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SUBGROUP SEPARABILITY OF CERTAIN HNN EXTENSIONS OF FINITELY GENERATED ABELIAN GROUPS

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ABSTRACT. In this note we give a characterization for certain HNN extensions of finitely generated abelian groups to be subgroup separable.

1. A group G is called subgroup separable if, for each finitely generated subgroup M and for each $x \in G\backslash M$, there exists a normal subgroup N of finite index in G such that $x \notin MN$. It is well known that free groups and polycyclic groups (and hence finitely generated abelian groups) are subgroup separable [**6**, **7**].

Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension where K is a finitely generated abelian group, A and B are the associated subgroups and φ is the associated isomorphism $\varphi : A \to B$. Suppose A and B have finite index in K. In this note we give a characterization for the HNN extension G to be subgroup separable. We shall prove the following:

Theorem 1. *Let* $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ *be an* HNN *extension where* K *is a finitely generated abelian group and* A *and* B *have finite index in* K. *Then the following are equivalent*:

(i) G *is subgroup separable*;

(ii) *Either* $K = A = B$ *or there exists a subgroup* H *of finite index in* K *and* H *is normal in* G;

(iii) *There exists a finitely generated abelian group* X *such that* K *is a subgroup of finite index in* X *and an automorphism* $\bar{\varphi} \in \operatorname{Aut} X$ *with* $\bar{\varphi}|_A = \varphi$.

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Theorem 1 strengthens the results of Andreadakis, Raptis and Varsos in [**2, 3, 4**] and Wong in [**8**]. We shall prove Theorem 1 in Sections 2 and 3 and then give some applications in Section 4.

The notations used in this note are standard. In addition, the following will be used. Let G be a group.

$N <_f G$, respectively $N \triangleleft_f G$, means N is a subgroup, respectively normal subgroup, of finite index in G, f.g. means finitely generated, s.s. means subgroup separable.

2. In this section we prove (i) $\Leftrightarrow$ (ii) in Theorem 1. We begin with the following lemma.

Lemma 1. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ where K is a finite group. Then G is subgroup separable.*

Proof. See Wong [**8**]. □

Lemma 2. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ where K is a finitely generated abelian group and A and B have finite index in K. If either $K = A = B$ or there exists a subgroup H of finite index in K and H is normal in G, then G is subgroup separable.*

Proof. Suppose $K = A = B$. Then $G = \langle t, K; t^{-1}Kt = K, \varphi\rangle$ is a split extension of the finitely generated abelian group K by the infinite cyclic group $\langle t\rangle$. Hence, by Theorem 4 of [**1**], G is s.s.

Suppose there exists a subgroup H of finite index in K and H is normal in G. Let M be an f.g. subgroup of G and $x \in G\backslash M$. If $x \notin MH$, then $xH \notin MH/H$. Now $G/H \simeq \langle t, K/H; t^{-1}A/Ht = B/H, \bar{\varphi}\rangle$ where $\bar{\varphi} : A/H \to B/H$ is the isomorphism induced by φ. Therefore G/H is s.s. by Lemma 1. Thus, there exists $N/H \triangleleft_f G/H$ such that $xH \notin MH/H \cdot N/H$, namely, there exists $N \triangleleft_f G$ such that $x \notin MN$.

Suppose that $x \in MH$. Then $x = mh$, $m \in M$, $h \in H$ but $h \notin H \cap M$ (since $x \notin M$). Now H and $H \cap M$ are f.g. abelian. Since H is s.s., [**7**], there exists a characteristic subgroup R of H of finite index in it such that $h \notin (H \cap M)R$. If $xR \in MR/R$, then $x = mh = m_1 r$, $m_1 \in M$, $r \in R$. Hence, $hr^{-1} = m^{-1}m_1 \in H \cap M$ (since $R < H$) and so $h \in (H \cap M)R$, a contradiction. So $xR \notin MR/R$. Now, again by Lemma 1, the group G/R is s.s. So we can argue as before, with R in place of H and find $N \lhd_f G$ such that $x \notin MN$. This completes the proof of the lemma. □

To prove the converse of Lemma 2, we need the next lemma.

Lemma 3. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ where K is a finitely generated abelian group. If $K = A \neq B$ or $K = B \neq A$, then G is not subgroup separable.*

Proof. We prove only the case $K = A \neq B$. The other case is similar. Let $X = \{a, b, \dots, f\}$ be a minimal generating set for $K = A$ such that $a \notin B$ (since $K = A \neq B$). Then $t^{-n}at^n = a\varphi^n \in B$, for all $n \in \mathbf{Z}^+$, since $K\varphi = A\varphi = B$. Let $G\theta$ denote a homomorphic image of G of order r. Then $a\theta = t^{-r}\theta a\theta t^r\theta = (t^{-r}at^r)\theta = (a\varphi^r)\theta \in B\theta$. Thus G is not s.s. □

Lemma 4. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ where K is a finitely generated abelian group and A, B have finite index in K. If G is subgroup separable, then either $K = A = B$ or there exists a subgroup H of finite index in K and H is normal in G.*

Proof. Suppose G is s.s. Then G is residually finite and so, by Theorem 1 of [**2**], either $K = A$ or $K = B$ or there exists a subgroup H of finite index in K and H is normal in G. However, Lemma 3 shows that, if $K = A \neq B$ or $K = B \neq A$, then G is not s.s. Hence, the result follows. □

3. In this section we prove (ii) $\Leftrightarrow$ (iii) in Theorem 1. First we quote the following two lemmas from Andreadakis, Raptis and Varsos [**4**].

Lemma 5. *Let K be a free abelian group of finite rank $r(K) = n$. Let A and B be subgroups of finite index in K and $\varphi : A \to B$ an isomorphism. Suppose there exists a subgroup H of finite index in K, $H < A \cap B$ and $H\varphi = H$. Then there exists a free abelian group X*

with $r(K) = r(X)$ such that K is a subgroup of finite index in X and an automorphism $\bar{\varphi} \in \operatorname{Aut} X$ with $\bar{\varphi}|_A = \varphi$.

Proof. This is Proposition 1 in [**4**]. $\square$

Lemma 6. *Let K be a finite abelian group. Let A and B be subgroups of K and $\varphi : A \to B$ an isomorphism. Then there exists a finite abelian group X such that K is a subgroup of X and an automorphism $\bar{\varphi} \in \operatorname{Aut} X$ with $\bar{\varphi}|_A = \varphi$.*

Proof. This is Proposition 3 in [**4**]. $\square$

Lemma 7. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ where K is a finitely generated abelian group and A and B have finite index in K. If either $K = A = B$ or there exists a subgroup H of finite index in K and H is normal in G then there exists a finitely generated abelian group X such that K is a subgroup of finite index in X and an automorphism $\bar{\varphi} \in \operatorname{Aut} X$ with $\bar{\varphi}|_A = \varphi$.*

Proof. If $K = A = B$, then φ is an automorphism of K. We take $X = K$ and $\bar{\varphi} = \varphi$. Suppose that there exists a subgroup H of finite index in K and H is normal in G.

If T is the torsion subgroup of K, then $A \cap T$ and $B \cap T$ are the torsion subgroups of A and B, respectively. Then $K \simeq (K/T) \times T$, $A \simeq (AT/T) \times (A \cap T)$ and $B \simeq (BT/T) \times (B \cap T)$. Also the isomorphism φ induces the isomorphism $\varphi_1 : AT/T \to BT/T$ with $(aT)\varphi_1 = (a\varphi)T$ and the restriction $\varphi_2 : A \cap T \to B \cap T$.

Let $K_1 = K/T$, $A_1 = AT/T$, $B_1 = BT/T$ and $H_1 = HT/T$. Clearly, K_1, A_1, B_1, H_1 and φ_1 satisfy the hypothesis of Lemma 5. Hence, there exists a free abelian group X_1 such that K_1 is a subgroup of finite index in X_1 and an automorphism $\bar{\varphi}_1 \in \operatorname{Aut} X_1$ with $\bar{\varphi}_1|_{A_1} = \varphi_1$.

For the finite abelian groups T, $A_2 = A \cap T$, $B_2 = B \cap T$ and the isomorphism φ_2, by Lemma 6, there exists a finite abelian group X_2 such that T is a subgroup of X_2 and an automorphism $\bar{\varphi}_2 \in \operatorname{Aut} X_2$ with $\bar{\varphi}_2|_{A_2} = \varphi_2$.

We set $X = X_1 \times X_2$, $\bar\varphi = (\bar\varphi_1, \bar\varphi_2)$. Then $K_1 \times T$ is a subgroup of finite index in X, $\bar\varphi$ is an automorphism of X and, furthermore, $\bar\varphi|_{A_1 \times A_2} = (\varphi_1, \varphi_2)$. The lemma now follows with the obvious identification of K with $K_1 \times T$, A with $A_1 \times A_2$ and B with $B_1 \times B_2$. $\square$

To prove the converse of Lemma 7, we need the next lemma.

Lemma 8. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ where K is a finitely generated abelian group and A and B have finite index in K. If φ comes from an automorphism of K, then there exists a subgroup H of finite index in K and H is normal in G.*

Proof. Let $S = A \cap B$. Then S has finite index, say s, in K. Since K is f.g., there exists a finite number of subgroups of index s in K. Let H be the intersection of all these subgroups of index s in K. Then H is characteristic in K and H has finite index in K. Since φ is an automorphism of K and $H < A \cap B$, we have $H \triangleleft G$. $\square$

Lemma 9. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ where K is a finitely generated abelian group and A and B have finite index in K. If there exists a finitely generated abelian group X such that K is a subgroup of finite index in X and an automorphism $\bar\varphi \in \operatorname{Aut} X$ with $\bar\varphi|_A = \varphi$, then there exists a subgroup H of finite index in K and H is normal in G.*

Proof. Let $G^* = \langle t, X; t^{-1}At = B, \varphi\rangle$. Now φ comes from the automorphism $\bar\varphi$ of X and hence, by Lemma 8, there exists a subgroup H of finite index in X, $H < A \cap B$ and H is normal in G^*. This implies that H has finite index in K (since $K < X$) and H is normal in G (since $G < G^*$). $\square$

Theorem 1 now follows from Lemmas 2, 4, 7 and 9.

4. We give some applications of our theorem. We restate Lemma 8 and compare it with Lemma 4.4 of B. Baumslag and Tretkoff [**5**], which states that $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ is residually finite if K is residually finite, K is A-separable and B-separable and φ comes from an automorphism of K.

Corollary 1. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ where K is a finitely generated abelian group and A and B have finite index in K. If φ comes from an automorphism of K, then G is subgroup separable.*

Corollary 2. *Let*

$$G = \langle t, a_1, a_2, \dots, a_n; t^{-1}a_i^{h_i}t = a_i^{d_i}, \quad i = 1, 2, \dots, n, [a_i, a_j] = 1\rangle$$

and $K = \langle a_1, a_2, \dots, a_n; [a_i, a_j] = 1\rangle$.

Then the following are equivalent:

(i) *G is subgroup separable*;

(ii) $|h_i| = |d_i|$, $i = 1, 2, \dots, n$;

(iii) *the map φ which sends $a_i^{h_i}$ to $a_i^{d_i}$, $i = 1, 2, \dots, n$ comes from an automorphism of K.*

Proof. Let $K = \langle a_1, a_2, \dots, a_n; [a_i, a_j] = 1\rangle$ be the free abelian group of rank n and $A = \langle a_1^{h_1}, a_2^{h_2}, \dots, a_n^{h_2}\rangle$, $B = \langle a_1^{d_1}, a_2^{d_2}, \dots, a_n^{d_n}\rangle$ and $a_i^{h_i}\varphi = a_i^{d_i}$, $i = 1, 2, \dots, n$. Then $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ and G satisfy the hypothesis of Theorem 1.

We show (i) $\Rightarrow$ (ii). Suppose G is s.s. Then, by Theorem 1, either $K = A = B$ or there exists a subgroup H of finite index in K and H is normal in G. If $K = A = B$, then trivially $|h_i| = |d_i| = 1$, $i = 1, 2, \dots, n$ and we are done. Now suppose that there exists a subgroup of H of finite index in K and H is normal in G. Since $H < A \cap B$, we can write $H = \langle a_1^{c_1}, a_2^{c_2}, \dots, a_n^{c_n}\rangle$ with $h_i|c_i$, $d_i|c_i$. But

$$\begin{aligned}(a_i^{c_i})\varphi &= (a_i^{c_ih_i/h_i})\varphi = (a_i^{h_i}\varphi)^{c_i/h_i}\\ &= a_i^{d_ic_i/h_i} = (a_i^{c_i})^{d_i/h_i}.\end{aligned}$$

Since $H\varphi = H$, we have $|h_i| = |d_i|$, $i = 1, 2, \dots, n$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) follows from Corollary 1.

From Corollary 2 we obtain the fact that the Baumslag-Solitar groups $G = \langle t, a; t^{-1}a^ht = a^k\rangle$ is subgroup separable if and only if $|h| = |k|$ (see Wong [**8**]). □

Corollary 2 also extends Corollary 3 of Andreadakis, Raptis and Varsos [**2**].

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