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NORM INEQUALITIES WITH POWER WEIGHTS FOR HÖRMANDER TYPE MULTIPLIERS

RANDY COMBS

1. Introduction. Let $m(x)$ be a bounded, measurable function on $\mathbf{R}^n$. The operator $T_m f$ defined by the Fourier transform equation

$$(T_m f)^\wedge(x) = m(x)\hat{f}(x)$$

is called a multiplier operator with multiplier m. Denote by λ a nonnegative real number, s a number greater than or equal to 1, $|x| \sim R$ the annulus $\{x : R < |x| < 2R\}$, and $\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index of nonnegative integers α_j with norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We say $m \in M(s, \lambda)$ if

(1)
$$B(m, s, \lambda) = ||m||_\infty + \sup_{R>0, |\alpha| \le \lambda} \left(R^{s|\alpha|-n} \int_{|x|\sim R} |D^\alpha m(x)|^s \, dx \right)^{1/s} < \infty$$

when λ is a positive integer. For the case where λ is not an integer, let 1 be the integer part of λ and let $\gamma = \lambda - l$. We say $m \in M(s, \lambda)$ if

$$B(m, s, \lambda) = B(m, s, l) + \sup_{R>0, 0<|z|<R/2} I(R, z) < \infty \tag{2}$$

where

$$I(R, z) = \sup_{|\alpha| = l} \left((R/|z|)^{\gamma s} R^{s|\alpha|-n} \times \int_{|x|\sim R} |D^\alpha m(x) - D^\alpha m(x-z)|^s \, dx \right)^{1/s}.$$

If λ is an integer, then those multipliers belonging to $M(2, \lambda)$ are the classical Hörmander-Mikhlin multipliers. The definition given here appears in [**4**].

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This paper contains generalizations to higher dimensions of the results contained in [**2**]. We refer the reader to that paper for further historical remarks.

We denote by $S_{0,0}$ the space of Schwartz functions whose Fourier transforms have compact support not including the origin. It should be noted that functions belonging to $S_{0,0}$ have vanishing moments of all orders. Given a real number σ, we define

$$||f||_{p,\sigma} = \bigg(\int_{R^n} |f(x)|^p |x|^\sigma \, dx \bigg)^{1/p}.$$

The main result of this paper is the following theorem.

Theorem 1.1. *Assume* $1 \leq s \leq 2$, $n/s < \lambda$, $m \in M(s, \lambda)$ *and* $1 < p < \infty$. *If* σ *is a real number that satisfies*

i) $\max(-n, -p\lambda) < \sigma < \min(p\lambda, -n + p(\lambda + n - n/s))$ *and*

ii)
$$0 < n\bigg(\frac{\sigma + n}{np} - l\bigg) < 1$$

where l *is the integer part of* $(\sigma + n)/(np)$, *then for each* $f \in S_{0,0}$,

$$||T_m f||_{p,\sigma} \leq CB_s ||f||_{p,\sigma}$$

where C *is independent of* m *and* f.

1.1. We now make some observations about the $M(s, \lambda)$ class and set some notation. First, the $M(s, \lambda)$ condition is monotonic in s and λ; that is, if $s \geq s_1$ and $\lambda \geq \lambda_1$, then $M(s, \lambda) \subset M(s_1, \lambda_1)$ and $B(m, s_1, \lambda_1) \leq B(m, s, \lambda)$.

For a positive real number t, define $\tau_t f(x) = f(tx)$. Let $1 \leq s \leq 2$, $\sigma \geq 0$, and $\lambda \geq 0$. If $m \in M(s, \lambda)$ with norm $B(m, s, \lambda)$,then from the definition of $M(s, \lambda)$ and an appropriate substitution, we see that the function $\tau_t m \in M(s, \lambda)$ with $B(\tau_t m, s, \lambda) \leq B(m, s, \lambda)$ for each $t > 0$. Furthermore, if ϕ is a Schwartz function supported in an annulus and $m \in M(s, \lambda)$, then the product function $(\tau_t \phi)m \in M(s, \lambda)$ with norm bounded by $CB(m, s, \lambda)$ for each $t > 0$, where the constant C depends on ϕ.

Following Hörmander [**1**], we fix a nonnegative $\phi \in C^\infty(\mathbf{R}^n)$ that has support contained in $\{x : 1/2 < |x| < 2\}$ and satisfies

$$\sum_{-\infty}^{\infty} \phi(2^{-j}x) = 1$$

for $x \neq 0$. With this ϕ given, we fix the following notation:

$$m_j(x) = \phi(2^{-j}x)m(x) \tag{3}$$

$$k_j(x) = \breve{m}_j(x) \tag{4}$$

$$M_N(x) = \sum_{-N}^{N} m_j(x) \tag{5}$$

$$K_N(x) = \sum_{-N}^{N} k_j(x). \tag{6}$$

We will decompose the function $m(x)$ as $m(x) = \sum_{-\infty}^{\infty} m_j(x)$ for $x \neq 0$ and note that $K_N * f(x)$ converges pointwise to $T_m f(x)$ for $f \in S_{0,0}$. Also, if $1 \leq s \leq 2$, $\lambda \geq 0$, and $m \in M(s,\lambda)$, then $M_N \in M(s,\lambda)$ and $B(M_N, s, \lambda) \leq CB(m, s, \lambda)$, where C is independent of N and m.

As a consequence of the above remarks, we have the following lemma.

Lemma 1.2. *Suppose* $1 \leq s \leq 2$, $\lambda \geq 0$ *and* $w(x) \geq 0$ *is a weight function. Let* $A \subset L^2$. *If, for some* C *independent of* f *and* $m \in M(s,\lambda)$,

$$||T_m f||_{p,w} \leq CB(m,s,\lambda)||f||_{p,w}$$

for each $m \in M(s,\lambda)$ *and for each* $f \in A$, *then there is a* C' *independent of* f, m, *and* N *such that*

$$||K_N * f||_{p,w} \leq C'B(m,s,\lambda)||f||_{p,w}$$

for all $f \in A$.

The method of proof of Theorem 1.1 depends upon the following lemma proved in the one-dimensional case in [**2**]. The n-dimensional case follows *mutatis mutandis.*

Lemma 1.3. *Suppose that* $K(x,y)$ *is a function defined on* $\mathbf{R}^n \times \mathbf{R}^n$ *and* $U(x)$ *and* $W(x)$ *are nonnegative functions defined on* $\mathbf{R}^n$. *Let* a *be a real number, and let* t *be in* $\mathbf{R}^n$. *Set*

$$Tf(x) = \int_{R^n} K(x,y)f(y)\,dy.$$

Suppose that

$$\int_{\{x:R<|x-t|<2R\}} |Th(x)|^p |x-t|^a U(x)\,dx \leq A \int_{R^n} |h(x)|^p |x-t|^a W(x)\,dx \tag{7}$$

for all $h \in C^\infty$ *with support in* $\{x : R/8 \leq |x-t| \leq 16R\}$, *where* A *is independent of* h *and* R. *If* $f \in C^\infty$, *then* $||Tf||_{p,u}^p$ *is bounded by the sum of*

$$CA||f||_{p,w}^p \tag{8}$$

$$C\int_0^\infty \left(\int_{B(t,r/4)} \left(\int_{\{x:r/2<|x-t|<2r\}} |K(x,y)|^p U(x)\,dx\right)^{1/p} \times |f(y)|\,dy\right)^p \frac{dr}{r}, \tag{9}$$

and

$$C\int_0^\infty \left(\int_{B(t,4r)^c} \left(\int_{\{x:r/2<|x-t|<2r\}} |K(x,y)|^p U(x)\,dx\right)^{1/p} \times |f(y)|\,dy\right)^p \frac{dr}{r} \tag{10}$$

where C *is independent of* f, K *and* W.

Also, we will use the following proposition found in [**4**].

Proposition 1.4. *If* $1 < p < \infty$, $-n < \sigma < n(p-1)$, $1 \leq s \leq 2$, $\lambda \geq n$, $m \in M(s,\lambda)$, *and* $f \in S$, *then*

$$\int_{R^n} |T_m f(x)|^p |x|^\sigma\,dx \leq CB_s \int_{R^n} |f(x)|^p |x|^\sigma\,dx$$

where C is independent of f and m.

To conclude this section, we state a variation of the Hardy inequalities found in [**3**, page 196]. The proof of these lemmas follow from a change to polar coordinates and the original Hardy inequalities.

Lemma 1.5. *If g is defined on $\mathbf{R}^n$, $q \geq 1$ and $t > 0$, then*

$$\int_0^\infty \left(\int_{|y|<r} |g(y)|\, dy \right)^q r^{-t-1}\, dr \leq C \int_{R^n} |g(y)|^q |y|^{nq-t-n}\, dy$$

where C depends only on t, q and n.

Lemma 1.6. *If g is defined on $\mathbf{R}^n$, $q \geq 1$ and $t > 0$, then*

$$\int_0^\infty \left(\int_{|y|>r} |g(y)|\, dy \right)^q r^{t-1}\, dr \leq C \int_{R^n} |g(y)|^q |y|^{nq+t-n}\, dy$$

where C depends only on t, q and n.

2. Preliminaries. Throughout this section and the following sections, p' will denote the exponent conjugate to p. Also, we will denote by B_s the norm $B(m, s, \lambda)$ when no confusion arises.

Lemma 2.1. *Suppose that $1 \leq s \leq 2$ and $1 \leq p < \infty$. Set $t = \min(p', s)$, and let λ be a nonnegative real number. Let α be a multi-index such that $0 \leq |\alpha| \leq \lambda$. If m is in $M(s, \lambda)$ and $2^j R > 1$, then*

$$(11) \quad \left(\int_{|x|\sim R} |D^\alpha k_j(x)|^p\, dx \right)^{1/p} \leq C B_s (2^j R)^{(|\alpha| - \lambda + n/t)} R^{n/p - (|\alpha| + n)}$$

where C depends only on λ and n.

Proof. We first consider the case where λ is an integer. Note that for x in the annulus $\{x : R < |x| < 2R\}$ and each multi-index β,

$$\sum_{|\beta|=\lambda} |x^\beta| \geq C R^\lambda.$$

Hence,

$$\left(\int_{|x|\sim R}|D^\alpha k_j(x)|^p\,dx\right)^{1/p}$$
$$\leq CR^{-\lambda}\sum_{|\beta|=\lambda}\left(\int_{|x|\sim R}|x^\beta D^\alpha k_j(x)|^p\,dx\right)^{1/p}.$$

By hypothesis, $p\leq t'$, so by Hölder's inequality and the Hausdorff Young inequality, we have the bound

$$CR^{-\lambda}R^{n(1/p-1/t')}\sum_{|\beta|=\lambda}\left(\int_{R^n}|D^\beta x^\alpha\phi(2^{-j}x)m(x)|^t\,dx\right)^{1/t}. \tag{12}$$

Note that for each j, $(2^{-j}x)^\alpha\phi(2^{-j}x)m(x)$ is in $M(t,\lambda)$ with a norm that is less than or equal to $CB(m,t,\lambda)$. Also note that the function is supported in $|x|\sim 2^j$. Consequently, with these facts, the $M(t,\lambda)$ condition, $B_t\leq CB_s$, and $1/p-1/t'=1/p-1+1/t$ we have (12) is

$$\leq CB_tR^{-\lambda}R^{n(1/p-1+1/t)}2^{j|\alpha|}2^{j(n/t-\lambda)}$$
$$\leq CB_s(2^jR)^{(|\alpha|-\lambda+n/t)}R^{n/p-(|\alpha|+n)}.$$

This concludes the proof where λ is an integer.

If λ is not an integer, set $\lambda=l+\gamma$, where l is the integer part of λ. Let β be a multi-index with $\beta_1+\cdots+\beta_n=l$, and set $z_\beta=\beta/(4R|\beta|)$, where $|\beta|$ is the Euclidean norm of β. Then $|\pm z_\beta|=1/(4R)\leq 2^j/4<2^{j-1}$. Also

$$\sum_{|\beta|=l}|x^\beta\sin(x\cdot z_\beta)|\geq CR^l>0$$

for x in the annulus $\{x:R<|x|<2R\}$. Note that multiplying by sine factors corresponds to taking differences on the Fourier transform side. Thus, we have the following inequalities

$$\left(\int_{|x|\sim R}|D^\alpha k_j(x)|^p\,dx\right)^{1/p}$$
$$\leq CR^{-l}\sum_{|\beta|=l}\left(\int_{|x|\sim R}|x^\beta\sin(x\cdot z_\beta)D^\alpha k_j(x)|^p\,dx\right)^{1/p}.$$

Hölder's inequality and the Hausdorff Young inequality imply this is

$$\leq CR^{-l+n(1/p-1/t')}\left(\int_{R^n}|D^\beta F_j(x+z_\beta)-D^\beta F_j(x-z_\beta)|^t\,dx\right)^{1/t}$$

where $F_j(x)=x^\alpha\phi(2^{-j}x)m(x)$. As before, $2^{-j|\alpha|}F_j(x)$ is in $M(t,\lambda)$ for j with a norm that is less than or equal to $CB(m,s,\lambda)$. Also note that the functions in the integrand are supported in $|x|\sim 2^j$ and, as shown above, $|z_\beta|<2^j/2$. Consequently, we have by the $M(t,\lambda)$ condition that the above is

$$\begin{aligned}&\leq CB_tR^{-l}R^{n(1/p-1+1/t)}2^{j|\alpha|}\left(\frac{|z_\beta|}{2^j}\right)^\gamma 2^{j(n/t-l)}\\&\leq CB_s(2^jR)^{|\alpha|-\lambda+n/t)}R^{n/p-(|\alpha|+n)}.\end{aligned}$$

This concludes the proof of the lemma. □

Theorem 2.2. *Suppose $1\leq s\leq 2$, $1\leq p<\infty$, and $R>0$. Set $t=\min(p',s)$. Let λ be a real number such that $\lambda>n/t$. If $m\in M(s,\lambda)$ and α is a multi-index such that $0\leq|\alpha|<\lambda-n/t$, then*

$$\left(\int_{|x|\sim R}|D^\alpha K_N(x)|^p\,dx\right)^{1/p}\leq CB_sR^{n/p-(|\alpha|+n)} \tag{13}$$

where C depends only on λ and n.

Proof. We have by Minkowski's inequality,

$$\left(\int_{|x|\sim R}|D^\alpha K_N(x)|^p\,dx\right)\leq\sum_{j=-N}^{N}\left(\int_{|x|\sim R}|D^\alpha k_j(x)|^p\,dx\right)^{1/p}. \tag{14}$$

We will dominate the sum on the right by an infinite series and thus obtain a bound for the lefthand side that is independent of N.

Let J be the first integer such that $2^jR>1$ for $j\geq J$. Consequently, $2^jR\leq 1$ for $j<J$ and

$$\begin{aligned}|D^\alpha k_j(x)|&=|(x^\alpha\phi(2^{-j}x)m(x))\check{}\,|\\&\leq C||m||_\infty 2^{j(|\alpha|+n)}\\&\leq CB_s2^{j(|\alpha|+n)}.\end{aligned}$$

Hence,

$$\left(\int_{|x|\sim R}|D^\alpha k_j(x)|^p\,dx\right)^{1/p}\le CB_s(2^jR)^{(|\alpha|+n)}R^{n/p-(|\alpha|+n)}$$

for $j<J$.

If $j\ge J$, then Lemma 2.1 implies

$$\left(\int_{|x|\sim R}|D^\alpha k_j(x)|^p\,dx\right)^{1/p}\le CB_s(2^jR)^{|\alpha|-\lambda+n/t}R^{n/p-(|\alpha|+n)}.$$

Now set $\varepsilon=|\alpha|+n$ and $\delta=|\alpha|-\lambda+n/t$. With these values, the sum on the right in (14) is dominated by $CB_sR^{n/p-(|\alpha|+n)}$ times

$$\sum_{-\infty}^{J-1}(2^jR)^\varepsilon+\sum_{j=J}^{\infty}(2^jR)^\delta\le 2.$$

This completes the proof of the theorem. $\square$

The following two lemmas will be used to prove Theorem 2.5.

Lemma 2.3. *Suppose* $1\le s\le 2$ *and* $1\le p<\infty$. *Set* $t=\min(p',s)$. *Let* λ *be a real number an* L *an integer such that* $0\le L<\lambda-n/t<L+1$. *Let* $R>0$ *and* $y\in\mathbf{R}^n$ *with* $|y|<R/2$. *If* $m\in M(s,\lambda)$ *and* j *is an integer such that* $2^j|y|>1$, *then there exists a* C *such that*

$$(15)\quad \left(\int_{|x|\sim R}|k_j(x-y)-\sum_{|\alpha|\le L}\frac{(-y)^\alpha}{|\alpha|!}D^\alpha k_j(x)|^p\,dx\right)^{1/p}\le CB_s(2^j|y|)^{L-\lambda+n/t}\left(\frac{|y|}{R}\right)^{\lambda-n/t}R^{n(1/p-1)}$$

where C *is independent of* y, R *and* j.

Proof. To prove (15), note that $|y|<R/2$ and Lemma 2.1 with $|\alpha|=0$ imply

$$(16)\quad \left(\int_{|x|\sim R}|k_j(x-y)|^p\,dx\right)^{1/p}\le CB_s(2^j|y|)^{(n/t-\lambda)}\left(\frac{|y|}{R}\right)^{(\lambda-n/t)}R^{n(1/p-1)}.$$

Also, by the same lemma we have

$$(17)\quad \left(\int_{|x|\sim R} |(-y)^\alpha D^\alpha k_j(x)|^p\,dx\right)^{1/p} \le CB_s(2^j|y|)^{(n/t+|\alpha|-\lambda)}\left(\frac{|y|}{R}\right)^{(\lambda-n/t)} R^{n(1/p-1)}.$$

Since $|\alpha| \le L$ and $2^j|y| > 1$, (16) and (17) are bounded by the righthand side of (15). This concludes the proof of the lemma. □

Lemma 2.4. *Suppose that* $1 \le s \le 2$ *and* $1 \le p < \infty$. *Set* $t = \min(p', s)$. *Let* λ *be a real number and* L *an integer such that* $0 \le L < \lambda - n/t < L+1$. *Let* $R > 0$ *and* $y \in \mathbf{R}^n$ *with* $|y| < R/2$. *If* $m \in M(s,\lambda)$ *and* j *is an integer such that* $2^j|y| \le 1$, *then there exists a* C *such that*

$$(18)\quad \left(\int_{|x|\sim R} \Big|k_j(x-y) - \sum_{|\alpha|\le L} \frac{(-y)^\alpha}{|\alpha|!} D^\alpha k_j(x)\Big|^p\,dx\right)^{1/p} \le CB_s(2^j|y|)^{L+1-\lambda+n/t}\left(\frac{|y|}{R}\right)^{\lambda-n/t} R^{n(1/p-1)}$$

where C *is independent of* y, R *and* j.

Proof. To prove (18), we consider the two cases when λ is an integer and when λ is a noninteger.

We first consider the case when λ is an integer. As in the proof of Lemma 2.1, on the annulus $\{x : R < |x| < 2R\}$ we have $\sum_{|\beta|=\lambda} |x^\beta| \ge CR^\lambda$. Hence, the lefthand side of (18) is bounded by

$$CR^{-\lambda}\left(\int_{|x|\sim R}\left(\sum_{|\beta|=\lambda}\Big|x^\beta(k_j(x-y) - \sum_{|\alpha|\le L}\frac{(-y)^\alpha}{|\alpha|!}D^\alpha k_j(x))\Big|\right)^p dx\right)^{1/p}.$$

By Hölder's inequality and the Hausdorff Young inequality, this is

$$(19)\qquad \leq C \sum_{|\beta|=\lambda} R^{-\lambda+n(1/p-1/t')} \times \left(\int_{R^n} \left| D^\beta \left[\left(e^{-ix\cdot y} - \sum_{|\alpha|\leq L} \frac{(-i)^{|\alpha|} x^\alpha y^\alpha}{|\alpha|!} \right) m_j(x) \right] \right|^t dx \right)^{1/t}.$$

Note that the support of m_j is in $|x| \sim 2^j$. By the Leibnitz formula and the fact that $m_j \in M(s,\lambda)$ with norm bounded by $CB(m,s,\lambda)$, we have the integral in (19) is equal to

$$\left(\int_{|x|\sim 2^j} \left| \sum_{\eta+\kappa=\beta} C_{\eta,\kappa} D^\eta \left[e^{-ix\cdot y} - \sum_{|\alpha|\leq L} \frac{(-i)^{|\alpha|} x^\alpha y^\alpha}{|\alpha|!} \right] D^\kappa m_j(x) \right|^t dx \right)^{1/t}$$

$$\leq C \sum_{\eta+\kappa=\beta} \left(\int_{|x|\sim R} |D^\eta g(x,y) D^\kappa m_j(x)|^t dx \right)^{1/t}$$

where $g(x,y) = e^{-ix\cdot y} - \sum_{|\alpha|\leq L} (-i)^{|\alpha|} x^\alpha y^\alpha / |\alpha|!$. Taylor's theorem then implies this is

$$\leq C \sum_{\eta+\kappa=\beta} |y|^{L+1} 2^{(L+1-|\eta|)} \left(\int_{|x|\sim 2^j} |D^\kappa m_j(x)|^t \, dx \right)^{1/t}$$
$$\leq CB_t |y|^{L+1} 2^{j(L+1-\lambda+n/t)}.$$

From this and (19) it follows that the lefthand side of (18) is bounded by

$$CB_t R^{-\lambda} R^{n(1/p-1/t')} |y|^{L+1} 2^{j(L+1-\lambda+n/t)}.$$

When we rearrange terms, we have the righthand side of (18). This concludes the proof when λ is an integer.

For the case where λ is a noninteger, we assume $2^j R \leq 1$. If $2^j R > 1$, the proof is similar to Lemma 2.1.

Let $\lambda = l + \gamma$ with l the integer part of λ. To avoid confusion, let $|\cdot|_E$ denote the Euclidean norm and $|\cdot|_M$ the multi-index norm. For

β such that $|\beta|_M = l$, set

$$z_\beta = \frac{2^j |y|_E}{R} \frac{\beta}{|\beta|_E}.$$

Note that for $R < |x| < 2R$, we have

$$\sum_{|\beta|=l} |x^\beta \cos(x \cdot z_\beta)| \geq CR^l.$$

We multiply by these cosine factors noting that this corresponds to taking sums on the Fourier transform side. Hence, the lefthand side of (18) is bounded by

$$\begin{aligned}
&\sum_{|\beta|=l} CR^{-l} \\
&\quad \times \left(\int_{|x|\sim R} \left| x^\beta \cos(x \cdot z_\beta)(k_j(x-y) - \sum_{|\alpha|\leq L} \frac{(-y)^\alpha}{|\alpha|!} D^\alpha k_j(x)) \right|^p dx \right)^{1/p} \\
&\leq \sum_{|\beta|=l} CR^{-l+n(1/p-1/t')} \\
&\quad \times \left(\int_{R^n} |D^\beta [g(x+z_\beta, y) m_j(x+z_\beta) + g(x-z_\beta, y) m_j(x-z_\beta)]|^t \, dx \right)^{1/t},
\end{aligned}$$

with $g(x,y)$ defined as above. By definition, $|\pm z_\beta| < 2^j/2$ and the functions in the integrand are supported in $|x| \sim 2^j$. For these x, $|x \pm z_\beta| \leq C 2^j$. Hence, by Taylor's theorem,

$$|D^\eta g(x \pm z_\beta, y) \leq C|y|^{L+1} 2^{j(L+1-|\eta|)} \tag{20}$$

for each multi-index η.

By the Liebnitz formula for derivatives we have the above bounded by

(21)

$$\sum_{|\beta|=l} CR^{-l+n(1/p-1/t')} \left(\int_{|x|\sim 2^j} \left| \sum_{\eta+\kappa=\beta} C_{\eta,\kappa} \Psi_{\eta,\kappa}(x, z_\beta, y) \right|^t dx \right)^{1/t}$$

where

$$\Psi_{\eta,\kappa}(x, z_\beta, y) = D^\eta g(x+z_\beta, y) D^\kappa m_j(x+z_\beta) \\ + D^\eta g(x-z_\beta, y) D^\kappa m_j(x-z_\beta).$$

If we add and subtract $m_j(x)$ in the argument of D^κ and use Minkowski's inequality, we get that (21) is bounded by

$$C \sum_{|\beta|=l} R^{-l+n(1/p-1/t')} \sum_{\eta+\kappa=\beta} (I_1 + I_2 + I_3) \tag{22}$$

where

$$I_1 = \left(\int_{|x|\sim 2^j} |D^\eta g(x+z_\beta, y) D^\kappa (m_j(x+z_\beta) - m_j(x))|^t \, dx \right)^{1/t}$$

$$I_2 = \left(\int_{|x|\sim 2^j} |D^\eta g(x-z_\beta, y) D^\kappa (m_j(x-z_\beta) - m_j(x))|^t \, dx \right)^{1/t}$$

$$I_3 = \left(\int_{|x|\sim 2^j} |D^\eta (g(x+z_\beta, y) + g(x-z_\beta, y)) D^\kappa m_j(x)|^t \, dx \right)^{1/t}.$$

The $M(t, |\kappa| + \gamma)$ condition and (20) imply that I_1 and I_2 have the bound

$$CB(m, t, |\kappa| + \gamma) |y|^{L+1} 2^{j(L+1-|\eta|)} \left(\frac{|z_\beta|}{2^j} \right)^\gamma 2^{j(n/t-|\kappa|)}.$$

The $M(t, l)$ condition and (20) implies that I_3 has the bound

$$CB(m, t, l) |y|^{L+1} 2^{j(L+1-|\eta|)} 2^{j(n/t-|\kappa|)}.$$

Hence, we have (22) is bounded by

$$CB_s R^{-l+n(1/p-1+1/t)} |y|^{L+1} (2^{j(L+1-l-\gamma+n/t)} |z_\beta|^\gamma + 2^{j(1-l+n/t)}).$$

However,

$$|z_\beta|^\gamma = \left(\frac{2^j |y|}{R} \right)^\gamma \le R^{-\gamma}$$

since $2^j |y| \le 1$. Altogether, we have the bound

$$CB_s R^{-l+n(1/p-1+1/t)} |y|^{L+1} 2^{j(L+1-l-\gamma+n/t)} R^{-\gamma} \\ = CB_s (2^j |y|)^{(L+1-\lambda+n/t)} \left(\frac{|y|}{R} \right)^{(\lambda-n/t)} R^{n(1/p-1)}.$$

This completes the proof of the lemma. □

The proof of the following theorem is similar to that of Theorem 2.2.

Theorem 2.5. *Suppose that* $1 \le s \le 2$ *and* $1 \le p < \infty$. *Set* $t = \min(p', s)$ *and let* L *be a nonnegative integer. Let* λ *be a real number such that* $0 \le L < \lambda - n/t < L+1$. *If* $m \in M(s,\lambda)$, *then there exists a* C *such that for each* $R > 0$ *and* $|y| < R/2$

$$(23) \quad \left(\int_{|x|\sim R} |K_N(x-y) - \sum_{|\alpha|\le L} \frac{(-y)^\alpha}{|\alpha|!} D^\alpha K_N(x)|^p \, dx \right)^{1/p} \le C B_s \left(\frac{|y|}{R} \right)^{\lambda - n/t} R^{n(1/p-1)}$$

where C *is independent of* y, R *and* N.

Proof. As in the proof of Theorem 2.2, the integral on the left in (23) is dominated by

$$\sum_{j=-N}^{N} \left(\int_{|x|\sim R} \left| k_j(x-y) - \sum_{|\alpha|\le L} \frac{(-y)^\alpha}{|\alpha|!} D^\alpha k_j(x) \right| dx \right)^{1/p}.$$

The terms in the sum are estimated by considering the two cases $2^j|y| > 1$ and $2^j|y| \le 1$ and then applying Lemmas 2.3 and 2.4. The proof is then finished as in Theorem 2.2. □

The proof of the following lemmas and theorems are similar to those in [**2**] and are provided here for completeness.

Lemma 2.6. *Assume that* $1 \le s \le 2$, $n/s < \lambda < n$, *and* $m \in M(s,\lambda)$. *If* $1 < p < n/(n-\lambda)$ *and* $p(n-\lambda) - n < \sigma < n(p-1)$ *and* f *is integrable, then*

$$||K_N * f||_{p,\sigma} \le C B_s ||f||_{p,\sigma}$$

where C *is independent of* m, N *and* f.

Proof. We apply Lemma 1.3 with $K(x,y) = K_N(x-y)$, $a = -\sigma$, $b = 0$ and $U(x) = W(x) = |x|^\sigma$. By Lemma 1.2 and Proposition 1.4, (7) is satisfied. Thus, we want to show that (9) and (10) have the bound $CB_s^p||f||_{p,\sigma}^p$.

For (9) we have

$$\int_0^\infty \left(\int_{|y|<r/4} \left(\int_{r/2<|x|<2r} |K_N(x-y)|^p |x|^\sigma \, dx \right)^{1/p} \times |f(y)| \, dy \right)^p \frac{dr}{r}. \tag{24}$$

Theorem 2.2 and the bounds on $|x|$ and $|y|$ imply that this is bounded by

$$CB_s^p \int_0^\infty r^{\sigma-1+n(1-p)} \left(\int_{|y|<r/4} |f(y)| \, dy \right)^p dr.$$

But, since $\sigma < n(p-1)$, Lemma 1.5 applies to give the bound

$$CB_s^p \int_{R^n} |f(y)|^p |y|^{np-(n(p-1)-\sigma)-n} \, dy = CB_s^p \int_{R^n} |f(y)|^p |y|^\sigma \, dy$$

which is the desired bound for (24).

We now turn to the estimate of (10). We have by hypothesis that $\sigma > p(n-\lambda)$. Hence,

$$\frac{np}{\sigma+n} < \frac{n}{n-\lambda},$$

and we can choose q such that

$$\max\left(p, \frac{np}{\sigma+n}\right) < q < \frac{n}{n-\lambda}.$$

With this q, we apply Hölder's inequality on the inner integral, and the bounds on $|x|$ and $|y|$ to obtain (10) are less than or equal to a constant times

$$\int_0^\infty \left(\int_{|y|>4r} \left(\int_{|y|/2<|x-y|<2|y|} |K_N(x-y)|^q \frac{dx}{r^n} \right)^{1/q} \times |f(y)| \, dy \right)^p r^{\sigma+n-1} \, dr.$$

By Theorem 2.2, this is

$$\leq C \int_0^\infty \left(\int_{|y|>4r} |y|^{-n+n/q} |f(y)|\, dy \right)^p r^{\sigma+n-np/q-1}\, dr. \tag{25}$$

We set $t = \sigma + n - np/q > 0$ and apply Lemma 1.6 to obtain (25) is

$$\leq CB_s^p \int_{R^n} |f(y)|^p |y|^\sigma\, dy.$$

This completes the proof of the lemma. □

By duality, we have

Lemma 2.7. *Assume that* $1 \leq s \leq 2$*,* $n/s < \lambda < n$*, and* $m \in M(s, \lambda)$*. If* $n/\lambda < p < \infty$ *and* $-n < \sigma < p\lambda - n$ *and* f *is integrable, then*

$$||K_N * f||_{p,\sigma} \leq CB_s ||f||_{p,\sigma}$$

where C *is independent of* m*,* N *and* f*.*

Theorem 2.8. *Assume that* $1 \leq s \leq 2$*,* $n/s < \lambda < n$*,* $m \in M(s, \lambda)$ *and* $1 < p < \infty$*. If* σ *is a real number such that*

$$\max(-n, -p\lambda) < \sigma < \min(n(p-1), p\lambda)$$

and f *is a Schwartz function, then*

$$||T_m f||_{p,\sigma} \leq CB_s ||f||_{p,\sigma}$$

where C *is independent of* m *and* f*.*

Proof. We fix p and σ satisfying the hypothesis of the theorem and observe that

$$\frac{np}{np - \sigma} < \min\left(p, \frac{n}{n - \lambda}\right).$$

Thus, we can choose a $\tilde{p}_0$ such that

$$\frac{np}{np - \sigma} < \tilde{p}_0 < \min\left(p, \frac{n}{n - \lambda}\right)$$

that also satisfies

$$\frac{n}{\tilde{p}_0} < n - \frac{\sigma}{p}.$$

Hence, there is an $\varepsilon > 0$ such that

$$n - \frac{\sigma}{p} - \lambda < \frac{n}{\tilde{p}_0 - \varepsilon} < n - \frac{\sigma}{p}$$

and

$$\tilde{p}_0 - \varepsilon < \min\left(p, \frac{n}{n - \lambda}\right).$$

We set $p_0 = \tilde{p}_0 - \varepsilon$ and observe

$$p_0(n - \lambda) - n < \frac{\sigma p_0}{p} < n(p_0 - 1)$$

with

$$1 < p_0 < \min\left(p, \frac{n}{n - \lambda}\right).$$

Thus, p_0 and $\sigma p_0/p$ satisfy the hypothesis of Lemma 2.6 from which we have

$$\int_{R^n} |K_N * f(x)|^{p_0} |x|^{\sigma p_0/p}\, dx \le C B_s^{p_0} \int_{R^n} |f(x)|^{p_0} |x|^{\sigma p_0/p}\, dx.$$

Similarly, choose p_1 such that

$$\max\left(\frac{n}{\lambda}, p\right) < p_1 < \infty$$

and

$$-n < \frac{\sigma p_1}{p} < p_1 \lambda - n.$$

Lemma 2.7 implies

$$\int_{R^n} |K_N * f(x)|^{p_1} |x|^{\sigma p_1/p}\, dx \le C B_s^{p_1} \int_{R^n} |f(x)|^{p_1} |x|^{\sigma p_1/p}\, dx.$$

Consequently, by the Riesz convexity theorem, we have

$$||K_N * f||_{p,\sigma} \le C B_s ||f||_{p,\sigma}.$$

The conclusion of the theorem then follows from Fatou's lemma. □

3. Main result. We now turn to the proof of Theorem 1.1.

We observe that if $\lambda > n/s$, $s > 1$, and $p\lambda < n(p-1)$, then the theorem is a consequence of Theorem 2.8. For $\lambda \geq n$, $s = 1$ and $\sigma < n(p-1)$, the theorem follows from Proposition 1.4.

To complete the proof, it suffices to consider the case for σ, p such that

$$\min(p\lambda, -n + p(\lambda + n - n/s) > \sigma > n(p-1)$$

with

$$\frac{\sigma + n}{np} = l + \gamma$$

where l is the integer part of $(\sigma + n)/(np)$ and $0 < n\gamma < 1$.

We fix $p, s, \lambda > n/s$ and $\sigma > n(p-1)$ satisfying the hypothesis and let $t = \min(p', s)$. Then

$$n(lp - 1) = \sigma - np\gamma < \sigma$$

and

$$n(lp - 1) + p = \sigma p(1 - n\gamma) > \sigma$$

since $1 - n\gamma > 0$. Hence,

$$n(lp - 1) < \sigma < n(lp - 1) + p. \tag{26}$$

Also, since $s \leq t$, we have by hypothesis

$$\sigma < -n + p(n + \lambda - n/t)$$

from which we obtain

$$n(l - 1 + 1/t) < \lambda.$$

Furthermore, by the monotonicity of the $M(s, \lambda)$ condition, we can assume without loss of generality that

$$n(l - 1 + 1/t) < \lambda < n(l - 1 + 1/t) + 1.$$

With this inequality, λ satisfies the hypothesis of Theorem 2.5.

Now let

$$K(x,y) = K_N(x-y) - \sum_{|\beta| \le n(l-1)} \frac{(-y)^\beta}{|\beta|!} D^\beta K_N(x)$$

$a = -\sigma$, $b = 0$, and $U(x) = W(x) = |x|^\sigma$ in Lemma 1.3.

Since f is in $S_{0,0}$ and thus has vanishing moment of all orders

$$\int_{R^n} K(x,y) f(y)\, dy = \int_{R^n} K_N(x-y) f(y)\, dy,$$

and the inequality (7) holds by Lemma 1.2 and Proposition 1.4. Hence, we need to show that (9) and (10) have the bound $CB_s^p ||f||_{p,\sigma}^p$, i.e.,

$$\int_0^\infty \left(\int_{|y|<r/4} \left(\int_{r/2<|x|<2r} |K_N(x-y)|^p |x|^\sigma \, dx \right)^{1/p} \times |f(y)|\, dy \right)^p \frac{dr}{r} \tag{27}$$

and

$$\int_0^\infty \left(\int_{|y|>4r} \left(\int_{r/2<|x|<2r} |K_N(x-y)|^p |x|^\sigma \, dx \right)^{1/p} \times |f(y)|\, dy \right)^p \frac{dr}{r} \tag{28}$$

have the bound $CB_s^p ||f||_{p,\sigma}^p$.

For (27), replace $|x|^\sigma$ by Cr^σ. Then Theorem 2.5 implies (27) has the bound

$$CB_s^p \int_0^\infty \left(\int_{|y|<r/4} \left(\frac{|y|}{r} \right)^{\lambda - n/t} |f(y)|\, dy \right)^p r^{\sigma + n(1-p) - p(\lambda - n/t) - 1}\, dr.$$

Since $0 < -\sigma - n + p(n + \lambda - n/t)$, Lemma 1.5 implies the latter is

$$\le CB_s^p \int_{R^n} |f(y)|^p |y|^{p(\lambda - n/t)} |y|^{np + \sigma + n(1-p) - p(\lambda - n/t) - n}\, dy = CB_s^p ||f||_{p,\sigma}^p.$$

We now consider (28). Note that the inner integral is bounded by a constant times the sum of

$$\int_{r/2<|x|<2r} |K_N(x)|^p r^\sigma \, dx \tag{29}$$

and

$$\sum_{|\beta|\le n(l-1)} \int_{r/2<|x|<2r} |y|^{|\beta|p} |D^\beta K_N(x)|^p r^\sigma \, dx. \tag{30}$$

By Theorem 2.2, (29) and (30) are bounded by

$$CB_s^p r^{\sigma+n(1-p)} \left(\frac{|y|}{r}\right)^{n(1-p)}$$

and

$$CB_s^p r^{\sigma+n(1-p)} \left(\frac{|y|}{r}\right)^{|\beta|p},$$

respectively.

However, since $|y|/r > 1$, these are bounded by

$$Cr^{\sigma+n(1-p)} \left(\frac{|y|}{r}\right)^{np(l-1)}.$$

Hence, (28) is bounded by

$$CB_s^p \int_0^\infty \left(\int_{|y|>4r} |y|^{n(l-1)} |f(y)| \, dy\right)^p r^{\sigma+n-npl-1} \, dr.$$

As observed above, $\sigma > n(pl-1)$. Hence we can apply Lemma 1.6 to obtain the bound

$$CB_s^p \int_{R^n} |f(y)|^p |y|^{pn(l-1)} |y|^{np+\sigma+n-npl-n} \, dy = CB_s^p ||f||_{p,\sigma}^p.$$

Thus, $||K_N * f||_{p,\sigma} \le CB_s ||f||_{p,\sigma}$ and an application of Fatou's lemma obtains the theorem. $\square$

4. Applications. We have the following definition for the $S^k_{1,0}$ symbol class of pseudo-differential operators.

Definition 4.1. Let Ω be an open set of $\mathbf{R}^n$ and $k \in \mathbf{R}$. We define the symbol class $S^k_{1,0}$ to consist of the set of $p \in C^\infty(\Omega \times \mathbf{R}^n)$ with the property that, for any compact $A \subset \Omega$, and multi-indices α, β, there exists a constant $C_{A,\alpha,\beta}$ such that

$$|D^\beta_x D^\alpha_\xi p(x,\xi)| \le C_{A,\alpha,\beta}(1+|\xi|)^{k-|\alpha|} \tag{31}$$

for all $x \in A$ and $\xi \in \mathbf{R}^n$.

We may assume, without loss of generality, that p has compact support in the x variable.

For each symbol $p \in S^k_{1,0}$, we have an associated operator, $p(x,D)$, defined by

$$p(x,D)f = \int_{R^n} p(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi. \tag{32}$$

Also, we define $p_\eta(\xi)$ to be the inverse Fourier transform of p in the x variable, i.e.,

$$p_\eta(\xi) = \frac{1}{(2\pi)^n}\int_{R^n} p(x,\xi)e^{ix\cdot\eta}\,dx.$$

The following lemma shows that p_η belongs to $M(s,\lambda)$ for each s and λ.

Lemma 4.2. *Let $1 \le s \le 2$ and $\lambda > 0$. If $k \ge 0$ and $p \in S^k_{1,0}$, then $p_\eta \in M(s,\lambda)$ for each fixed η, and moreover,*

$$B(p_\eta, s, \lambda) \le \frac{C}{1+|\eta|^{2n}} \tag{33}$$

where C is independent of η.

Proof. We will show that the lemma holds whenever λ is a positive integer, and the general case will follow from the monotonicity of the $M(s,\lambda)$ condition.

Let λ be a positive integer. Given a multi-index α and $r > 0$, we prove that for each η,

$$\left(\int_{r<|\xi|<2r} |D_\xi^\alpha p_\eta(\xi)|^s \, d\xi\right)^{1/s} \leq \frac{Cr^{n/s-|\alpha|}}{1+|\eta|^{2n}}. \tag{34}$$

Equation (33) will then follow from the definition of the $M(s,\lambda)$ condition.

We observe that for an arbitrary multi-index β,

$$|\eta^\beta D_\xi^\alpha p_\eta(\xi)| \leq C \int_{R^n} |D_x^\beta D_\xi^\alpha p(x,\xi)| \, dx.$$

Thus, since we have assumed that p has compact support in the x variable and β is arbitrary, we have

$$|D_\xi^\alpha p_\eta(\xi)| \leq C \frac{(1+|\xi|)^{k-|\alpha|}}{1+|\eta|^{2n}}. \tag{35}$$

The righthand side of (34) follows readily from (35). This concludes the proof of the lemma. □

Theorem 4.3. *Let σ be a real number satisfying the hypothesis of Theorem 1.1. Let $k \leq 0$ and assume that $1 < p < \infty$. If $P \in S_{1,0}^k$, then*

$$||P(\cdot, D)f||_{p,\sigma} \leq C||f||_{p,\sigma} \tag{36}$$

for $f \in S_{0,0}$ with C independent of f.

Proof. All of the functions involved in the definition of $P(x,D)f$ are absolutely integrable. Hence, we may switch the order of integration to obtain

$$P(x,D)f = C \int_{R^n} e^{-ix\cdot\eta} \left[\int_{R^n} e^{ix\cdot\xi} P_\eta(\xi) \hat{f}(\xi) \, d\xi\right] d\eta.$$

We note that the inner integral is precisely $T_{P_\eta} f(x)$. Consequently, the lefthand side of (36) is equal to

$$\left(\int_{R^n} \left|\int_{R^n} e^{-ix\cdot\eta} T_{P_\eta} f(x) \, dx\right|^p |x|^\sigma \, dx\right)^{1/p}$$

and by Minkowski's integral inequality, this is bounded by

$$\int_{R^n} \left(\int_{R^n} |T_{P_\eta} f(x)|^p |x|^\sigma \, dx \right)^{1/p} d\eta.$$

Lemma 4.2 and Theorem 1.1 imply that this is bounded by the righthand side of (36). This concludes the proof of the theorem. □

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Department of Mathematics, Physical Sciences and Engineering Technology, West Texas A&M University, Canyon, TX 79016.