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A NATURAL EXTENSION OF A NONSINGULAR ENDOMORPHISM OF A MEASURE SPACE

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0. Introduction. Let (M, Σ, m) be a measure space. An endomorphism of M is a surjective map $S : M \to M$ such that $S^{-1}\Sigma \subseteq \Sigma$. An automorphism is a bijective map S such that S and S^{-1} are endomorphisms.

Since automorphisms are simple kinds of endomorphisms, establishing their properties can be easier than establishing those of general endomorphisms. In certain cases, an endomorphism S has a related automorphism T such that S and T have certain of the same properties. The sense in which T is related to S will be made more precise later. Such an automorphism T will be called a natural extension of S.

An endomorphism S is called measure-preserving if $m(S^{-1}E) = m(E)$ for every E in Σ. Rohlin [**6**, pp. 22–24] established that a measure-preserving endomorphism of a Lebesgue space has a natural extension. Implicit in his proof is the use of some kind of theorem on extension of measures. Cornfeld-Fomin-Sinai [**1**, pp. 239–240] proved Rohlin's result using the Daniell-Kolmogorov theorem, which needs the measure m to have a compact approximation property. Silva [**7**, pp. 8–11] extended Rohlin's result by constructing a natural extension of a nonsingular endomorphism of a standard Borel space. An endomorphism S is called nonsingular when $m(S^{-1}E) = 0$ if and only if $m(E) = 0$ for every E in Σ. Silva uses the skew-product construction to reduce to the measure-preserving case, and from that extension builds a natural extension for the nonsingular endomorphism. Lambert [**3**] claims to have a natural extension of an endomorphism of a general measure space. However, he assumes in addition to nonsingularity that $m(E) = 0$ implies $m(SE) = 0$. This condition is somewhat undesirable since it does not hold even in the measure-preserving case, as shown in the following example suggested by Choksi:

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Let (M, Σ, m) be the product of countable copies of the unit interval together with the Borel sets and Lebesgue measure. Define a map S on M by $S(x_0, x_1, \dots) = (x_1, x_2, \dots)$. If $A = A_0 \times A_1 \times \cdots$, then $S^{-1}A = [0,1] \times A_0 \times A_1 \times \cdots$, so S is easily seen to be measurable and measure-preserving. On the other hand, let $E = \{0\} \times [0,1] \times [0,1] \times \cdots$. Then $m(E) = 0$, but $SE = [0,1] \times [0,1] \times \cdots$ and $m(SE) = 1$.

Also, in [**3**, Theorem 5], to prove countable additivity of a certain set function m, Lambert states that $m(SA \cap S^2B) = m(S(A \cap SB))$ for measurable sets A and B, which does not hold when S is not one-to-one.

I will give a proof of Silva's result without using the skew-product construction and without assuming Lambert's extra hypothesis.

I begin with a description of factor spaces as developed in [**6**].

1. Factor spaces. Let (M, Σ, m) be a measure space and T an automorphism of M. Let ζ be a partition of M, i.e., ζ is a collection of disjoint sets in Σ whose union is M. Note that $T^{-1}\zeta$ is another partition of M. We write $T^{-1}\zeta \le \zeta$ to mean that ζ is a finer partition than $T^{-1}\zeta$. We say then that ζ is invariant with respect to T. In that case, we have a sequence of partitions that are getting finer and finer:

$$\zeta \le T\zeta \le T^2\zeta \le \cdots .$$

Define $\prod_{n=0}^{+\infty} T^n\zeta$ to be the least fine partition that is finer than $T^n\zeta$ for each $n = 0, 1, 2, \dots$. Define ζ to be exhaustive with respect to T if $\prod_{n=0}^{+\infty} T^n\zeta$ is the decomposition of M into individual points. Given such a partition ζ, we build another measure space $(M|\zeta, \Sigma', \mu)$, the factor space of M with respect to ζ: put $M|\zeta \equiv \zeta$. Define $H_\zeta : M \to M|\zeta$ as $H_\zeta(x) = [x]$ where $[x]$ is the unique set in $M|\zeta$ containing x. Define Σ' by putting $A \in \Sigma'$ if and only if $H_\zeta^{-1}A \in \Sigma$. Put $\mu(A) = m(H_\zeta^{-1}A)$ for $A \in \Sigma'$. Then $(M|\zeta, \Sigma', \mu)$ is a measure space and H_ζ is a measure-preserving homomorphism.

I now want to define an endomorphism $T_\zeta : M|\zeta \to M|\zeta$ such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{T} & M \\ {\scriptstyle H_\zeta}\big\downarrow & & \big\downarrow{\scriptstyle H_\zeta} \\ M|\zeta & \xrightarrow[T_\zeta]{} & M|\zeta \end{array}$$

Define $T_\zeta(C) = D$ if and only if $T(C) \subseteq D$. Since $T^{-1}\zeta \leq \zeta$, this makes T_ζ a well-defined map such that $T_\zeta \circ H_\zeta = H_\zeta \circ T$. This last commutativity relation makes checking that T_ζ is measurable easy. Moreover, if T is measure-preserving, so is T_ζ. The automorphism T is called the natural extension of T_ζ. More generally, we have the following definition:

Let S be an endomorphism of a measure space $(X, \mathcal{B}, \nu)$. S is said to have a natural extension T if there exists a measure space (M, Σ, m), an automorphism T of M, and an exhaustive decomposition ζ of M such that $M|\zeta \simeq X$ and $T_\zeta \simeq S$. This last means that there exists a measure-preserving isomorphism $\varphi : M|\zeta \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} M|\zeta & \xrightarrow{T_\zeta} & M|\zeta \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ X & \xrightarrow[T]{} & X \end{array}$$

This definition does not determine the natural extension or its properties and is therefore somewhat vague. In the nonsingular case one may wish to add conditions on the Radon-Nikodym derivative, as is done in Theorem 5.9 (ii) of [**8**]. Dajani and Hawkins discuss cohomologous measures in [**2**]. These conditions make the choice more canonical. They point out that equivalent but noncohomologous measures may have different natural extensions. In this paper the Radon-Nikodym derivative of the natural extension built will be obtained explicitly. We come now to our main result.

2. A natural extension of a nonsingular endomorphism. We have the following:

Theorem. *Let $(X, \mathcal{B})$ be a standard Borel space, ν a finite continuous measure on X with $\nu(X) = 1$, and S a nonsingular endomorphism of X. Then S admits a natural extension.*

Note that it has been shown, for example in [**4**], that it is enough to assume that S is onto almost everywhere.

Proof. Let $M = \{(x_0, x_1, x_2, \dots) : x_i \in X,\ S_{x_{i+1}} = x_i, i = 0, 1, 2, \dots\}$. M is a subset of the product of a countable number of copies of X. I want to define a measure m on the product space that is concentrated on M. To do so, I must first make sure that M is a measurable subset of the product space with the product σ-algebra.

Lemma. *M is a measurable subset of the product σ-algebra.*

Proof. Put $Y^N = \{(x_0, x_1, \dots, x_N) \in \prod_{n=0}^{N} X : S_{x_{i+1}} = x_i, i = 0, \dots, N-1\}$. Let $Z^N = Y^N \times \prod_{n=N+1}^{+\infty} X$. Then $M = \cap_{N=0}^{+\infty} Z^N$. Therefore, to show that M is measurable, it is enough to show that Y^N is a measurable subset of the finite product space. I will prove this by induction on N.

Note that $Y^0 = X$, a measurable subset of X.

$Y^1 = \{(x_0, x_1) \in X \times X : Sx_1 = x_0\}$, i.e., $Y^1 = \{(Sx, x) : x \in X\}$. That is, Y^1 is essentially the graph of a measurable function, and one can show that Y^1 is a measurable subset of $X \times X$.

Now let $N \geq 2$. Note that

$$\begin{aligned} Y^N &= \{(S^N x, S^{N-1} x, \dots, Sx, x) : x \in X\} \\ &= \{(S^{N-1} z, S^{N-2} z, \dots, z, w) : z, w \in X\} \\ &\quad \cap \{(y, S^{N-1} x, \dots, Sx, x) : y, x \in X\}. \end{aligned}$$

That is, $Y^N = (Y^{N-1} \times X) \cap (X \times Y^{N-1})$. By the induction hypothesis, Y^{N-1} is a measurable subset of $\prod_{n=0}^{N-1} X$, so Y^N is a measurable subset of $\prod_{n=0}^{N} X$. Therefore M is a measurable subset of the infinite product σ-algebra, as required.

Now since S is nonsingular, $\nu \circ S^{-1} \ll \nu$. Let h be the Radon-Nikodym derivative. Then h is a measurable function on X, and $0 < h < +\infty$ for almost every (ν), since $\nu \ll \nu \circ S^{-1}$. In [**3**], Lambert proceeds in the following way. He defines a sequence of functions:

$$H_0 \equiv 1$$

$$H_n(x) = \frac{1}{(h \circ S)(x) \cdots (h \circ S^n)(x)}$$

for $n \geq 1$. H_n is defined almost everywhere since $h > 0$ almost everywhere and is measurable since h and S are. Note that

$$H_{n+1} = \frac{H_n \circ S}{h \circ S}$$

for all n. Lambert proves the following:

Lemma.
$$\int_A H_n \, d\nu = \int_{S^{-k}A} H_{n+k} \, d\nu$$
for $A \in \mathcal{B}$ *and for each* n *and each* $k = 0, 1, 2, \ldots$.

Proof. By induction, it is enough to show the lemma for $k = 1$. We have:

$$\int_{S^{-1}A} H_{n+1} \, d\nu = \int_{S^{-1}A} \frac{H_n \circ S}{h \circ S} \, d\nu = \int_A \frac{H_n}{h} \, d\left(\nu \circ S^{-1}\right)$$
$$= \int_A \frac{H_n}{h} \, h d\nu = \int_A H_n \, d\nu$$

as desired. Also, note that

$$\int_X H_n \, d\nu = 1$$

for every n. This is easy to show by induction:

$$\int_X H_0 \, d\nu = \int_X 1 \, d\nu = 1.$$

Furthermore,

$$\begin{aligned} \int_X H_{n+1} \, d\nu &= \int_{S^{-1}X} H_{n+1} \, d\nu \\ &= \int_X H_n \, d\nu \quad \text{(by the lemma above)} \\ &= 1 \quad \text{(by the induction hypothesis).} \end{aligned}$$

Lambert defines a set function m on certain cylindrical sets in M and tries to show directly that m extends to a measure on M. I choose to define m on the whole product space in the following way: define a measure m_N on the finite product space $\prod_{n=0}^{N} X$ by putting

$$m_N(A_0 \times A_1 \times \cdots \times A_N) = m_N\left((A_0 \times A_1 \times \cdots \times A_N) \cap Y^N\right)$$
$$= \int_X \chi_{S^{-N}A_0 \cap S^{-(N-1)}A_1 \cap \cdots \cap A_N} H_N \, d\nu$$

for $A_0, A_1, \ldots, A_N$ in $\mathcal{B}$. I want to show that m_N is a measure on $\prod_{n=0}^{N} X$. To do so, it is enough to show that m_N is countably additive on rectangles. Suppose that

$$A_0 \times A_1 \times \cdots \times A_N = \bigcup_{i=1}^{+\infty} \left(A_0^i + A_1^i \times \cdots \times A_N^i\right)$$

is a disjoint union. Then

$$(A_0 \times A_1 \times \cdots \times A_N) \cap Y^N = \bigcup_{i=1}^{+\infty} \left(A_0^i \times A_1^i \times \cdots \times A_N^i\right) \cap Y^N.$$

This union is also disjoint. Note that

$$\left(S^N x_N, S^{N-1} x_N, \ldots, x_N\right) \in A_0 \times A_1 \times \cdots \times A_N$$

if and only if

$$x_N \in S^{-N} A_0 \cap S^{-(N-1)} A_1 \cap \cdots \cap A_N.$$

We therefore have that

$$S^{-N} A_0 \cap S^{-(N-1)} A_1 \cap \cdots \cap A_N$$
$$= \bigcup_{i=1}^{+\infty} \left(S^{-N} A_0^i \cap S^{-(N-1)} A_1^i \cap \cdots \cap A_N^i\right)$$

and this union is disjoint. Therefore, by the monotone convergence theorem,

$$m_N(A_0 \times A_1 \times \cdots \times A_N) = \int_X \chi_{S^{-N}A_0 \cap S^{-(N-1)}A_1 \cap \cdots \cap A_N} H_N \, d\nu$$
$$= \int_X \sum_{i=1}^{+\infty} \chi_{S^{-N}A_0^i \cap \cdots \cap A_N^i} H_N \, d\nu$$
$$= \sum_{i=1}^{+\infty} \int_X \chi_{S^{-N}A_0^i \cap \cdots \cap A_N^i} H_N \, d\nu$$
$$= \sum_{i=1}^{+\infty} m_N\left(A_0^i \times A_1^i \times \cdots \times A_N^i\right).$$

Therefore, m_N is countable additive on rectangles and thus extends to a measure on $\prod_{n=0}^{N} X$.

Moreover, note that the m_N form a compatible sequence of measures, that is,

$$m_{N+1}(A_0 \times \cdots \times A_N \times X) = m_N(A_0 \times \cdots \times A_N).$$

This is because

$$\begin{aligned} m_{N+1}(A_0 \times \cdots \times A_N \times X) &= \int_X \chi_{S^{-(N+1)}A_0 \cap \cdots \cap S^{-1}A_N \cap X} H_{N+1}\, d\nu \\ &= \int_X \chi_{S^{-1}(S^{-N}A_0 \cap \cdots \cap A_N)} H_{N+1}\, d\nu \\ &= \int_X \chi_{S^{-N}A_0 \cap \cdots \cap A_N} H_N\, d\nu \end{aligned}$$

(by the lemma, since S is measurable)

$$= m_N(A_0 \times \cdots \times A_N)$$

Therefore, by the Daniell-Kolmogorov theorem [**5**, Theorem 5.1], since a product of standard Borel spaces is standard Borel, the m_N extend to a unique measure m on the infinite product space.

From the way m is constructed, we see that m is zero outside of M. We have, moreover, that $m(M) = 1$, since:

$$\begin{aligned} m(Z^N) &= m\biggl(Y^N \times \prod_{n=N+1}^{+\infty} X\biggr) = m_N\left(Y^N\right) \\ &= m_N\left((X \times \cdots \times X) \cap Y^N\right) \\ &= \int_X \chi_{S^{-N}X \cap \cdots \cap X} H_N\, d\nu = \int_X H_N\, d\nu = 1. \end{aligned}$$

Therefore $m(M) = \lim_{N \to +\infty} m(Z^N) = 1$. Let Σ be the σ-algebra of measurable sets of the product space intersected with M. Then I have a measure space (M, Σ, m).

Now I need to define T on M so that T is an automorphism extending S. Put

$$T(x_0, x_1, x_2, \dots) = (Sx_0, x_0, x_1, \dots).$$

Then T is easily seen to be bijective. Note that

$$T^{-1}(y_0, y_1, y_2, \dots) = (y_1, y_2, \dots) \quad \text{for } (y_0, y_1, y_2, \dots) \in M.$$

I claim that T is an automorphism. To show this, begin by noting that Σ is generated by sets of the form

$$\begin{aligned}(A)_n &= (X \times \cdots \times X \times A \times X \times \cdots) \cap M \\ &= \{x = (x_0, x_1, \dots) \in M : x_n \in A\},\end{aligned}$$

where $A \in \mathcal{B}$ and $n = 0, 1, 2, \dots$. Therefore, to show that T is an automorphism, it is enough to show that T^{-1} and T send sets of this form to measurable sets. We have, for $n \geq 1$,

$$\begin{aligned}T^{-1}(A)_n &= \left\{T^{-1}x : x_n \in A\right\} = \{(x_1, x_2, \dots) : x_n \in A\} \\ &= \{y = (y_0, y_1, \dots) : y_{n-1} \in A\} = (A)_{n-1}.\end{aligned}$$

For $n = 0$,

$$T^{-1}(A)_n = \{(x_1, \dots) : x_0 = Sx_1 \in A\} = \left(S^{-1}A\right)_0.$$

Also, for any n,

$$\begin{aligned}T(A)_n &= \{Tx : x_n \in A\} = \{(Sx_0, x_0, x_1, \dots) : x_n \in A\} \\ &= \{x \in M : x_{n+1} \in A\} = (A)_{n+1}.\end{aligned}$$

This shows that T^{-1} and T are measurable. Moreover, T is actually nonsingular, since for $n \geq 1$,

$$m\left(T^{-1}(A)_n\right) = m((A)_{n-1}) = \int_A H_{n-1}\, d\nu.$$

On the other hand,

$$m((A)_n) = \int_A H_{n-1}\, d\nu.$$

Since the H_n are strictly positive almost everywhere, either of the above integrals is zero if and only if $\nu(A)$ is zero which in turn forces the other integral to be zero. Since S is nonsingular, the same argument holds for $n = 0$. This shows that T is nonsingular when restricted to

(M, Σ_n), where Σ_n is the σ-algebra $\Sigma_n = \{(A)_n : A \in \mathcal{B}\}$. To show nonsingularity for any measurable set, proceed as follows. Fix $n \geq 1$ and consider the measures $m \circ T^{-1}$ and m restricted to Σ_n. We already know that

$$m\left(T^{-1}(A)_n\right) = m((A)_{n-1}) = \int_A H_{n-1}(t)\, d\nu(t)$$

(by the lemma). On the other hand,

$$m((A)_n) = \int_A H_n(t)\, d\nu(t).$$

Now think of ν as a measure on (M, Σ_n); that is, put $\nu_n((A)_n) = \nu(A)$, and for $x = (x_0, x_1, \dots) \in M$ put $H_n(x) = H_n(x)$. Then ν_n is a measure on (M, Σ_n), and it is easy to see that

$$\begin{aligned} m\left(T^{-1}(A)_n\right) &= \int_A H_{n-1}(x_n)\, d\nu(x_n) \\ &= \int_{(A)_n} \tilde{H}_{n-1}(x)\, d\nu_n(x), \end{aligned}$$

where $x = (x_0, x_1, \dots) \in M$. Also,

$$m((A)_n) = \int_{(A)n} \tilde{H}_{n-1}(x)\, d\nu_n(x).$$

Therefore, we have three equivalent measures $m \circ T^{-1}$, m, ν_n on (M, Σ_n) and

$$\begin{aligned} \frac{d\left(m \circ T^{-1}\right)}{dm}(x) &= \frac{d\left(m \circ T^{-1}\right)}{d\nu_n}(x) \cdot \frac{d\nu_n}{dm}(x) \\ &= \frac{\tilde{H}_{n-1}(x)}{\tilde{H}_n(x)} = \frac{H_{n-1}(x_n)}{H_n(x_n)} \\ &= \frac{1/(h(x_{n-1}) \cdots h(x_1))}{1/(h(x_{n-1}) \cdots h(x_1) h(x_0))} \\ &\qquad\qquad \text{(by definition of } H_{n-1} \text{ and } H_n\text{)} \\ &= h(x_0) \end{aligned}$$

For $n = 0$, we have

$$\begin{aligned} m\left(T^{-1}(A)_0\right) &= m\left(\left(S^{-1}A\right)_0\right) \\ &= \int_{S^{-1}A} H_0(x_0)\,d\nu(x_0) = \int_A h(x_0)\,d\nu(x_0). \end{aligned}$$

On the other hand,

$$m((A)_0) = \int_A H_0(x_0)\,d\nu(x_0) = \int_A 1\,d\nu(x_0).$$

Therefore we have

$$\frac{d\left(m\circ T^{-1}\right)}{dm}(x) = h(x_0)$$

on (M, Σ_0) as well. This shows that, for any n,

$$\left(m\circ T^{-1}\right)((A)_n) = \int_{(A)_n} h(\pi_0(x))\,dm(x)$$

($\pi_0 : M \to X$ is the projection onto the first coordinate). Since the $(A)_n$ generate Σ and $h(\pi_0(x)) \geq 0$, by approximation we get that, for any $E \in \Sigma$,

$$\left(m\circ T^{-1}(E)\right) = \int_E h(\pi_0(x))\,dm(x).$$

Therefore, $m \circ T^{-1} \ll m$, and $(d(m\circ T^{-1})/dm)(x) = h(\pi_0(x))$. Similarly, for fixed n,

$$\begin{aligned} m(T(A)_n) &= \int_A H_{n+1}(x_n)\,d\nu(x_n) \\ &= \int_A \frac{1}{h(x_{n-1})\cdots h(x_0)h(Sx_0)}\,d\nu(x_n). \end{aligned}$$

Also,

$$\begin{aligned} m((A)_n) &= \int_A H_n(x_n)\,d\nu(x_n) \\ &= \int_A \frac{1}{h(x_{n-1})\cdots h(x_0)}\,d\nu(x_n). \end{aligned}$$

Therefore,

$$\frac{d(m \circ T)}{dm}(x) = \frac{1}{h(Sx_0)} = \frac{1}{(h \circ S)(\pi_0(x))}.$$

The above Radon-Nikodym derivative is independent of n and so again by approximation, for any $E \in \Sigma$,

$$m(T(E)) = \int_E \frac{1}{(h \circ S)(\pi_0(x))}\, dm(x).$$

That is, $m \circ T \ll m$, or $m \ll m \circ T^{-1}$. Therefore, T is nonsingular. Note that if S is measure-preserving, the H_n are all equal to 1. In that case, T is also measure-preserving, by the above.

What is the desired decomposition ζ? Define an equivalence relation on M by putting $x \sim x'$ if and only if $x_0 = x'_0$. This relation gives rise to a partition ζ. Then

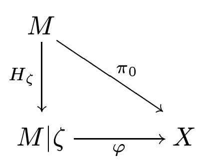
$$M|\zeta = \{[(x_0, \dots)] : x_0 \in X\}.$$

I claim that ζ is exhaustive with respect to T. One way of proving this is to show that any point of M can be obtained as an intersection $\cap_{n=0}^{+\infty} T^n C_n$ where $C_n \in \zeta$. Let $z = (z_0, z_1, \dots) \in M$. Let $C_0 = [(z_0, \dots)]$. Then $z \in C_0$. Let $C_1 = [(z_1, \dots)]$. Then

$$TC_1 = \{y = (y_0, y_1, \dots) : y_0 = z_0, y_1 = z_1\}.$$

Note that $C_0 \supseteq TC_1$ and that $z \in TC_1$. In general, put $C_n = [(z_n, \dots)]$. Then $T^n C_n = \{y \in M : y_0 = z_0, \dots, y_n = z_n\}$. This gives that $T^n C_n \in \zeta$ and $\cap_{n=0}^{+\infty} T^n C_n = \{z\}$ as desired.

Now define $\varphi : M|\zeta \to X$ as $\varphi[(x_0, \dots)] = x_0$. φ is clearly bijective. Note that



where $\pi_0(x_0, x_1, \dots) = x_0$. φ is indeed an isomorphism. Given $A \in \mathcal{B}$, we have that $\varphi^{-1}A$ is measurable if and only if $H_\zeta^{-1}\varphi^{-1}A$ is measurable in M. But $H_\zeta^{-1}\varphi^{-1}A = \pi_0^{-1}A = (A)_0 \in \Sigma$. Also,

$$\mu(\varphi^{-1}A) = m\left(H_\zeta^{-1}\varphi^{-1}A\right) = m((A)_0)$$
$$= \int_A H_0 \, d\nu = \nu(A).$$

On the other hand, let A be measurable in $M|\zeta$, that is, $H_\zeta^{-1}A$ is measurable in M. We have $\pi_0(H_\zeta^{-1}A) = \varphi A$. Since m is concentrated on M, $\varphi(A) \times X \times X \times \cdots$ differs from the measurable set $(\varphi(A) \times X \times X \times \cdots) \cap M = H_\zeta^{-1}A$ by a set of measure zero and hence is measurable in the product space. Therefore, $\varphi(A)$ is measurable in X and

$$\nu(\varphi(A)) = m((\varphi A)_0) = m\left(H_\zeta^{-1}A\right) = \mu(A).$$

So φ is a measure-preserving isomorphism. That is, $M|\zeta \simeq X$. Finally, note that

$$T[(x_0, \dots)] \subseteq [(Sx_0, \dots)].$$

Therefore, $T_\zeta[(x_0, \dots)] = [(Sx_0, \dots)]$, making the following diagram commutative:

$$\begin{array}{ccc} M|\zeta & \xrightarrow{T_\zeta} & M|\zeta \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ X & \xrightarrow[S]{} & X \end{array}$$

That is, $T_\zeta \simeq S$. Therefore, T is a natural extension of S, and the theorem is proved. $\square$

It is to be noted that the original measures and the choice of the H_n play an important role in the nature of the extension, although once the choice is made, the extension measure is unique. As we discussed earlier, Dajani and Hawkins [**2**] point out that even equivalent but noncohomologous measures can give rise to different natural extensions.

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REFERENCES

1. J.P. Cornfeld, S.V. Fomin and Ya.G. Sinai, *Ergodic theory*, Springer-Verlag, New York, 1982.

2. K.G. Dajani and J.M. Hawkins, *Examples of natural extensions of nonsingular endomorphisms*, Proc. Amer. Math. Soc. **120** (1994), 1211–1217.

3. A. Lambert, *A bi-measurable transformation generated by a non-measure preserving transformation*, Rocky Mountain J. Math. **20** (1990), 460–474.

4. D. Maharam, *Invertible liftings of measurable transformations*, Israel J. Math. **73** (1991), 225–245.

5. K.R. Parthasarathy, *Probability measures on the metric spaces*, Academic Press, New York, 1967.

6. V.A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Amer. Math. Soc. **39** (1964), 1–36.

7. C.E. Silva, *On μ-recurrent non-singular endomorphisms*, Israel J. Math. **61** (1988), 1–13.

8. C.E. Silva and P. Thieullen, *The subadditive ergodic theorem and recurrence properties of Markovian transformations*, J. Math. Anal. Appl. **154** (1991), 83–99.

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