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A LATTICE PROOF OF A MODULAR IDENTITY

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ABSTRACT. We give a lattice rearrangement proof of a six-parameter identity whose terms have the form $x^\alpha T(k_1,l_1) \cdot T(k_2,l_2)$, where $T(k,l) = \sum_{-\infty}^{\infty} x^{kn^2+ln}$. A new balanced Q^2 identity is then established through its use.

1. Introduction. In this paper we give a new proof (Theorem 1) of a variant of a fundamental identity we published earlier in [**5**]. This proof is accomplished by an adroit rearrangement of the indexing lattices in the identity.

The formula in Theorem 1 is also shown (Theorem 3) to be equivalent to the earlier formula and is employed here since its form is convenient for carrying out the rearrangement. The formula is then used in Theorem 4 to prove a new balanced trinomial Q^2 identity (see [**5**] for the meaning of this terminology), where Q is the familiar single-variable quintuple product. The proof itself consists of assigning sets of values to the six parameters in the formula, thereby producing a small family of identities, and then showing that the Q^2 identity is equal to a certain linear combination of these identities.

2. The fundamental identity. Throughout this paper we will use the single-variable function T (cf. [**3**, (2)]) defined by

$$\begin{aligned} T(k,l) &\stackrel{\text{def}}{=} \sum_{-\infty}^{\infty} x^{kn^2+ln} \\ &= \prod_{n=1}^{\infty} (1-x^{2kn})(1+x^{2kn-k+l})(1+x^{2kn-k-l}). \end{aligned}$$

We call an identity a "T^2 identity" if each of its terms has the form $x^\alpha T(k_1,l_1)T(k_2,l_2)$. We also say that a T^2 identity is "balanced" if the first component pair (k_1,k_2) in each of its terms is the same.

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In working with T-functions, it is often important to re-index their sums. This can be accomplished by simple transformation rules, which are useful in putting $T(k,l)$ in "reduced form," i.e., where $0 \le l \le k$ (cf. [**3**, p. 780]). These rules are: the *negative rule* [**3**, (13)],

$$T(k-l) = T(k,l), \tag{2.1}$$

the *single-step formula* [**3**, (14)],

$$T(k,l) = x^{k-l}T(k,2k-l), \tag{2.2}$$

and the *general transformation formula*: If $l = 2kq + r$, where $q \in Z$, then

$$T(k,l) = x^{-q^2k-qr}T(k,r). \tag{2.3}$$

The proof of (2.3) consists of observing that

$$\begin{aligned} T(k,l) &= \sum_{-\infty}^{\infty} x^{kn^2+(2kq+r)n} \\ &= \sum_{-\infty}^{\infty} x^{k(n-q)^2+(2kq+r)(n-q)} \\ &= x^{-q^2k-qr}T(k,r). \end{aligned}$$

Note that when we use (2.3) to put $T(k,l)$ into reduced form, we take r so that $-k < r \le k$ and then use (2.1) if necessary. Throughout the rest of this paper, we will give all T-functions with numerical arguments in reduced form.

The following general identity is of importance in generating special sets of identities from which proofs of other identities can be made, as we will see in Theorem 4.

Theorem 1. *Suppose that* $m, k, u, v \in \mathbf{Z}^+$ *are such that* $uv < 2m$, *and let* $e, f \in (1/2)\mathbf{Z}$. *If the polynomials in* n *in* (2.5) *and* (2.6) *have integer coefficients, then we have*

$$\sum_{n\in M} x^{\alpha_n}T(k_1,l_{1n})T(k_2,l_{2n}) = \sum_{n\in M'} x^{\alpha_n}T(k_1,l'_{1n})T(k_2,l'_{2n}), \tag{2.4}$$

where

$$\alpha_n = \frac{2u^2vk}{m}n^2 + 2uen, \qquad (2.5)$$
$$k_1 = uk, \qquad k_2 = (2m-uv)vk,$$

$$\begin{aligned} l_{1n} &= (2u^2vk/m)n + ue + f \\ l'_{1n} &= (2u^2vk/m)n + ue - f \\ l_{2n} &= (2m-uv)((2uvk/m)n + e) - vf \\ l'_{2n} &= (2m-uv)((2uvk/m)n + e) + vf, \end{aligned} \qquad (2.6)$$

and M and M' are any complete residue systems $\pmod{m}$.

Proof. We begin by showing that the two polynomials

$$L_n(x,y) = L_n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = k_1x^2 + k_2y^2 + l_{1n}x + l_{2n}y + \alpha_n$$

and

$$R_n(x,y) = R_n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = k_1x^2 + k_2y^2 + l'_{1n}x + l'_{2n}y + \alpha_n$$

are related by an affine map, viz.

$$L_n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = R_{n'}\left(A\begin{bmatrix} x \\ y \end{bmatrix} + B\right), \qquad (2.7)$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{m}\begin{bmatrix} uv-m & (2m-uv)v \\ u & m-uv \end{bmatrix} \qquad (2.8)$$

and

$$B = \begin{bmatrix} g \\ h \end{bmatrix} = \frac{1}{m}\begin{bmatrix} uv(n-n') \\ u(n-n') \end{bmatrix}.$$

This is verified by routine calculation using (2.5), (2.6) and (2.8) as follows:

$$(2.9)\quad R_{n'}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g \\ h \end{bmatrix}\right)$$
$$\begin{aligned}
&= k_1(ax+by+g)^2 \\
&\quad + k_2(cx+dy+h)^2 + l'_{1n'}(ax+by+g) \\
&\quad + l'_{2n'}(cx+dy+h) + \alpha_{n'} \\
&= (k_1a^2+k_2c^2)x^2 + 2(k_1ab+k_2cd)xy + (k_1b^2+k_2d^2)y^2 \\
&\quad + (2k_1ag + 2k_2ch + l'_{1n'}a + l'_{2n'}c)x \\
&\quad + (2k_1bg + 2k_2dh + l'_{1n'}b + l'_{2n'}d)y + R_{n'}(g,h) \\
&= ukx^2 + (2m-uv)vky^2 + \left[\frac{2u^2vk}{m}n + ue + f\right]x \\
&\quad + \left[(2m-uv)\left(\frac{2uvk}{m}n + e\right) - vf\right]y \\
&\quad + \frac{2u^2vk}{m}n^2 + 2uen \\
&= k_1x^2 + k_2y^2 + l_{1n}x + l_{2n}y + \alpha_n \\
&= L_n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).
\end{aligned}$$

We next specialize (2.7) to a sub-lattice by setting

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v & m \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} + \begin{bmatrix} n'-n \\ 0 \end{bmatrix},$$

so

(2.10)

$$\begin{aligned}
A\begin{bmatrix} x \\ y \end{bmatrix} + B &= \frac{1}{m}\begin{bmatrix} uv-m & (2m-uv)v \\ u & m-uv \end{bmatrix}\left(\begin{bmatrix} v & m \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} + \begin{bmatrix} n'-n \\ 0 \end{bmatrix}\right) \\
&\quad + \frac{1}{m}\begin{bmatrix} uv(n-n') \\ u(n-n') \end{bmatrix} \\
&= \begin{bmatrix} v & uv-m \\ 1 & u \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} + \begin{bmatrix} n-n' \\ 0 \end{bmatrix}.
\end{aligned}$$

Thus, (2.7) becomes

$$(2.11)\quad L_n\left(\begin{bmatrix} v & m \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} + \begin{bmatrix} n'-n \\ 0 \end{bmatrix}\right) = R_{n'}\left(\begin{bmatrix} v & uv-m \\ 1 & u \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} + \begin{bmatrix} n-n' \\ 0 \end{bmatrix}\right)$$

or, in its nonmatrix form,

$$(2.12)\quad L_n(vs+mt+n'-n, s) = R_{n'}(vs+(uv-m)t+n-n', s+ut).$$

In what follows, the symbol $\sum_i$ means $\sum_{i=-\infty}^{\infty}$. We also take $M = M' = \{0, 1, \dots, m-1\}$. Note that the affine maps that are used (designated in the parentheses at the right) are one-to-one between the respective sublattices in $\mathbf{Z}^2$.

We can now prove (2.4) by transforming the sums on the left into the sums on the right.

$$\begin{aligned}
&\sum_{n=0}^{m-1} x^{\alpha_n} T(k_1, l_{1n}) T(k_2, l_{2n}) \\
&= \sum_{n=0}^{m-1}\sum_i\sum_j x^{L_n(i,j)} \\
&= \sum_{n=0}^{m-1}\sum_s\sum_r x^{L_n(vs+r-n,s)} && \left(\begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} - \begin{bmatrix} n \\ 0 \end{bmatrix}\right) \\
&= \sum_{n'=0}^{m-1}\sum_s\sum_t\sum_{n=0}^{m-1} x^{L_n(vs+mt+n'-n,s)} && (r = mt+n') \\
&= \sum_{n'=0}^{m-1}\sum_s\sum_t\sum_{n=0}^{m-1} x^{R_{n'}(vs+(uv-m)t+n-n',s+ut)} && \text{by (2.12)} \\
&= \sum_{n'=0}^{m-1}\sum_j\sum_t\sum_{n=0}^{m-1} x^{R_{n'}(vj-(mt-n)-n',j)} && \left(\begin{bmatrix} j \\ t \end{bmatrix} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}\right) \\
&= \sum_{n'=0}^{m-1}\sum_j\sum_r x^{R_{n'}(vj-r-n',j)} && (r = mt-n)
\end{aligned}$$

$$= \sum_{n'=0}^{m-1} \sum_i \sum_j x^{R_{n'}(i,j)} \qquad \left(\begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} -1 & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ j \end{bmatrix} - \begin{bmatrix} n' \\ 0 \end{bmatrix} \right)$$

$$= \sum_{n'=0}^{m-1} x^{\alpha_{n'}} T(k_1, l'_{1n'}) T(k_2, l'_{2n'}).$$

In order to show that the indexing sets M and M' can be taken to be any complete residue system $(\mathrm{mod}\ m)$, we prove that each term of the identity actually remains invariant when index values are increased by m.

On the lefthand side set $S(n) \stackrel{\text{def}}{=} x^{\alpha_n} T(k_1, l_{1n}) T(k_2, l_{2n})$. To prove the invariance on this side, it suffices to show that

$$S(n+m) = S(n), \qquad \forall\, n \in \mathbf{Z}. \tag{2.13}$$

To do this, we have

$$\begin{aligned} S(n+m) &= x^{(2u^2vk/m)(n+m)^2 + 2ue(n+m)} T\left(k_1, \frac{2u^2vk}{m}(n+m) + ue + f\right) \\ &\quad \cdot T\left(k_2, (2m - uv)\left(\frac{2uvk}{m}(n+m) + e\right) - vf\right) \\ &= x^{a_n + 4u^2vkn + 2u^2vkm + 2uem} T(k_1, 2u^2vk + l_{1n}) \\ &\quad \cdot T(k_2, 2(2m - uv)uvk + l_{2n}) \\ &= x^{\alpha_n + 4u^2vkn + 2u^2vkm + 2uem} T(k_1, 2k_1uv + l_{1n}) \\ &\quad \cdot T(k_2, 2k_2u + l_{2n}) \\ &= x^{\beta} S(n), \end{aligned}$$

where the final term is reduced using (2.3). The final exponent,

$$\beta = 4u^2vkn + 2u^2vkm + 2uem - (uv)^2k_1 - uvl_{1n} - u^2k_2 - ul_{2n},$$

is readily shown to be 0, using (2.5) and (2.6).

On the righthand side, set $S'(n) \stackrel{\text{def}}{=} x^{\alpha_n} T(k_1, l'_{1n}) T(k_2, l'_{2n})$. That this remains invariant for $n \in M'$ follows from the fact that $S'(n)$ is obtained from $S(n)$ by merely changing the sign of f in (2.6), so

$$S'(n+m) = S'(n), \qquad \forall\, n \in \mathbf{Z}. \qquad \square \tag{2.14}$$

Remarks. 1. To see the combinatorial nature of the proof of Theorem 1, note that each of the T^2 terms in equation (2.4) can be written as a doubly-indexed sum of the form

$$\sum_{(i,j)\in\mathbf{Z}^2} x^{k_1 i^2 + k_2 j^2 + l_1 i + l_2 j + \alpha}.$$

Because all the coefficients in such a sum are 1, equation (2.4) asserts that the powers of x on the two sides are the same, so the right side is merely a rearrangement of the left. Equation (2.11) shows just how the powers of x in the m sums on the left are rearranged and collected together to form the m sums on the right. In particular, the terms of, say, the nth sum on the right are generated on the left by letting the index point (i,j) range over certain lattices in the indexing planes of the m sums on the left. Specifically, the lattices

$$\mathcal{L}_{n,n'} : \begin{bmatrix} i \\ j \end{bmatrix} = s \begin{bmatrix} v \\ 1 \end{bmatrix} + t \begin{bmatrix} m \\ 0 \end{bmatrix} + \begin{bmatrix} n'-n \\ 0 \end{bmatrix}$$

and

$$\mathcal{R}_{n,n'} : \begin{bmatrix} i' \\ j' \end{bmatrix} = s \begin{bmatrix} v \\ 1 \end{bmatrix} + t \begin{bmatrix} uv-m \\ u \end{bmatrix} + \begin{bmatrix} n-n' \\ 0 \end{bmatrix},$$

associated with sum n on the left and n' on the right, respectively, satisfy

$$L_{n,n'}\left(\begin{bmatrix} i \\ j \end{bmatrix} \right) = R_{n,n'}\left(\begin{bmatrix} i' \\ j' \end{bmatrix} \right).$$

For each n, the m lattices $\mathcal{L}_{n1}, \dots, \mathcal{L}_{nm}$ form a partition of the nth indexing plane on the left and for each n', the m lattices $\mathcal{L}'_{1n'}, \dots, \mathcal{L}'_{mn'}$ form a partition of the n'th indexing plane on the right.

2. When u is even or v is odd, it is worth mentioning that (2.4) is still a correct formula when each T is replaced by T_1, where $T_1(k,l) \stackrel{\text{def}}{=} \sum_{-\infty}^{\infty} (-1)^n x^{kn^2+ln} = \prod_{n=1}^{\infty} (1-x^{2kn})(1-x^{2kn-k+l})(1-x^{2kn-k-l})$.

It is interesting to compare Theorem 1 with the following theorem, which was proved in a very different way in [**5**, Theorem 1].

Theorem 2. *Suppose that $m,k,u,v \in \mathbf{Z}^+$ are such that $uv < 2m$, and let $e, f \in (1/2)\mathbf{Z}$. If the polynomials in n in* (2.16) *and* (2.17) *have* *integer coefficients, then we have*

$$\sum_{n\in M} x^{\alpha_n} T(k_1, l_{1n}) T(k_2, l_{2n}) = \sum_{n\in M'} x^{\alpha_n} T(k_1, l'_{1n}) T(k_2, l'_{2n}), \tag{2.15}$$

where

$$\begin{gathered} \alpha_n = \frac{2vk}{m} n^2 + 2en, \\ k_1 = uk, \qquad k_2 = (2m - uv)vk, \end{gathered} \tag{2.16}$$

$$\begin{aligned} l_{1n} &= \frac{2uvk}{m} n + uef \\ l'_{1n} &= \frac{2uvk}{m} n + ue - f \\ l_{2n} &= (2m - uv)\left(\frac{2vk}{m} n + e\right) - vf \\ l'_{2n} &= (2m - uv)\left(\frac{2vk}{m} n + e\right) + vf, \end{aligned} \tag{2.17}$$

and M and M' are any complete residue systems $\pmod m$.

The relationship between these two theorems is expressed in the next theorem.

Theorem 3. *Theorem* 1 *is equivalent to Theorem* 2.

Proof. Let Theorem 1* and Theorem 2*, respectively, be Theorem 1 and Theorem 2 with the additional hypothesis that $(m, u) = 1$. We will show that

$$\text{Theorem 1} \Longleftrightarrow \text{Theorem 1}^* \Longleftrightarrow \text{Theorem 2}^* \Longleftrightarrow \text{Theorem 2}.$$

It is easy to see that Theorem 1* $\Longleftrightarrow$ Theorem 2*, for if u and m are relatively prime, then un runs through a complete residue system mod m as n does. Replacing n by un in (2.16) and (2.17) yields the values in (2.5) and (2.6). Conversely, we can absorb the u and u^2 into n and n^2 in (2.5) and (2.6) to obtain (2.16) and (2.17). It is also clear that Theorem 1 $\Longrightarrow$ Theorem 1* and Theorem 2 $\Longrightarrow$ Theorem 2*, since a starred theorem is the original theorem with the $(m, u) = 1$ condition added. Thus, it remains to prove that Theorem 1* $\Longrightarrow$ Theorem 1 and Theorem 2* $\Longrightarrow$ Theorem 2.

We begin the proof of the first implication by assuming that m, k, u, v, e and f are numbers that satisfy the hypothesis of Theorem 1, with $\alpha_n, k_1, k_2, l_{1n}, l_{2n}, l'_{1n}$ and l'_{2n} being defined as in (2.5) and (2.6). Let $d = \gcd(m, u)$, and set

$$\bar{m} = \frac{m}{d}, \qquad \bar{k} = dk, \qquad \bar{u} = \frac{u}{d}, \qquad \bar{v} = v, \\ \bar{e} = dk \quad \text{and} \quad \bar{f} = f, \tag{2.18}$$

with $\bar{\alpha}_n, \bar{k}_1, \dots, \bar{l}'_{2n}$ being defined in terms of the $\bar{m}, \dots, \bar{f}$ as in (2.5) and (2.6). Since $(\bar{m}, \bar{u}) = 1$ and we have assumed that Theorem 1* is true, Theorem 1 holds in the barred letters. A straightforward calculation shows that $\bar{\alpha}_n = (2\bar{u}^2\bar{v}\bar{k}/\bar{m})n^2 + \bar{u}\bar{e} + \bar{f} = \alpha_n$, using (2.18). Similarly, $\bar{k}_1 = k_1, \dots, \bar{l}'_{2n} = l'_{2n}$. Consequently, equation (2.4) also holds in unbarred letters as n runs through a complete residue system $\pmod{\bar{m}}$. To see that this implies that (2.4) holds as n runs over the full residue system modulo m, simply observe that a complete residue system $\pmod{m}$ can be partitioned into d complete residue systems $\pmod{\bar{m}}$. Thus (2.4), for the full modulus m, is just the sum of d copies of (2.4) with modulus $\bar{m}$.

To show that Theorem 2* implies Theorem 2, assume that Theorem 2 holds when $(m, u) = 1$ and let $d = \gcd(m, u)$. Unlike the previous case with Theorem 1, it is not necessarily true that an application of Theorem 2 is a sum of d copies of the same equation using Theorem 2*. It is true, however, that any equation obtained using Theorem 2 can be obtained by adding d applications of Theorem 2* for correctly chosen parameters. To do this, assume that r lies in the interval $0 \le r \le d-1$ and consider the parameters

$$\bar{m} = \frac{m}{d}, \qquad \bar{k} = dk, \qquad \bar{u} = \frac{u}{d}, \\ \bar{v} = v, \qquad \bar{e} = d\left(e + \frac{2vkr}{m}\right) \quad \text{and} \quad \bar{f} = f. \tag{2.19}$$

Since $(\bar{m}, \bar{u}) = 1$, Theorem 2* holds for the six parameters in (2.19) where the variables in (2.16) and (2.17) are all barred as before. We find that $\bar{k}_1 = k_1$, $\bar{k}_2 = k_2$,

$$l_{1,dn+r} = \bar{l}_{1n}, \qquad l_{2,dn+r} = \bar{l}_{2n}, \qquad \alpha_{dn+r} = \bar{\alpha}_n + \frac{2vkr^2}{m} + 2er,$$

and similarly for $l'_{1,dn+r}$ and $l'_{2,dn+r}$, since these are obtained from $l_{1,dn+r}$ and $l_{2,dn+r}$ by sending f to $-f$. The equation obtained in (2.15), as n runs through a complete residue system mod m, is a sum of d equations of the form

$$\sum_{n=0}^{\bar{m}-1} x^{\alpha_{dn+r}} T(k_1, l_{1,dn+r}) T(k_2, l_{2,dn+r}) = \sum_{n=0}^{\bar{m}-1} x^{\alpha_{dn+r}} T(k_1, l'_{1,dn+r}) T(k_2, l'_{2,dn+r}).$$

For each r, this equation is just $x^{(2vkr^2/m)+2er}$ times the equation obtained by using Theorem 2* with the parameters in (2.19). This completes the proof. $\square$

3. A balanced Q^2 identity. In this section we prove a new identity using Theorem 1. In this proof a small family of identities is derived from (2.4) by giving sets of values to its parameters. The identity in question is then verified by showing it is a certain linear combination of these identities.

Within the set of balanced T^2 identities there is a special and interesting subset—the balanced Q^2 identities. Here Q stands for the usual quintuple product

$$Q(m,k) \stackrel{\text{def}}{=} \prod_{n \in S} (1 - x^n) = \sum_{-\infty}^{\infty} x^{m(3n^2+n)/2} (x^{-3kn} - x^{3kn+k}),$$

where $S = \{n \in Z^+ : n \equiv 0, \pm k, \pm(m-2k), \pm(m-k), m \pmod{2m}\}$.

The Q^2 identity presented here was first discovered by a computer search and was then proved by a computer-assisted, theoretical argument. In [**5**, Theorems 3 and 4], we proved two other balanced trinomial Q^2 identities. Compared with the abundance of balanced T^2 identities, such identities seem to be rare.

Theorem 4.

$$Q(7,3)Q(35,5) + x^3Q(7,2)Q(35,15) = Q(7,1)Q(35,10). \tag{3.1}$$

Proof. Since (3.1) is a Q^2 identity (balanced at (7, 35)), we can routinely rewrite it as a T^2 identity (balanced at (21/2,105/2)) by first transforming each Q into T terms by the following formula [3, (24)]:

$$Q(m,k) = T\left(\frac{3m}{2}, \frac{m}{2} - 3k\right) - x^kT\left(\frac{3m}{2}, \frac{m}{2} + 3k\right). \tag{3.2}$$

We then have, after reducing, that

$$\begin{aligned} Q(7,3) &= T\left(\frac{21}{2}, -\frac{11}{2}\right) - x^3T\left(\frac{21}{2}, \frac{25}{2}\right) \\ &= T\left(\frac{21}{2}, \frac{11}{2}\right) - xT\left(\frac{21}{2}, \frac{17}{2}\right) \end{aligned}$$

and

$$Q(35,5) = T\left(\frac{105}{2}, \frac{5}{2}\right) - x^5T\left(\frac{105}{2}, \frac{65}{2}\right).$$

(This second equation could also be obtained by writing $Q(7,1)$ as a difference of T's and then sending $x \to x^5$.) Thus, the first term in (3.1) becomes

$$\begin{aligned} Q(7,3)Q(35,5) = T\left(\frac{21}{2}, \frac{11}{2}\right)&T\left(\frac{105}{2}, \frac{5}{2}\right) \\ - x^5T\left(\frac{21}{2}, \frac{11}{2}\right)&T\left(\frac{105}{2}, \frac{65}{2}\right) \\ - xT\left(\frac{21}{2}, \frac{17}{2}\right)&T\left(\frac{105}{2}, \frac{5}{2}\right) \\ + x^6T\left(\frac{21}{2}, \frac{17}{2}\right)&T\left(\frac{105}{2}, \frac{65}{2}\right). \end{aligned} \tag{3.3}$$

In what follows it will be convenient to work with integer components, so we replace x by x^2 throughout. For brevity, we write

$$S(\alpha, l_1, l_2) = x^\alpha T(21, l_1)T(105, l_2).$$

Then (3.3) becomes

$$Q(14,6)Q(70,10) = S(0,11,5) - S(10,11,65) - S(2,17,5) + S(12,17,65).$$

Similarly, the second and third terms in (3.1) (with $x \to x^2$) become

$$x^6 Q(14,4)Q(70,30) = S(6,5,55) - S(16,5,85) - S(10,19,55) + S(20,19,85)$$

and

$$Q(14,2)Q(70,20) = S(0,1,25) - S(20,1,95) - S(2,13,25) + S(22,13,95).$$

(Note that multiplying an S by x^n increases its first argument by n.) Thus (3.1) holds if and only if

$$(3.4)\quad \begin{aligned} &S(0,11,5) + S(2,13,25) + S(6,5,55) + S(12,17,65) \\ &\quad + S(20,1,95) + S(20,19,85) \\ &= S(0,1,25) + S(2,17,5) + S(10,11,65) + S(10,19,55) \\ &\quad + S(16,5,85) + S(22,13,95). \end{aligned}$$

If we next set $m = 3$, $k = 21$ and $u = v = 1$ in Theorem 1, we obtain the special formula

$$(3.5)\quad \sum_{n=-1}^{1} S(14n^2 + 2en, 14n + e + f, 70n + 5e - f) = \sum_{n=-1}^{1} S(14n^2 + 2en, 14n + e - f, 70n + 5e + f).$$

Substituting $(e,f) = (1,10), (4,5)$ and $(-2,15)$ into (3.5), respectively, gives the three identities:

$$(3.6)\quad S(0,11,5) + S(12,3,75) + S(12,17,65) = S(0,9,15) + S(10,19,55) + S(16,5,85),$$

$$(3.7)\quad S(0,9,15)+S(6,5,55)+S(20,19,85) = S(0,1,25)+S(6,15,45)+S(22,13,95),$$

and

$$(3.8)\quad S(0,13,25)+S(4,15,45)+S(18,1,95) = S(0,17,5)+S(8,11,65)+S(10,3,75)$$

Then the linear combination of these equations, (3.6) + (3.7) $+x^2\times$ (3.8), produces (3.4) when the common sum $S(0,9,15)+S(12,3,75)+S(6,15,45)$ is cancelled from its two sides. □

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