ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 4, Fall 1996

APPLICATION OF CATLIN'S BOX CONSTRUCTION TO SUBELLIPTICITY OF $n-1$ FORMS

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1. Introduction. Subelliptic estimates for the $\bar{\partial}$-Neumann problem is the main tool in establishing local regularity for the solution of the $\bar{\partial}$-Neumann problem, which has lots of applications in several complex variables. On strictly pseudoconvex domains, subelliptic estimates hold with $\varepsilon = 1/2$. The question is much harder on weakly pseudoconvex domains. The most outstanding works on weakly pseudoconvex domains are by Kohn [**12**], Catlin [**1**] and [**2**]. Catlin used D'Angelo's [**4**] notion of finite type to establish necessary and sufficient conditions for subellipticity. Catlin [**3**] constructed a plurisubharmonic function to prove that in $\mathbf{C}^2$ if a point $z_0 \in b\Omega$ is of finite type m, then a precise subelliptic estimate of order $\varepsilon = 1/m$ holds at z_0. This result is established earlier in Kohn [**12**] and also in Fornaess and Sibony [**7**].

In this note we establish a result similar to the above-mentioned theorem in $\mathbf{C}^2$. We will prove a result for $(p, n-1)$ forms for non-pseudoconvex domains in $\mathbf{C}^n$. Some previous results in non-pseudoconvex domains are done in Hörmander [**11**], Ho [**8, 9**] and [**10**]. Hörmander dealt mainly with the case where the Levi-form is nondegenerate. Ho dealt with the case that the Levi-form of a vector field is bounded below by a certain nonnegative function. Here we deal with the case where there is a vector field whose Levi-form is nonnegative and with 'finite type' m, then we show that a subelliptic estimate of order $\varepsilon = 1/m$ holds for $(p, n-1)$ forms. This is essentially the best result we can expect with the existence of such a single vector field. (See Catlin [**1**] and Ho [**9**].) We use extensively the 'box construction' and the method of constructing plurisubharmonic functions in Catlin [**3**].

Though it seems that the theorem is not a surprising one, we should note that the question of subelliptic estimate is much more subtle for non-pseudoconvex domains. We given an example in the last part

Received by the editors on June 7, 1994, and in revised form on January 24, 1995.

that there is a domain in which there is a three-dimensional cross section being pseudoconvex and of finite type, but there is no subelliptic estimate for all $(p,1)$ up to $(p,n-1)$ forms for the original domain.

Let us first introduce some definitions and notations.

Definition 1. Let Ω be a smooth domain in $\mathbf{C}^n$ and $z_0 \in b\Omega$. Then a subelliptic estimate is said to hold for (p,q) forms at z_0 if there is a neighborhood U of z_0, $\varepsilon > 0$, $C > 0$, such that

$$|||u|||_\varepsilon^2 \leq C(||\bar{\partial}u||^2 + ||\bar{\partial}^* u||^2 + ||u||^2) \tag{1}$$

for all smooth (p,q) forms u compactly supported in U and which lies in the domain of $\bar{\partial}^*$. Here $|||\cdot|||_\varepsilon$ denotes the *tangential Sobolev norm of order* ε.

Let r be a smooth defining function of Ω, $L_1, \dots, L_n$ an orthonormal basis of smooth holomorphic vector fields in which $L_1, \dots, L_{n-1}$ are tangential to $b\Omega$, and $\omega_1, \dots, \omega_n$ are (1,0) forms dual to $L_1, \dots, L_n$. Then the Levi-form is

$$c_{ij} = \partial\bar{\partial} r(L_i, \overline{L}_j), \qquad i,j = 1, \dots, n-1.$$

If λ is a smooth function on Ω, then we write

$$\lambda_{ij} = \partial\bar{\partial}\lambda(L_i, \overline{L}_j), \qquad i,j = 1, \dots, n-1.$$

$||\cdot||_\phi^2$ denotes the L^2 norm with weight $e^{-\phi}$ and $\delta_i u = e^{\phi} L_i(e^{-\phi}u)$. We denote

$$S_\delta = \{z \in \Omega \mid -\delta < r(z) < 0\}.$$

To simplify the notations we will often prove theorems for $(0,n-1)$ forms though the theorems are stated for $(p,n-1)$ forms. C represents a constant that may vary from line to line.

2. Subelliptic estimates.

We will first give a theorem which is closely related to Theorem 2.2 in Catlin [**2**].

Theorem 1. *Let Ω be a smooth domain in $\mathbf{C}^n$ and V a neighborhood of a point $z_0 \in b\Omega$. Assume that in coordinates $(z_1, z_2, \dots, z_n)$ the* *tangential vector field* $L = r_{z_1}\partial/\partial z_2 - r_{z_2}\partial/\partial z_1$ *is nonvanishing in* V *and satisfies*:

1. $\partial\bar{\partial}r(L,\overline{L}) \geq 0$ *in* V,

2. *for all* $\delta > 0$ *sufficiently small, there is a* C^2 *bounded function* g_δ *in* V *such that*

(a) $\partial\bar{\partial}g_\delta(L,\overline{L}) \geq 0$ *in* V,

(b) *in* $V \cap S_\delta$ *we have* $\partial\bar{\partial}g_\delta(L,\overline{L}) \geq c\delta^{-2\varepsilon}$.

Then a subelliptic estimate (1) *of order* ε *holds for* $(p, n-1)$ *forms at* z_0.

Proof. Let L_1 be L in the statement of the theorem, and let u be a smooth $(0, n-1)$ form in Dom $(\bar{\partial}^*)$. Then

$$u = \sum_i u_i \bar{\omega}_i \wedge \cdots \wedge \hat{\bar{\omega}}_i \wedge \cdots \wedge \bar{\omega}_n$$

where $\wedge$ means that the term is missing. Since $u \in$ Dom $(\bar{\partial}^*)$, $u_i = 0$ on $b\Omega$ for $i < n$, and it follows that $||\,|u_i|\,||_1 \leq C(||\bar{\partial}u||^2 + ||\bar{\partial}^*u||^2 + ||u||^2)$ for $i < n$. It is not hard to see that we then need only to consider that $u = u_n\bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_{n-1}$. We will write $u_n = u$.

Expanding $||\bar{\partial}u||^2 + ||\bar{\partial}^*u||^2$ (the reader may refer to [**6**]) we see that

$$||\bar{\partial}u||^2 + ||\bar{\partial}^*u||^2 \approx ||L_1u||^2 + \cdots + ||L_{n-1}u||^2 + ||\overline{L}_nu||^2.$$

Now

$$\begin{aligned} ||L_1u||^2_\phi &= ||\delta_1u + (L_1\phi)u||^2_\phi \\ &\geq ||\delta_1u||^2_\phi - ||(L_1\phi)u||^2_\phi \\ &= ||\overline{L}_1u||^2_\phi + \int_{b\Omega\cap V} c_{11}|u|^2e^{-\phi}\,dS \\ &\quad + \int_{\Omega\cap V} \phi_{11}|u|^2e^{-\phi}\,dV - ||(L_1\phi)u||^2_\phi \\ &\quad + O(||\overline{L}_1u||_\phi||u||_\phi + ||\delta_1u||_\phi||u||_\phi) \end{aligned}$$

where in the third line we use proposition (3.1.3) of Hörmander [**11**] and the fact that the first order derivatives of $[\delta_1, \overline{L}_1]$ involve only

$\partial/\partial z_1, \partial/\partial z_2$ and its conjugates. It follows that

$$\begin{aligned}||\overline{L}_1 u||_\phi^2 + \sum_{i=2}^{n-1} ||L_i u||_\phi^2 + ||\overline{L}_n u||_\phi^2 + \int_{b\Omega\cap V} c_{11}|u|^2 e^{-\phi}\, dS \\ + \int_{\Omega\cap V} \phi_{11}|u|^2 e^{-\phi}\, dV - ||(L_1\phi)u||_\phi^2 \\ \leq C(||\bar{\partial}u||_\phi^2 + ||\bar{\partial}^* u||_\phi^2 + ||u||_\phi^2).\end{aligned}$$

If $|\psi| \leq 1$ and $\psi_{11} \geq 0$, we put $\phi = e^\psi/3$. Then $\phi_{11} - |L_1\phi|^2 \geq C\psi_{11}$. Hence

$$\begin{aligned}||\overline{L}_1 u||^2 + \sum_{i=2}^{n-1} ||L_i u||^2 + ||\overline{L}_n u||^2 + \int_{\Omega\cap V} \psi_{11}|u|^2 e^{-\phi}\, dV \\ \leq C(||\bar{\partial}u||^2 + ||\bar{\partial}^* u||^2 + ||u||^2).\end{aligned}$$

If we now put $\psi = g_\delta$, then we get the estimate

$$\begin{aligned}\delta^{2\varepsilon} \int_{S_\delta\cap V} |u|^2\, dV + ||\overline{L}_1 u||^2 + \sum_{i=2}^{n-1} ||L_i u||^2 + ||\overline{L}_n u||^2 \\ \leq C(||\bar{\partial}u||^2 + ||\bar{\partial}^* u||^2 + ||u||^2).\end{aligned}$$

This estimate is similar to (2.8) in Catlin [**2**] except that we have $\sum_{i=2}^{n-1} ||L_i u||^2$ instead of $\sum_{i=2}^{n-1} ||\overline{L}_i u||^2$. The proof of Theorem 2.2 in [**2**] goes through and we conclude that

$$||\,|u|\,||_\varepsilon^2 \leq C(||\bar{\partial}u||^2 + ||\bar{\partial}^* u||^2 + ||u||^2). \qquad \square$$

Let $z = (z_1, z_2, z_3, \dots, z_n)$. We write $z' = (z_1, z_2)$, $z'' = (z_3, \dots, z_n)$ and denote the intersection of the domain Ω with the complex planes in (z_1, z_2) at z'' as

$$\Omega_{z''}(z_1, z_2) = \{(z_1, z_2) \in \mathbf{C}^2 \mid r(z_1, z_2, z'') < 0\} \subset \mathbf{C}^2.$$

We now state the main theorem in this paper.

Theorem 2. *Let $\Omega \subset \mathbf{C}^n$ be a smooth domain, $w'' = (w_3, \dots, w_n)$ a point in $\mathbf{C}^{n-2}$. Assume that the defining function r satisfies*

1. *$\Omega_{z''}(z_1, z_2) \subset \mathbf{C}^2$ are pseudoconvex for all z'' in a neighborhood of w''.*
2. *$\Omega_{w''}(z_1, z_2) \subset \mathbf{C}^2$ is of finite type m at $(z_1, z_2) = (w_1, w_2)$.*

Then a subelliptic estimate of order $\varepsilon = 1/m$ holds for $(p, n-1)$ forms for Ω at (w_1, w_2, w'').

Proof. Since we will modify Catlin's argument in $\mathbf{C}^2$ to construct the function g_δ in Theorem 1 in $\mathbf{C}^n$, we will outline the argument in Catlin [**3**] and adopt the necessary changes here. The main ideas in [**3**] are as follows:

Let Ω be a domain in $\mathbf{C}^2$ and $z_0 \in b\Omega$. For each point $w = (w_1, w_2)$ in a neighborhood U of z_0, there are 'boxes' $Q_\delta(w)$ about w where the size is approximately $\tau(w, \delta)$ in the z_1 direction and δ in the z_2 direction. $T(w, \delta)$ is the δ-type at w. ($\tau(w, \delta)$ and $T(w, \delta)$ are defined in (1.6) and (1.25) in [**3**].) There is an $a > 0$ such that for each $w \in U \cap b\Omega$, there is a smooth function $g_{w,\delta}$ on $\overline{\Omega}$ which satisfies

1. If $g_{w,\delta}$ is not plurisubharmonic at z, then

$$T(\pi(z), a\delta) < T(w, \delta).$$

2. For $z \in Q_{a\delta}(w) \cap \Omega$ and $L = s_1 L_1 + s_2 L_2$,

(a) If $\partial\bar{\partial} g_{w,\delta}(L, \overline{L})(z) \geq C((\tau(w, \delta))^{-2}|s_1|^2 + \delta^{-2}|s_2|^2)$ fails at z, then

$$T(\pi(z), a\delta) < T(w, \delta).$$

(b) $\partial\bar{\partial} g_{w,\delta}(L, \overline{L}) \leq C((\tau(w, \delta))^{-2}|s_1|^2 + \delta^{-2}|s_2|^2)$.

Then $U \cap b\Omega$ is decomposed into $U \cap b\Omega \subset \cup_{l=2}^m K_l$ where

$$K_l = \{z \in U \cap b\Omega; T(z, a^{m-l}\delta) = T(z, a^{m-l+1}\delta) = l\}.$$

For each l define

$$\lambda_\delta^l(z) = \sum_{k=1}^n g_{z_k, \delta_l}(z), \qquad z \in \overline{\Omega}$$

where $\{z_k\}$ are points on K_l. Then $\lambda_\delta^l(z)$ satisfies

1. $\lambda_\delta^l(z)$ is a bounded function,

2. $\lambda_\delta^l(z)$ is plurisubharmonic at z if $l^*(\pi(z)) = \min_l\{T(\pi(z), \delta_l) \leq l\} > l$,

3.
$$\partial\bar\partial\lambda_\delta^l(L, \overline{L})(z) \geq C((\tau(w,\delta))^{-2}|s_1|^2 + \delta^{-2}|s_2|^2)$$
if $z \in \{z \in \Omega; r(z) \leq c\delta, \pi(z) \in K_l\}$.

Hence, by choosing $\varepsilon > 0$ small enough the function

$$\tilde\lambda_\delta(z) = \sum_{l=2}^{m} \lambda_\delta^l(z)\varepsilon^l$$

is the required function that $\partial\bar\partial\lambda_\delta^l(L, \overline{L})(z) \geq C\delta^{-2\varepsilon}|L|^2$ in $S_\delta \cap U$.

We now define the 'boxes' in our situation and see the necessary changes that need to be made. We refer the reader to [**3**] for the details. For the sake of simplicity, we assume the point of finite type (w_1, w_2, w'') is the origin. Let U be a neighborhood of the origin such that $\partial r/\partial z_2 \neq 0$ in U.

1. For each $(z', z'') \in U$, there is a change of coordinates $\Phi : \mathbf{C}^n \to \mathbf{C}^n$ with

$$\Phi_1(\zeta) = z_1 + \zeta_1$$
$$\Phi_2(\zeta) = z_2 + d_0(z)\zeta_2 + \sum_{k=1}^{m} d_k(z)\zeta_1^k$$
$$\Phi_i(\zeta) = z_i + \zeta_i, \qquad i = 3, \dots, n$$

where $d_0(z) \neq 0$ and $d_k(z)$ are smooth functions of z, $k = 0, \dots, m$. In coordinates $(\zeta_1, \dots, \zeta_n)$ the function $\rho(\zeta) = r(\Phi(\zeta))$ is of the form

$$\rho(\zeta) = r(z) + \operatorname{Re}\zeta_2 + \sum_{\substack{j+k\leq m\\ j,k>0}} a_{j,k}(z)\zeta_1^j\bar\zeta_1^k + O(|\zeta_1|^{m+1} + |\zeta_2|\,|\zeta| + |\zeta''|).$$

2. Define

$$A_l(z) = \max\{|a_{j,k}(z)|, j + k = l\}, \qquad l = 2, \dots, m,$$
$$\tau(z, \delta) = \min\left\{\left(\frac{\delta}{A_l(z)}\right)^{1/l}, 2 \leq l \leq m\right\}$$

and

$$T(z,\delta) = \min\left\{l; \left(\frac{\delta}{A_l(z)}\right)^{1/l} = \tau(z,\delta)\right\}.$$

$T(z,\delta)$ is the δ-type at z.

3. We can define 'boxes' in ζ and z coordinates as follows:

$$R_\delta(z) = \{\zeta \in \mathbf{C}^n; |\zeta_1| < \tau(z,\delta), |\zeta_i| < \delta, i = 2, \dots, n\}$$
$$Q_\delta(z) = \{z \in \mathbf{C}^n; z = \Phi(\zeta), \zeta \in R_\delta(z)\}.$$

4. It follows that the various estimates on derivatives of ρ hold in $R_\delta(z)$. For example, we have

$$|\rho(\zeta) - \rho(0)| \le \delta, \qquad \zeta \in R_\delta(z)$$
$$|D_1^l \rho(\zeta)| \le \delta(\tau(z,\delta))^{-l}, \qquad \zeta \in R_\delta(z),\ l = 1, \dots, m.$$

5. From the derivative estimates in (4), we get the following:

(a) For $w \in Q_\delta(z)$, we have

$$\tau(w,\delta) \approx \tau(z,\delta).$$

(b) There exists a small constant $d > 0$ so that if $z \in Q_{d\delta}(z)$, then

$$T(w,\varepsilon) \le T(z,\delta)$$

for all $\varepsilon \le d\delta$.

(c) There exists a constant C such that if $w \in Q_\delta(z)$, then

$$Q_\delta(w) \subset Q_{C\delta}(z)$$

and

$$Q_\delta(z) \subset Q_{C\delta}(w).$$

6. The following proposition is similar to Proposition 2.1 of [**3**]:

Proposition 1. *There exists an $a > 0$ such that for all points $w \in b\Omega \cap U$, there is a smooth function $g_{w,\delta}$ that satisfies*:

(a) $|g_{w,\delta}(z)| \le 1$ *for* $z \in \overline{\Omega}$ *and* $g_{w,\delta}$ *is supported in* $Q_\delta(z) \cap \overline{\Omega}$.

(b) *If* $\partial\bar{\partial} g_{w,\delta}(L, \overline{L})(z) < 0$, *the inequality*

$$\partial\bar{\partial} g_{w,\delta}(L, \overline{L})(z) \ge C(\tau(w,\delta))^{-2}$$

fails, then $T(\pi(z), a\delta) < T(w,\delta)$.

(c) *For all* $z \in Q_\delta(w) \cap \overline{\Omega}$,

$$\partial\bar{\partial} g_{w,\delta}(L, \overline{L})(z) \le C(\tau(w,\delta))^{-2}.$$

Proof. The proof is the same as in [**3**] (2.1), except that we only estimate in direction L_1. Also we replace $\psi(\zeta) = \chi(\tau^{-2}|\zeta_1|^2 + \delta^{-2}|\zeta_2|^2)$ by $\chi(\tau^{-2}|\zeta_1|^2 + \delta^{-2}(|\zeta_2|^2 + \cdots + |\zeta_n|^2))$. All the estimates in that paper hold because of the estimates in (5). □

7. Now we can follow the same procedure as in the outline of Catlin's proof to decompose $U \cap b\Omega$ into $\cup K_l$ and choose points $\{z_k\}$ on K_l. Then define

$$\lambda_\delta^l(z) = \sum_{k=1}^{n} g_{z_k,\delta_l}(z)$$

with $\delta_l = a^{m-l}\delta$. Finally, with $\varepsilon > 0$ small enough,

$$g_\delta(z) = \sum_{l=2}^{m} \lambda_\delta^l(z)\varepsilon^l$$

satisfies all the properties in Theorem 1. □

3. Counterexample. Let Ω be defined by the function

$$r = 2\text{Re }\, z_4 + |z_1|^4 + |z_2|^4 - |z_1|^2|z_3|^2 - |z_2|^2|z_3|^2 - |z_3|^4.$$

Then it is easily seen that the domain

$$\Omega' = \{(z_1, z_2, z_4); (z_1, z_2, 0, z_4) \in \Omega\} \subset \mathbf{C}^3$$

is pseudoconvex and finite type at the origin. In fact, a subelliptic estimate of order $\varepsilon = 1/4$ holds both for $(p,1)$ and $(p,2)$ forms at the origin for Ω'. But for the original Ω the Levi matrix is

$$\begin{pmatrix} 4|z_1|^2 - |z_3|^2 & 0 & -\bar{z}_1 z_3 \\ 0 & 4|z_2|^2 - |z_3|^2 & -\bar{z}_2 z_3 \\ -z_1\bar{z}_3 & -z_2\bar{z}_3 & -(|z_1|^2 + |z_2|^2 + 4|z_3|^2) \end{pmatrix}.$$

It is not hard to see that

1. Along $\{z \in b\Omega; z_1 = z_2 = z_3 \neq 0\}$, the Levi-form has two positive eigenvalues and one negative eigenvalue. Hence, by a theorem of Derridj [**5**], there is no subelliptic estimate for $(p,1)$ forms. (The theorem says that if the Levi-form has $n-q-1$ positive eigenvalues and q negative eigenvalues at $z \in b\Omega$, then there is no subelliptic estimate for (p,q) forms at z.)

2. Along $\{z \in b\Omega; z_2 = z_3 = 3z_1 \neq 0\}$, the Levi-form has one positive eigenvalue and two negative eigenvalues. Hence, same as above, there is no subelliptic estimate for $(p,2)$ forms.

3. Along $\{z \in b\Omega; z_3 = 3z_1 = 3z_2 \neq 0\}$, the Levi-form has three negative eigenvalues. There is no subelliptic estimate for $(p,3)$ forms.

This example in particular shows that the requirement that there is a family of two-dimensional cross sections of Ω being pseudoconvex is necessary. It is not sufficient to have only one single piece of cross section being pseudoconvex and of finite type.

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