Engineering, Univ. Niš, Niš, 1988.

4. ———, *On the remainder term for analytic functions of Gauss-Lobatto and Gauss-Radau quadratures*, Rocky Mountain J. Math. **21** (1991), 209–226.

5. W. Gautschi and S.E. Notaris, *An algebraic study of Gauss-Kronrod quadrature formulae for Jacobi weight functions*, Math. Comp. **51** (1988), 231–248.

6. W. Gautschi and S. Li, *Gauss-Radau and Gauss-Lobatto quadratures with double end points*, J. Comp. App. Math. **34** (1991), 343–360.

7. ———, *The remainder term for analytic functions of Gauss-Radau and Gauss-Lobatto quadrature rules with multiple end points*, J. Comp. App. Math. **33** (1990), 315–329.

8. J. Kautsky and S. Elhay, *Gauss quadrature and Jacobi matrices for weight functions not of one sign*, Math. Comp. **43** (1984), 543–550.

9. A.S. Kronrod, *Nodes and weights for quadrature formulae. Sixteen-place tables*, Izdat. "Nauka," Moscow, 1964 (Russian).

10. ———, *Integration with control of accuracy*, Dokl. Akad. Nauk SSSR **154** (1964), 283–286 (Russian).

11. G. Monegato, *A note on extended Gaussian quadrature rules*, Math. Comp. **30** (1976), 812–817.

12. ———, *Stieltjes polynomials and related quadrature rules*, SIAM Review **24** (1982), 137–158.

13. P. Rabinowitz, *Gauss-Kronrod integration rules for Cauchy principal value integrals*, Math. Comp. **41** (1982), 63–78.

14. G. Szegö, *Über gewisse orthogonale Polynome, die zu einer oszillierenden Belegungsfunktion geh ören*, Math. Ann. **110** (1934), 501–513.

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